**Numerical Linear Algebra**
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**Lecture - 44**
**Orthogonal Projections**

Hello friends, welcome to today's lecture, if you recall, in previous lecture, we have discussed the singular value decomposition of a rectangular matrix and we have seen, how to find out say the matrix u S and v which are known as the singular decomposition component of a matrix a and we also have seen certain example where we have calculated our orthogonal matrices u and v such that a can be written as u transpose S and v.

And we also have seen that the columns of the orthogonal matrix u is known as left singular vector of a and columns of v are known as right singular vector of matrix a and ah. So, we in this lecture, we continue our study and we will discuss certain geometrical properties and algebraic properties of this matrix a with the help of singular value decomposition.

So, in today's class, we start with the orthonormal projection. So, we need to understand; what is orthogonal projection first of all. So, let us say that let S be a subspace of R n.

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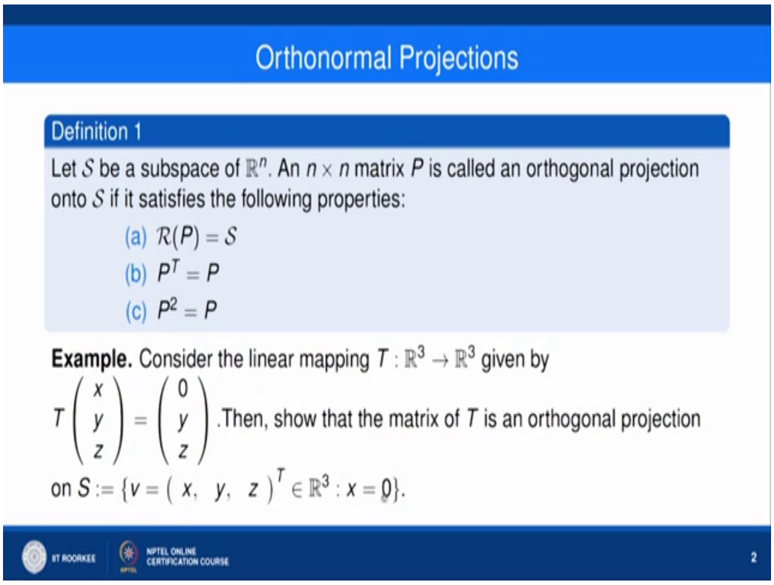

So, R n is the working vector space where we have working and S be a subspace of R n and any n cross n matrix P is called an orthogonal projection on to S if it satisfy the following three properties the first property that range of P is S. So, basically P is the map from R n to S and here it follows these condition that range of P is S and P is a symmetric matrix it means that P transpose equal to P and P square equal to P means, this P matrix is basically an idempotent matrix. So, if matrix P satisfy these three properties, we call this P as orthogonal projection on to S. So, let us take one simple example. So, here we have say linear mapping T form R 3 to R 3 given by T of x y z which map this vector x y z to 0 y z here.

So, here if you look at the first column is getting the 0 and rest it is as it is. So, we can say that it is basically mapping from R 3 to if you look at the image space image space is nothing, but y z space, then we want to show that the matrix of T is an orthogonal projection on S where S is defined as v as x y z transpose which is a element of R 3 such that x is equal to 0. So, it means that it is y z plane here to show that matrix of T is an orthogonal projection on S. So, we need to first find out say what is the matrix of T.

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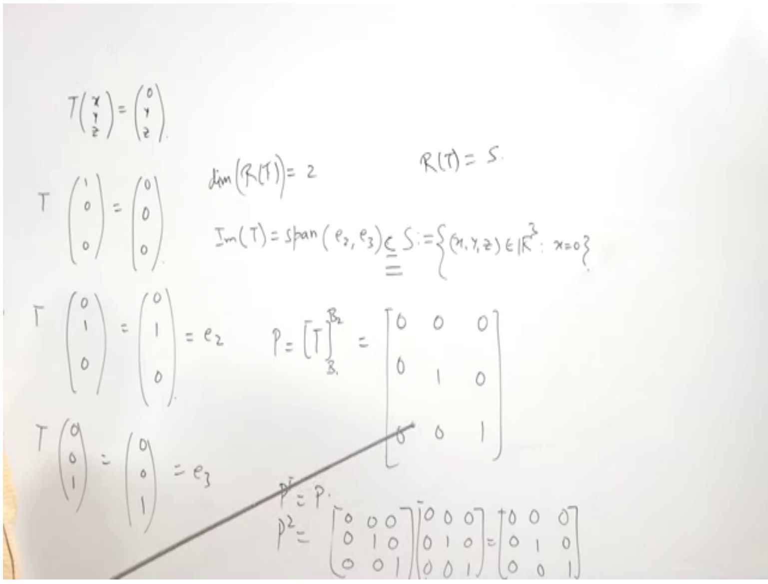

So, if you look at T is defined as this that T operating on x y z is giving a element 0 y z, it means that the first component is 0 the remaining two elements are same. So, to find out a matrix of T, we need to operate this T on the basis of R 3 and we have to write it again in their terms of basis of R 3.

So, let us take the standard basis here for R 3. So, let us T operate on e 1. So, e 1 is 1 0 0, if you look at if you follow this mapping, then it will give you 0 0 0 and T operating on e 2 that is 0 1 0, it will give you e 2 0 1 0 which is written here as e 2. Similarly, T operating on 0 0 1 which will give you 0 0 1 and it is nothing, but e 3. So, you can say that the image of T if you write the same in terms of e 1, e 2, e 3, you will get the same component e 2 and e 3. So, here you can say that image of T is nothing, but span of e 2 and e 3 because this is 0 vector. So, it will not span anything. So, image of T is span of e 2 and e 3. So, it means that it will span a set whose first component is 0 and y z, it can be any element in R. So, we can say that a span of this is a subset of this S you can say like this.

That span of S is a subset of this S. Now, if you look at here the, what is the dimension of image of T or dimension of range space of P. So, T here. So, it is T here. So, here if you look at that is these spanings vectors e 2 and e 3 are linearly independent. So, it will span say a subspace whose dimension is two and if you look at this subspace S its S dimension is 2 here. So, we can say that this equality this a is not a it is not a subset of this, but it is actually the same as this. So, it means that that image of T or we can say that range of T is nothing, but this vector space S. So, here the first condition follows that range of T is whole of S. Now to prove that it is an orthogonal projection we have to show that it is symmetric. So, if you look at the matrix T with standard bases b 1 and b 2 here, here, b 1 is same as the u 1, u 2 and b 2 is also the same standard basis.

So, here if you calculate the matrix P which is a matrix representation of this linear operator T, then 0 0 0; so, first column is this and second column is e 2 and third column is e 3. So, here this is the matrix of linear mapping T now we have to look at whether it satisfy the remaining two properties. So, first property is that it is a symmetric matrix. So, a it is quite easily, we can see that it is a triangular matrix.

So, P transpose is equal to P and here we to show the last property that is P is an idempotent matrix. So, look at P square. So, P square you calculate P into P and if you calculate it is coming out to be P again. So, it means that it satisfy all the properties listed as condition for orthogonal projection. So, we can say that here matrix of T is in orthogonal projection on to this vector subspace of R 3 where S is defined as all those x y z in R 3 such that the first component x is 0 here.

So, here we have seen one example now let us move to next. So, here we say that if ah.

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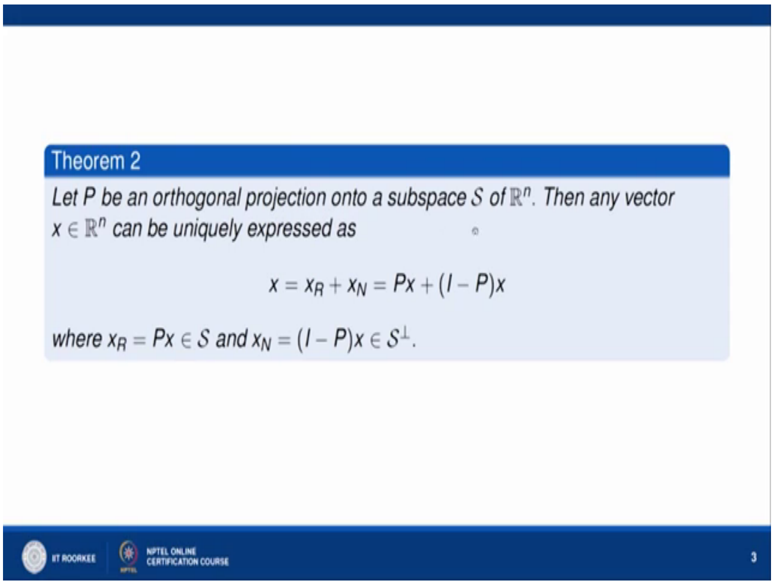


We have a an orthogonal projection on to a subspace S of R n then any vector x in R n can be uniquely expressed as x as x R plus x N where x R is given by P x and it belongs to the subspace S and x N which is defined as i minus P x which is going to be element of S perp. So, S perp is defined as set of all vectors orthogonal to elements of S all the element of S here.

So, we can say that every element x can be written in this form. So, let prove this theorem small theorem and why we are proving all this theorem because it will with the help of this theory we are going to find out say orthogonal projection on to say range space of a null space of a and range space of a transpose and null space of a transpose we with the help of this theory, we are finding out the orthogonal projection on these subspaces.

So, to prove this theorem let us look at here. So, to show that that if P is an orthogonal projection on to subspace of R n then any vector x of R n can be uniquely expressed as x R plus x N where x R is written as a P x where P P x will be the element of x and i minus P x where i minus P x is an element of S perp and this representation is a unique representation. So, that we wanted to show it here. So, for that we recall that T is a orthogonal projection from R n to s. So, it means that x goes to P of x.

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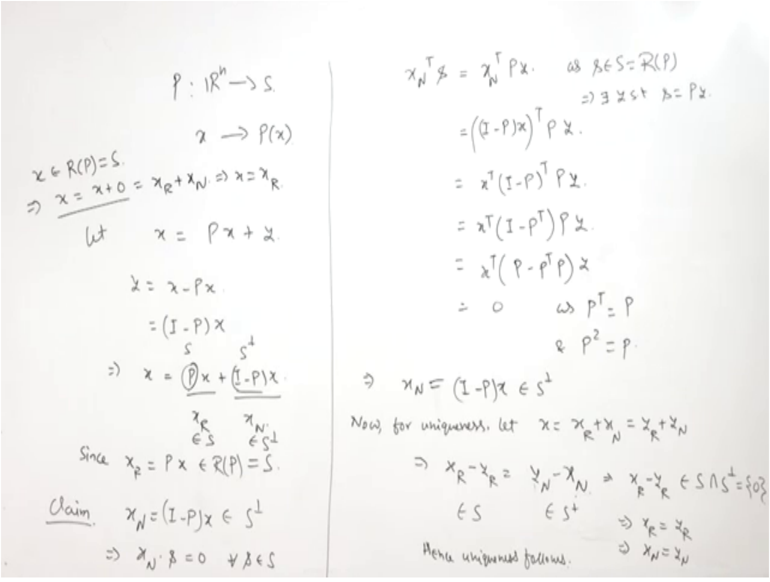

So, it means that a any element x in R n can be written as P x plus some element of y here. Now here y is an element R n here. So, to find out this y you just look at that y can be written as x minus P of x and this can be written as i minus P x.

So, it means that this suggest that if we write x as P x plus some y then y can be written as i minus P of x. So, it means that x can be written as P of x plus i minus P of x and we call this P x as x R and we call this as x N, we want to show that this x R is an S and x N is an S perp to show that we just recall that look at here x R is nothing, but P of x. So, P of x is means what that this x R is going to be the element of range space of P. Now what is range space of P range space of P is nothing, but your S. So, it means that x R is an element of S here. Now our claim is that this x N which is defined as i minus P into x we claim that this is an element of S perp. So, to show that it is an element of S perp we have to show that this that the dot product of x n dot S is going to be 0 for every S of S in a capitals S.

So, it means that this x N is going to be orthogonal to every element of this s. So, to show this look at this x N dot S x N dot S is written as x N transpose S. Now here we already know that if S belongs to this S where this capital S is nothing, but range space of P because we already know that P is an orthogonal projection on to this vector space s. So, it means that S belongs to range space of P. So, it means it means that there exist a y such that S can be written as P of y.

So, utilizing this expression for this small S we can say that x N transpose S can be written as x N transpose P of y now to further simplify let us utilize the utilize the the expression for x N where x N we are denoting as i minus P x. So, writing x N transpose as i minus P x transpose and here we apply the property of transpose. So, we can write it x transpose i minus P transpose P y.

Now, here this i minus P transpose is nothing, but i transpose minus P transpose now i transpose is simply i and minus P transpose P y now this can be written as x transpose P minus P transpose P y now here we utilize the property of ortho orthogonal projection here that P transpose is p. So, it this going to be the P square and P square we already know that it is same as p. So, it is what x transpose P minus P y which is nothing, but zero. So, it means that this x N transpose S is going to be 0 for every S in capital S here. So, it means that this x N dot S is going to be 0 for every element of this capital s. So, it means that x N will belongs to the this orthogonal orthogonal compliment of this S this means that x N is a member of S perp.

Now, so, here we have proved that x N is an S perp. So, it means that every element x can be written as element of S plus element of S perp. So, it means that here we can say that this x can be written as this x R plus x N where this x R belongs to S and x N belongs to S perp here now we need to show that this representation is a unique representation. So, it means that for every x we can have a we can represent in this form uniquely. So, for that to prove the uniqueness part, let us say that we have two representation of the same vector x. So, x can be written as say x R plus x N and similarly y R plus y n here now we want to show that x R is going to be y R and x N is going to be y n it is the same thing.

So, for that you simplify this can be written as x R minus y R equal to y n minus y, we simply subtracted this. So, because x is equal to this and x equal to this. So, we can simply say that x R minus y R is going to be y n minus y R now the this is y n this is x N here. So, here we have this is x N. So, it means that here we have x R minus y R and here is y n minus x N now the thing is that here x N x R minus y R both are the element of s. So, x R minus y R is again a an element of S similarly y n minus x N is element of S perp. So, now, these two are equal. So, it means that this x R minus y R is an element in S as well the element of S perp. So, it means that x R minus y R is an element of S intersection S perp now we already know that the only element they have common is this

0 element so that we can easily prove that the element which is common in S in an S perp is only 0.

So, it means that x R minus y R is nothing , but 0. So, which say that x R is nothing, but y of R and if x R is equal to y of R then y R is going to be x N here. So, x N is going to be y n. So, it means that x R is equal to y R and x N is going to be y n. So, it means that x can be written uniquely in terms of x R and x N and. So, it means that this representation that x can be written as x R plus x N where x x R is element of S and x N is an element of S perp this representation is a unique represent. So, so it means that if you know the orthono[rmal] orthogonal projection on to a subspace S S then every element of R n, you can write in terms of element of S and an element of S perp and this representation is going to be unique representation.

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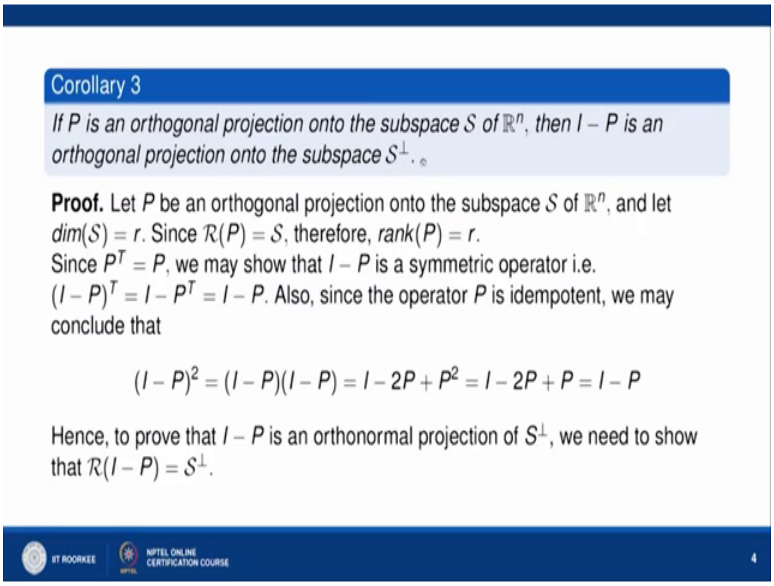


Now, with the help of this theorem we are going to prove one more corollary here which says that if P is an orthogonal projection on to the subspace S of R n, then this i minus P is an orthogonal projection on to the subspace S per. So, please try to what it means that if P is an orthogonal projection on S on to S then i minus P is going to be orthogonal projection on to the subspace S perp. So, here we have we have written this x as P x plus i minus P x. So, here this is is an orthogonal projection on to S. So, we have to we we what we want to show here that i minus P is an orthogonal projection on S perp. So, to show that it is an orthogonal projection on S perp we have to satisfy we have to show that it satisfy all the properties of orthogonal projection.

So, ah. So, let us say that let P be a an orthogonal projection on to the subspace S of R n that is what is given here and let dimension of S is r. Let us say some dimension of S is going to be some nonnegative number it may be 0, but if it is 0, then it is quite obvious then the P is nothing, but 0 and I can be written as the orthogonal projection on to the subspace S perp. Now S perp is in this case is whole of R n. So, here let us assume that R is a hm positive quantity. So, dimension of S is going to be r. Now we already know that that P is an orthogonal projection on to subspace S it means that range of P is going to be S. So, it means that rank rank of this matrix P is going to be R now we already know that P transpose is equal to P. So, with the help of this we want to show that i minus P is also symmetric operator.

For that we simply calculate the transpose of i minus p. So, if you look at the transpose of i minus P it is nothing, but i transpose minus P transpose i transpose is simply i. So, this can be written as i minus P because P transpose is same as p. So, with the help of symmetricity of this P matrix, we have shown that i minus P is also going to be a symmetric operator.

Now, we we want to show again that this i minus P is an idempotent matrix for that we recall that P is idempotent and let us calculate the square of this i minus P whole square and i minus P whole square is nothing, but i minus P into i minus P and if you simplify, it it is going to be i minus two P plus P square now P square is same as P because P is idempotent. So, it can be written as i minus two P plus p.

So, it is we can say that it is nothing, but i minus p. So, it means that i minus P is an idempotent matrix and we have already shown that i minus P is a symmetric operator. So, only thing we left out is that i minus P has a range which is nothing, but S perp. So, we have to show that range space of i minus P is nothing, but S perp.

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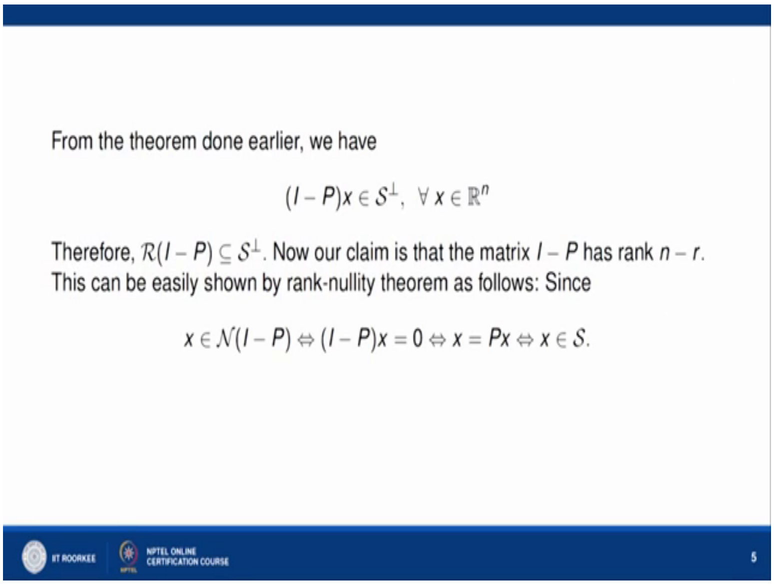


So, for that let say that i minus P x belongs to S perp that we have already seen that we have written x as x R and x N where x N is an element written like i minus P x and we have shown that this x N is an element of S perp. So, it means that any element of form i minus P into x is going to be element of S perp. So, it means that we have shown that i minus P x is an element of S perp for every x belongs to R n that we have already shown. So, it means that range space of i minus P is a subspace of S transpose. So, it is going to be a subset of S perp the only ah. So, to show the equality we have to show that it has same dimension as S perp.

So, if we show that these two are sharing the same dimension and hence they are going to be equal. So, for that we are going to use the rank nullity theorem. So, our claim is that the matrix i minus P has rank n minus r. So, this can be easily shown by rank nullity theorem as follows. So, let us take the element in null space of i minus p. So, if you take the element in null space of i minus P imply and implied by because these are equivalent statement that if you take x belongs to null space of i minus P it means that i minus P x is equal to 0 now this statement also imply that x is going to be the element in null space of i minus p. So, it means that both the statement are equivalent. So, here we are writing we are looking at the equivalent statement.

So, i minus P x equal to 0, if we simplify it is nothing, but x is equal to P of x. So, here we are simply multiplying. So, it is x minus P x equal to 0. So, that implies that x is equal to P of x. Now x equal to P of x means what if you look at P of x P x is element in range space of P it means that P x is going to be the element of S. So, it means that x is going to be element in S now again when x is an element of S, then we can write if you look at here this if x belongs to say range space of P that is S. So, it means that we can write x as x plus 0 here. So, it means that and we already know that this representation is unique. So, it means that x is also an element of x can be written as P of x. So, x can be written as P of x because here x is nothing, but x of r. So, here this is nothing, but x R r plus x of n.

So, it means that here your x is same as x R here right and x R is what x R is nothing, but projection of x on on to S. So, it means that x equal to P x. So, this implies this and this implies this. So, it means that if x belongs to null space of i minus P this implies that x belongs to S. So, it means that this null space of i minus P is element of is a subset of S and similarly element of S is an element of null space of i minus p. So, it means that both subspace share the same dimension or we can say that both subspace are same.

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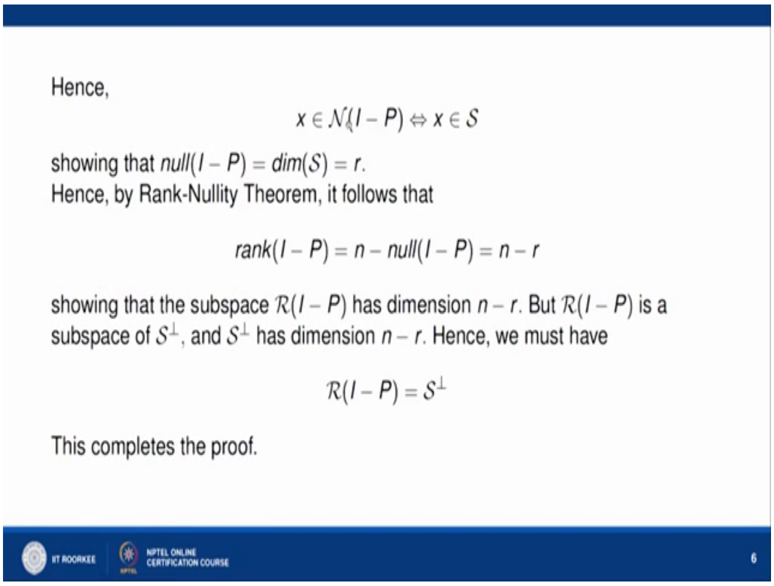


So, here we say that x belongs to null space of i minus P imply implied by that x belongs to S. So, it means that null null nullity of i minus P is same as dimension of S and we already know the dimension of S is R. So, it means that nullity of i minus P is going to be R.

Now, we are using the rank nullity theorem for this operator i minus p. So, it means that rank of i minus P is equal to n minus nullity of i minus P now nullity of i minus P, we already know that it is s. So, it means that rank of i minus P is going to be n minus r. So, it means that the range space of i minus P has dimension n minus R, but we already know that range space of i minus P is going to be is subspace of S plus that we have shown here that i minus P range of range space of i minus P is subset of S perp and we already we just shown that range space of i minus P has dimension n minus r. So, range space of i minus P is subspace of S perp and we know that S perp has dimension n minus R again the rank nullity theorem that if dimension of S is n the dimension of S perp is going to be the dimension n minus R.

Here we need not to use any rank nullity theorem it is simple observation that R n can be written as S direct sum with S perp and dimension of S is same as R then dimension of S perp is going to be n minus R only. So, it means that range space of i minus P is same as S perp. So, with this we have shown that this i minus P satisfy all the properties of orthogonal projection ah. So, it means that i minus P is going to be an orthogonal projection on on to S perp is it.

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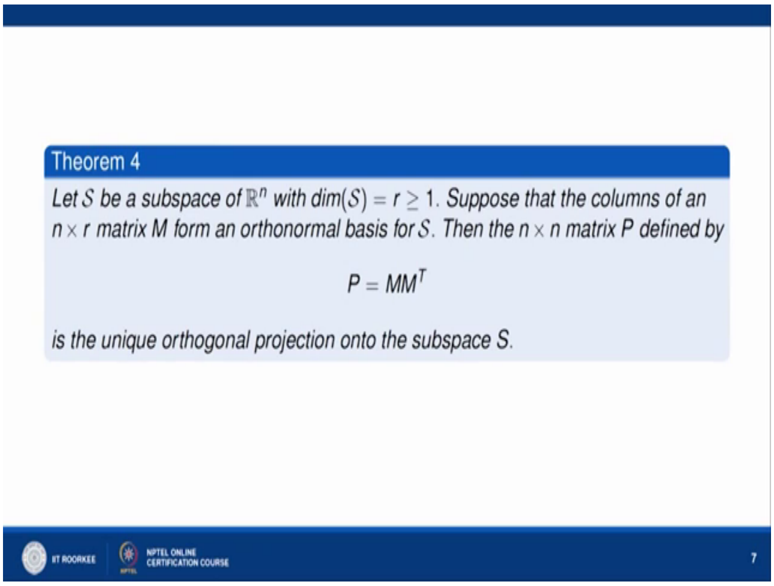


So, now with the help of whatever theory we have developed now let us say that if we have an vector subspace S now with the help of say orthonormal basis of subspace S, we can generate the orthogonal projection on to the vector space S.

So, this theorem four says this that let S be a subspace of R n with dimension of S is R where R is any number which is greater than or equal to one suppose that the columns of an n cross R matrix M form an orthonormal basis for S. So, we have orthonormal basis is given to us. So, it means that we have S whose dimension is R and orthonormal basis is given as S now with the help of orthonormal basis of S we can find out say M matrix M whose whose whose size is n cross n and columns of this M represent the base orthonormal basis for S then then with the help of this M we can find out a matrix P defined as M M transpose and we claim that with this theorem, we claim that this P is going to be unique orthogonal projection on to the subspace S.

So, it means that if we have the hm orthonormal basis for a vector space S then we can find out orthogonal projection on to the subspace s. So, it means that with the help of basis we are trying to find out say orthogonal projection onto that vector space S. So, that is we are going to prove here. So, let us prove this important theorem.

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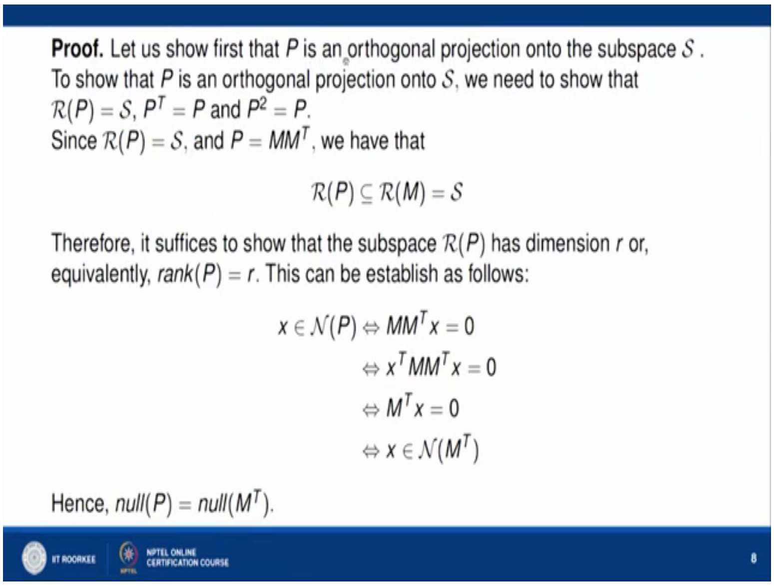


So, ah; so, let us show first that P is an orthogonal projection on to the subspace S. So, it means that with the notation of this P which is given as M M transpose we want to show that this actually is an orthogonal projection on to the subspace S. So, to show that P is an orthogonal projection on to S we need to show that range space of P is S P transpose is P and P square is equal to p. So, we already know that range space of M is going to be S and P we know that it is M M transpose. So, we can show that range space of P is sub[set] subset of range space of M. So, we can say that we arrange a space of M is given as s. So, it means that range space of P is a subset of this S.

Now, to show that to show that equality we have to show that they share the same dimension here. So, for that we have to show that the the subspace R P has dimension R or equivalently we can say that rank of P is going to be R. So, this can be established as follows. So, if you take any element in null space of P then it means that M M transpose x is equal to 0. So, this implies this and this implies this.

So, to now M M transpose x equal to 0, we can multiply x on both the side and we can say that x transpose M M transpose x is equal to 0 here and then we can say that with the help of this, we can say that norm of M transpose x is equal to 0 and now using the property of norm we can say that norm of M transpose x equal to 0 if and only if that M transpose x is going to be 0. So, it means that M transpose x is going to be 0.

So, it means that x belongs to null space of M transpose. So, we have shown that that x x is a member of null space of P then x is going to be the member of null space of n transpose and this relation is a equivalent relation. So, it means that this implies this and this implies this. So, it means that null space of P is same as null space of M transpose. So, it means that now what is the nullity of M transpose.

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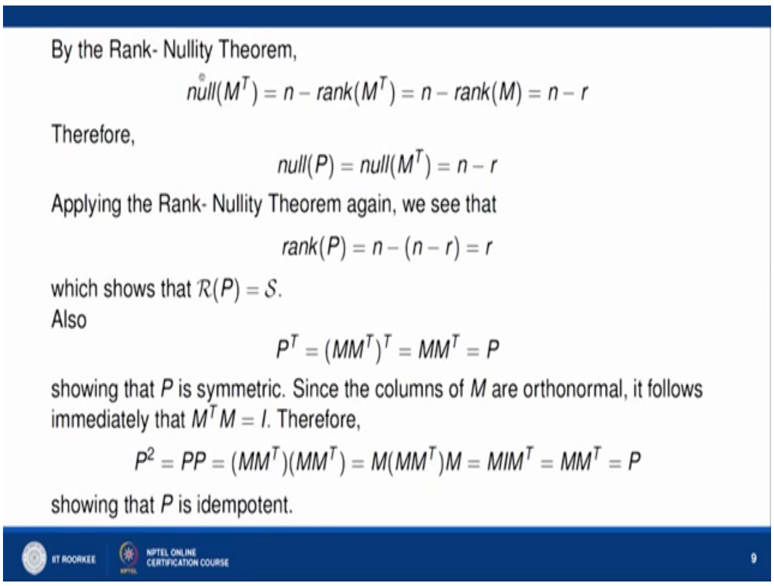

So, nullity of M transpose to we use rank nullity theorem. So, nullity of M transpose is nothing, but n minus rank of M transpose now rank of M transpose is same as the rank of M. So, it means that it is nothing, but n minus rank of M and we already know that rank of M is going to be r because M the columns of M is generating the subspace S and subspace S has a dimension r. So, we know that rank of M is going to be r. So, null space of M transpose is nothing, but n minus r.

Now, it means that nullity of P is going to be nullity of M transpose which is nothing, but n minus r. So, nullity of P is your n minus r. Now again use rank nullity theorem, but this time you use rank nullity theorem for this p. So, earlier we have used rank nullity theorem for M transpose now we are using the rank nullity theorem for this P. So, it means that rank of P is equal to n minus nullity of P now nullity of P is n minus r. So, we can say that rank of P is equal to r. So, it means that range space of P is subset of S now and dimension of S is r and rank of P is r it means that range space of P is also having the dimension r. So, it means that these two vectors subspaces are same as r equal.

So, it means that range space of P is equal to s. So, it means that first property is shown now to show the other property we look at the symmetricity of this matrix P where P transpose, we have to calculate. So, P is M M transpose and using the property of transpose it can be written as it is nothing, but M transpose same as M.

So, it is M M transpose and which is nothing, but p. So, it means that P which is defined as M M transpose is an symmetric is a symmetric matrix. Now to show that that P square is same as P here we have to observe that M the columns of M are orthonormal we have already assumed that the columns of M forms a orthonormal basis of S here look at here the columns of an n cross R matrix M form an orthonormal basis for s. So, if you calculate the M transpose M then we can say that it is i. So, how we can look at the that these are going to be i; you just look at here M.

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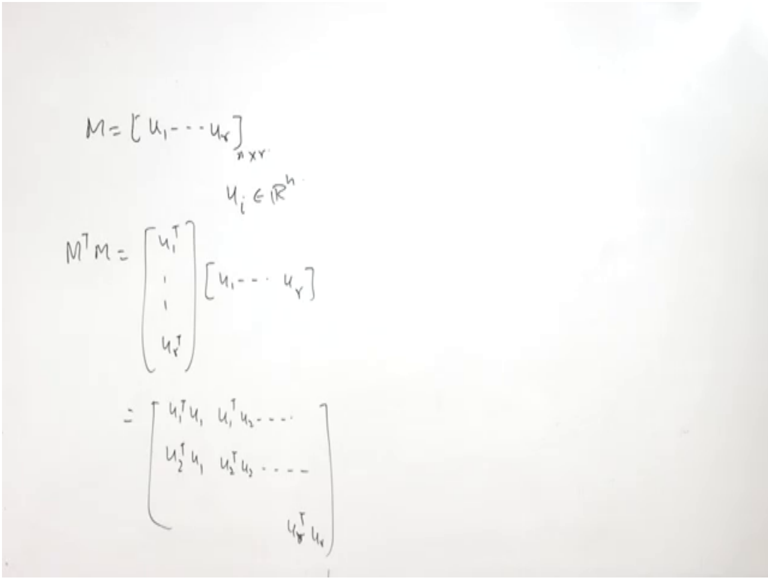

So, here we already know that the columns of M let us call these columns as u 1 to u r and these will form say orthonormal basis for s. So, it means that u 1 and u r are all orthogonal vectors here and norm of each u i is basically one.

So, if you calculate M transpose m. So, M transpose M can be written as this and if you form this product then we have u 1 transpose u 1 u one transpose u 2 and. So, on and u 2 transpose u 1 u 2 transpose u 2 and so on and last one, it is u r transpose u r now we know that u u 1 has norm one. So, it is going to be one here and u 1 u 2 are orthogonal to each other. So, it is going to be 0. So, all these off diagonal matrix off diagonal elements are going to be 0 the only element left is a diagonal element and because of that norm of each u i is one we can say that diagonal vectors are nothing, but one. So, it means that it is going to be a identity matrix. So, M transpose M is going to be identity matrix M so, using this information that M transpose M is an identity matrix calculate this P square. So, P square is nothing, but P P P into P and P is M M transpose into M M transpose. So, if you simplify it is M into M M transpose M now we know that M transpose M is i. So, we can say that M M transpose is also going to be i transpose. So, it is nothing, but i.

So, using this we can write it M i M transpose which is nothing, but M M transpose and M M transpose is nothing, but p. So, it means that P square is an idempotent matrix. So, we have shown that range space of P is S P is symmetric and P is idempotent. So, with the help of ah, this we have shown that P is going to be P is going to be orthogonal projection on to the subspace S the only thing we have to show is now that it has a unique that such a P is going to be unique. So, for that let us assume that we have 2.

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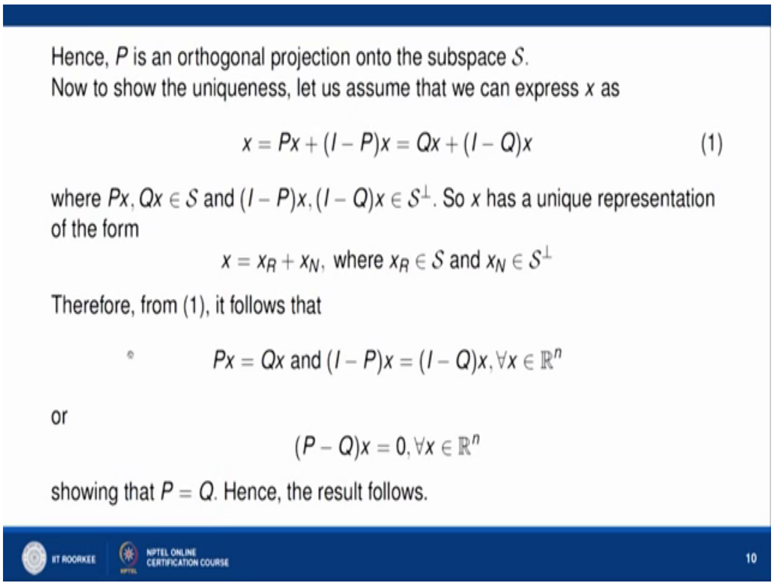


Orthogonal projection on to the subspace S let us say that P and Q. So, it means that x can be written as P x plus minus P x and Q x plus i minus Q x because we already know that with the help of orthogonal projection, we can write x has a element of S and element of S perp.

So, here we write x as P x plus i minus P x equal to Q x plus i minus Q x. So, since it is this representation is going to be unique. So, it means that P x is going to be ele[ment] P x is same as Q x P x is an element of S Q x is an element of S and we because of uniqueness we have P x equal to Q x and i minus P x equal to i minus Q x. So, we can say that P minus Q x equal to 0 for every x into x in R n now since it is true for every x in R n.

So, in particular we take x as e i's and we can show that every row is going to be 0 row here. So, it means that we have to we show that P minus Q is actually a 0 matrix or we can say that P is going to be Q and which shows the uniqueness of this orthogonal projection on to the subspace s. So, here the if you look at what we have achieved with the help of this theorem that with the help of only the orthonormal basis of S we are able to find out say orthogonal projection on to the vector subspace S here. So, that is the very important thing about this theorem.

So, now in now we have we will stop here and we will continue this study in next class where we find some example or we utilize this theorem and utilize the theory of S v d to find out say orthogonal projection on vector subspaces say range space of R null space of null space of a and all these thing we discuss in next class.

Thank you for listening us. Thank you.