Numerical Linear Algebra
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Lecture – 59
Jacobi Method- I

Hello friends, I welcome you to my lecture on Jacobi method. There will be two lectures on this topic we now this is my first lecture. Jacobi method is an identity method we can calculate the eigen values and eigen vectors of a real symmetric matrix by a process known as diagonalization. Carl Gustav Jacobi first proposed this method in 1846, but it became widely used in 1950 when the computers where available. If A is a real symmetric matrix, then all the eigen values of A are real. This is a very well known result.

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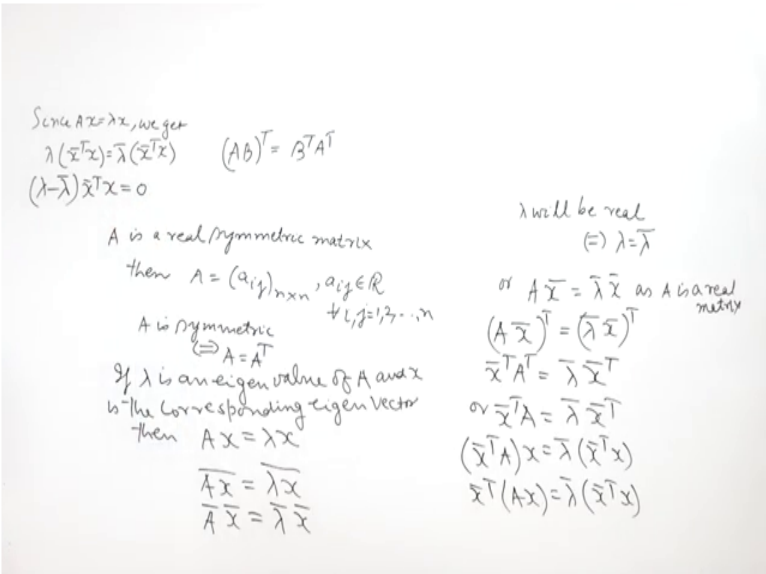

If A is a real symmetric matrix then A equal to a i j. So, it is n by n matrix then a i j belong to R, for all i comma j. From 1 to n and A is symmetric so, we have A equal to A transpose. So, if A is a real symmetric matrix then, all the eigen values of A are real.

So, if lambda is an eigen value of A and x is the corresponding eigen vector then, we have the matrix equation A x equal to lambda x. Now, we can easily see that the eigen values of A are real. So, for that we will need to show that a conjugate lambda conjugate equal to lambda ok. Lambda will be real if an only if lambda equal to lambda conjugate.

So, you can take the conjugate of both sides, you can take the conjugate of both sides then, we will have A conjugate n x conjugate and we have lambda conjugate, x conjugate ok.

Since A is a real matrix, A conjugate is equal to A. So, we have as A real matrix ok. Now, let us take the transport on both sides. So, we have now, you know that a if A and B are two matrices, then AB transpose is equal to B transpose A transpose.

So, here we shall have x conjugate transpose A transpose equal to lambda is a scalar. So, lambda conjugate is also a scalar. So, it is transpose will give you the same thing. So, we have x conjugate transpose; here, I have not changed the order because lambda is a scalar. So, we can bring it before the vector. So, now A transpose is equal to A. So, we have we can write like this.

Now, let us post multiply by x then, we have. Now, matrix multiplication is associative so, we can write like this but we have A x equal to lambda x. So, we have lambda times x conjugate transpose, x equal to lambda conjugate x conjugate transpose x, or we may write this is equal to 0.

Now, in order to prove that lambda is real, we had to show that lambda equal to lambda conjugate, but for that we must prove that x conjugate transpose x is not equal to 0. And let us see how we prove that.

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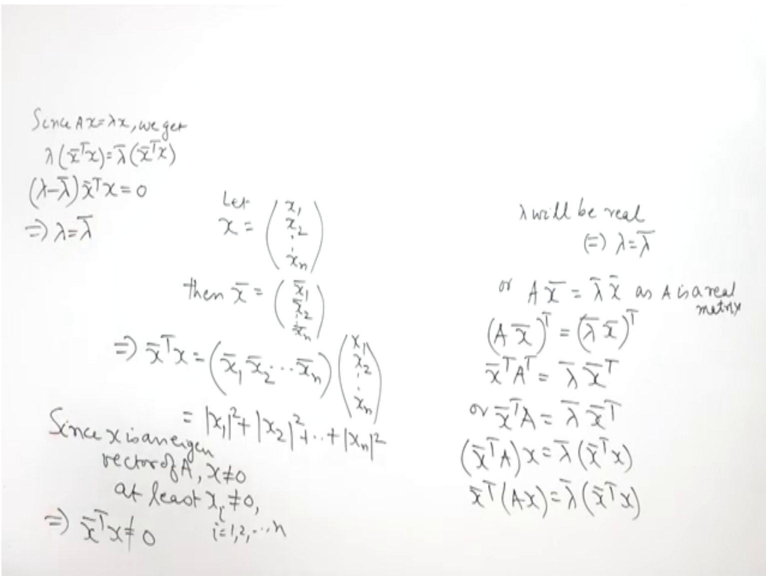

X conjugate transpose let us say let x be equal to the column matrix x 1, x 2, x n. Then x conjugate is equal to x 1 conjugate, x 2 conjugate, x n conjugate the column matrix or we can say lambda x conjugate transpose x is equal to x conjugate transpose will come rho matrix. So, x 1 conjugate, x 2 conjugate at x n conjugate, and x is column vector x 1, x 2 x n.

Now, this is 1 by n matrix, this is n by 1 matrix. When we multiply we get 1 by 1 matrix or we get a scalar, which is x 1 into x 1 conjugate which means mod of x 1 square then, we have similarly mod of x 2 square and so on mod of x n square. Now, since x is an eigen vector of the matrix A corresponding to eigen value lambda. X is not equal to 0.

Since, since x is an eigen vector of A, x is not equal to 0. X is not equal to 0 means; at least one of the components of x is not equal to 0. At least 1 x i is not 0, where i takes values and therefore, with this sum ok. Which is the sum of non-negative numbers mod of x 1 square, mod of x 2 square, mod of x n square cannot be 0. So, this implies this is not equal to 0. So, this is not equal to 0 means, lambda is equal to lambda conjugate and therefore, we have the eigen values of A real symmetric matrix as real.

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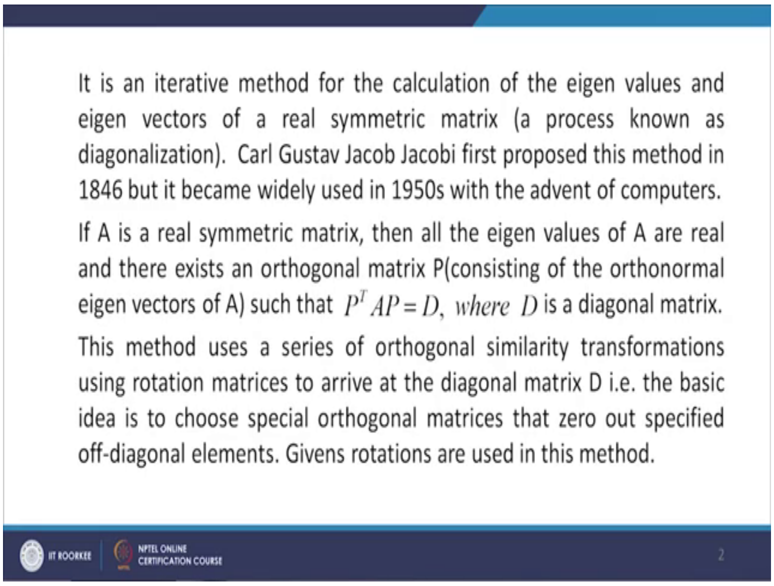

So, if A is a real symmetric matrix then, all the eigen values of a are real. And moreover it can be shown that there exists an orthogonal matrix P consisting of the orthonormal eigen vectors of A.

Now, what is an orthogonal matrix? A matrix is said to be orthogonal, if A A transpose is equal to identity matrix.

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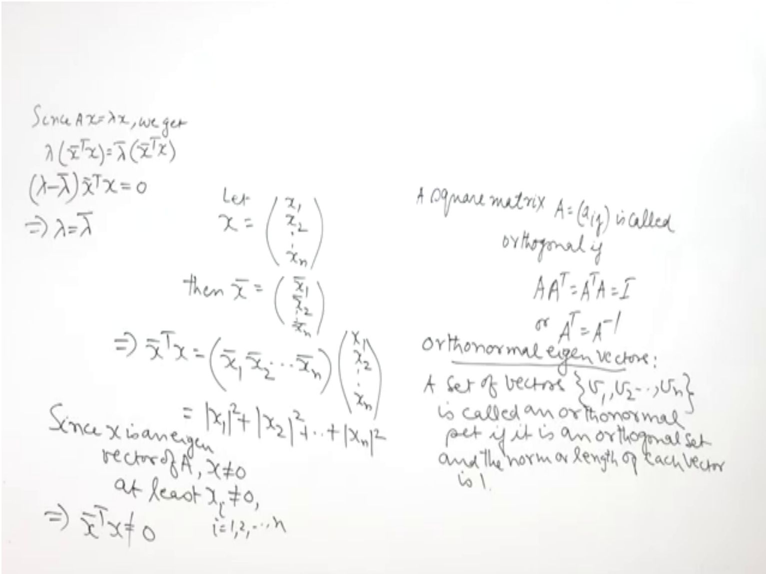

A square matrix A is called orthogonal, if A A transpose is equal to A transpose A equal to identity, or we can say A transpose equals A transpose equals A inverse.

Now, this orthogonal matrix consists of the orthonormal eigen vectors of A. What are ortho 2 normal eigen vectors? Let us see, orthonormal eigen vectors ok. So, a set of vectors a set of vectors say v 1, v 2 and so on v n is called an orthonormal set. If it is an orthogonal set and the norm of each element is 1, and the norm of norm or length of each vector is 1. Now, when do we call a set up vectors to be orthogonal? If be inner product of any two vectors distinct vectors of the set is equal to 0.

So, let us see, how we define an orthogonal set.

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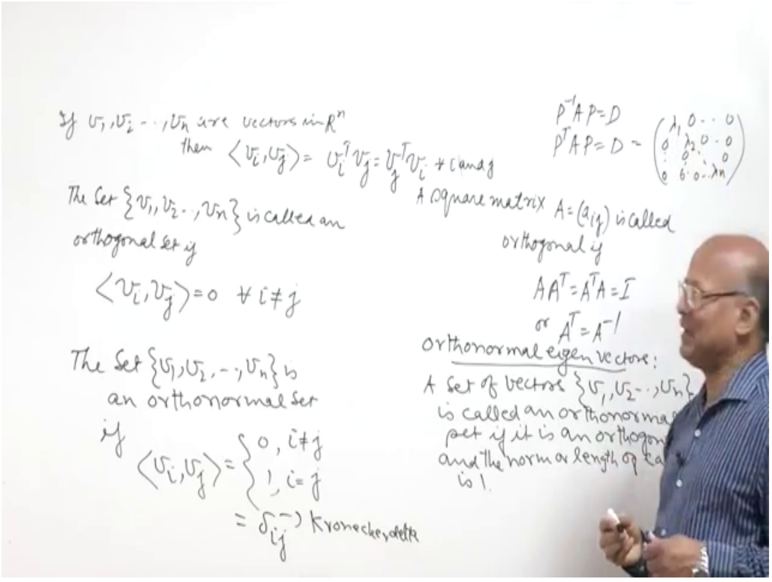

So, the set v 1, v 2, v n is called an orthogonal set, if the inner product of v i with v j is equal to 0 for all i not equal to j. So, here, we are considering a inner productive. A inner productive space a set up vectors v 1, v 2, v n is said to be an orthogonal set, if the inner product of any two distinct vectors of the set is equal to 0.

So, then it will be called orthogonal and then, it will become orthonormal provided the norm of each element say v i of the z is equal to 1. So, we can say that the set of v 1, v 2, v n is an orthonormal set, if in inner product of v i with v j is equal to 0 when i is not equal to j, and 1 when i is equal to j. Because, inner product of v i with v i is norm of v i square. So, norm v i square is equal to 1 means, norm of v i equal to 1.

So, this can also be written as this is equal to delta i j. Delta i j is the Kronecker delta this is the Kronecker delta, which takes value 1 when i is equal to j, and 0 when i is not equal to j.

Now, when v 1, v 2, v n are n tuples, that is they are vectors in R n, then we can write. If v ~~one~~ 1, v 2, v n are vectors in R to the power n then, the inner product the inner product v i, v j can be written as this is equal to v i transpose v j.

We can write as v i transpose v j or we can also write v j transpose v i because, v i v j is real ok. V i v j is a real number. So, we can write as v i transpose v j or v i we can write v j v j transpose v i for every i and j. So, here the eigen vectors of the matrix A will turn out to be orthogonal and moreover that, each eigen vector will have length 1. So, they will constitute a orthonormal set such that P transpose A P equal to D matrix ok.

When P when D matrix A is diagonalized by using the matrix orthogonal matrix P then, since P transpose is equal to P inverse, we have a we when A is diagonalized by using the matrix P, we have the equation P inverse A P equal to D, but since P is orthogonal. I can write P inverse as P transpose. So, there exists orthogonal matrix p such that P transpose A P equal to D, where D is a the D is a diagonal then matrix.

Now, this method uses a series of orthogonal similarity transformations, which are found using the rotation matrices to arrive at the diagonal matrix. The a basic idea of the method is to chose a special orthogonal matrices that zero out is specified of diagonal elements, and for this purpose givens rotation matrices are used.

Now, in the let us discuss the outline of the algorithm.

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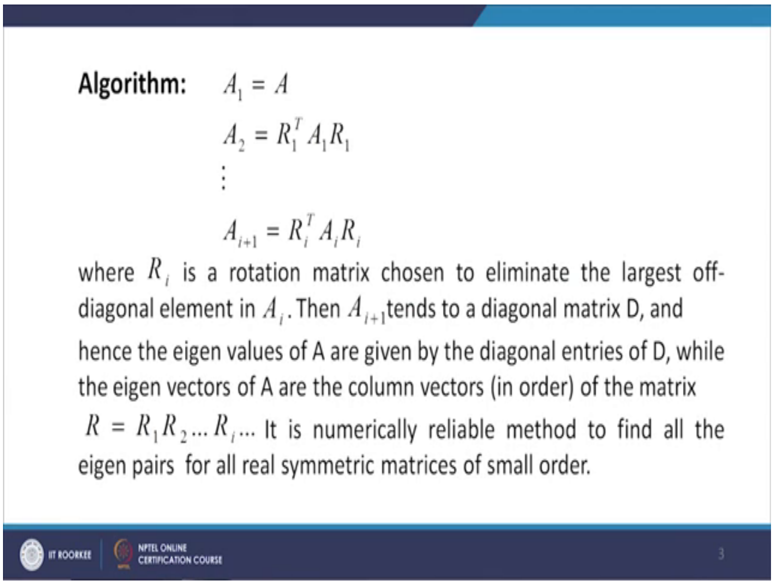

So, A 1 we take as the given matrix A, A 2 is R 1 transpose A 1 R 1 and so, on. A i plus 1 is R i transpose A i R i, where R i is a rotation matrix, which is chosen in such a way that we want to eliminate the largest off diagonal element in A i.

Now, A i plus 1 when we go on doing this A i plus 1 will tend to a diagonal matrix D. And we know that the eigen values of A, when A is similar to the diagonal matrix the D then, the eigen values of A R the diagonal entries of D. So, the eigen values of A R given by the diagonal entries of D, while the eigen vectors of A will be the column vectors of in order of the matrix R equal to R 1, R 2, R i and so on.

So, first if you write say suppose for example, say this is your lambda 1, lambda 2, lambda n and here we have sub zeros ok. So, in the first column we have eigen value lambda 1. So, the matrix R which is R 1 into R 2 and R i and so on will be the matrix. Where the first column will correspond to the eigen vector of A corresponding to the eigen value lambda 1.

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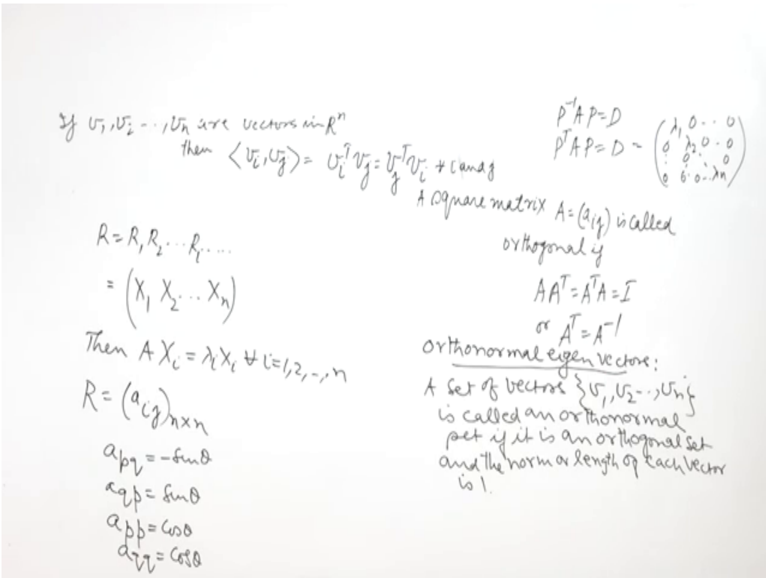

Suppose the first column of R matrix is x 1, second column is x 2 and nth column is x n then, we shall have a lambda i equal to A x i equal to lambda i, x i for every i, that is x i will be the eigen vector of a corresponding to the eigen value lambda i for every i; i equal to 1 2 3 and so on up to n.

So, the columns of the matrix R or the eigen vectors of a matrix in order as the eigen values are written in the diagonal matrix.

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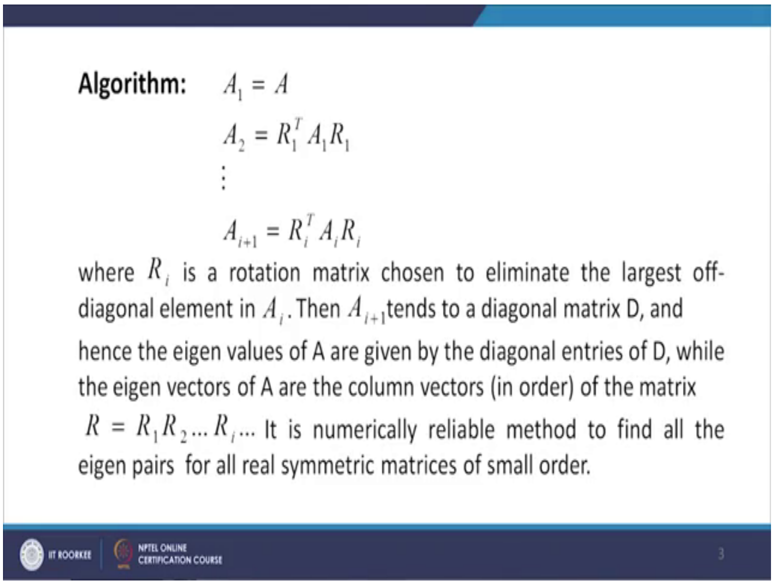

Now, this method is numerically reliable method to find all the eigen pairs, for all cl real symmetric matrices of a small order.

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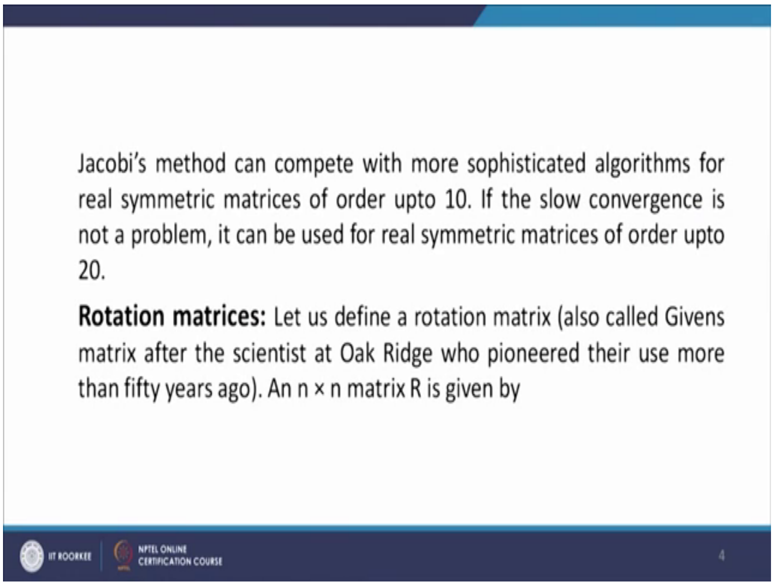

Jacobi's method can compete with more sophisticated algorithms for a real symmetric matrices of order up to 10. And if the slow convergence is not a problem, then it can be used for real symmetric matrices of order up to 20.

Now, let us define rotation matrix, rotation matrix is also called as a given matrix givens was a scientist at oak ridge who pioneered their use more than 50 years ago.

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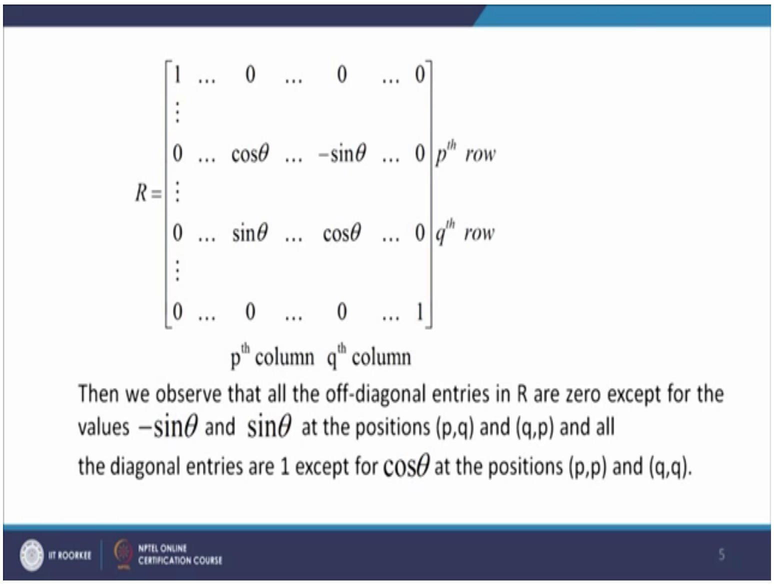

So, an n by n matrix R is given by this matrix. So, you can see in this matrix all the off-diagonal entries in R are 0 except for the values minus sin theta. Here, you can see minus sin theta which occurs in the p th row and q th column.

So, this is if you denote the entries of R by a i j notation then, a p q a p q is equal to minus sin theta because, it occurs in the p th column and p th row and q th column. And the entry in the q p th q p th entry a q p is equal to sin theta ok. Which occurs in the q th row and p th column, and the diagonal entries this this entry in the p th row and p th column that is a p p is equal to cos theta, a q q is equal to cos theta ok. All other diagonal entries are one and all other off diagonal entries are 0.

So, what we want to say that, the givens matrix are has p p th row q th element as minus sin theta, q th row p th column element as sin theta, and a p p element has cos theta, a q q element as cos theta. All the elements on the diagonal other than these two app and a q q are once, while all the off diagonal elements are 0 accepting a p q and a q p. So, we observed that all the off diagonal entry then RR0 except for the values minus sin theta and sin theta, which occur at the positions p q and q p, and all the diagonal entries are 1 except for cos theta which occurs at the position p p and q q.

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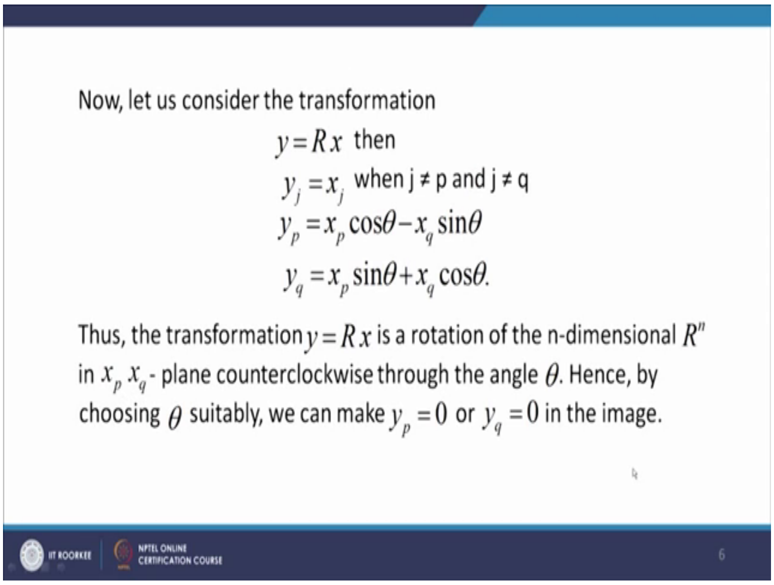

Now, let us consider the transformation y is equal to R x. So, let us consider y is equal to R x.

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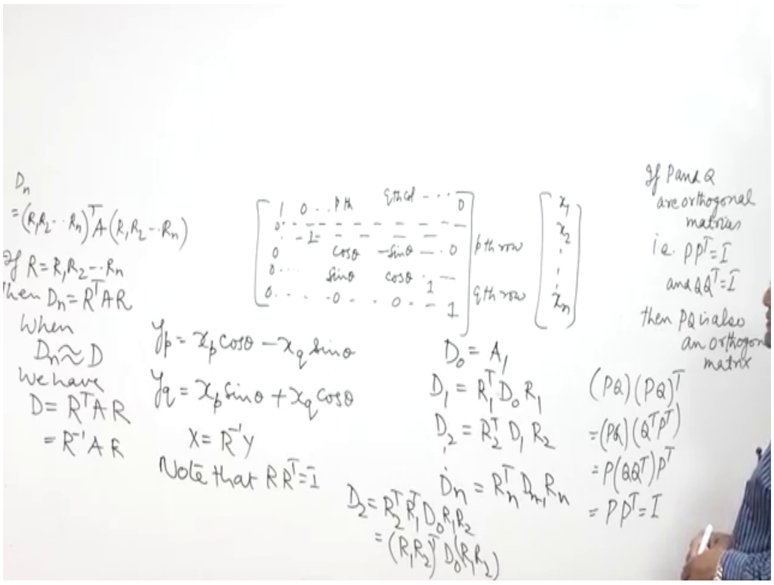

Y is y 1, y 2, y n and r is the given rotation matrix. So, 1 0 and then this is my p th column this q th column. And this is p th row and this is q th row so, this is 0 here. The element which occurs in the p th row p th column, which we take as let me write, this will be taken as cos theta a p p p th column. So, this will be taken as cos theta ok.

And here, we have the element which occurs in the p th row and q th column as minus sin theta, and this we have here 0. And then the element which occurs in p th q th row and p th column we take as sin theta and this element we take as cos theta. And all other elements are zeros and here, we have 0 0 and we have 1. So, all other entries are one on this diagonal ok. This matrix R and then the matrix x is x 1, x 2 and so on x n, if you multiply the R n by x then what do you notice?

J suppose y j y j means, j th row ok. Here y j, y j will be equal to x j when j is not equal to p and j is not equal to q because, the change will occur only in the row j when j becomes equal to p and j becomes equal to q because all other places we have diagonal entries as 1 while the off diagonal entries are 0. So, y j is equal to x j when j is not equal to p and j is not equal to k q. While y p y p will be equal to what let us row, when you multiply p th row by this vector you will get y p. So, y p will be equal to you see y p will be equal to you multiply the p th row here by x 1 x 2 x n. And you will get x p cos theta x p cos theta and then, minus x q sin theta.

So, when you multiply that column to this row we get y p equal to this and y q when you multiply the q th row by that vector we get this is a p th column. So, we get x p sin theta plus x q cos theta. So, we get this thus the transformation y is equal to R x is a rotation of b and dimensional space R to the power n in the x p x q plane. Counter clockwise through the angle theta and by choosing theta suitably, we can make y p equal to 0 or y q equal to 0 in the image.

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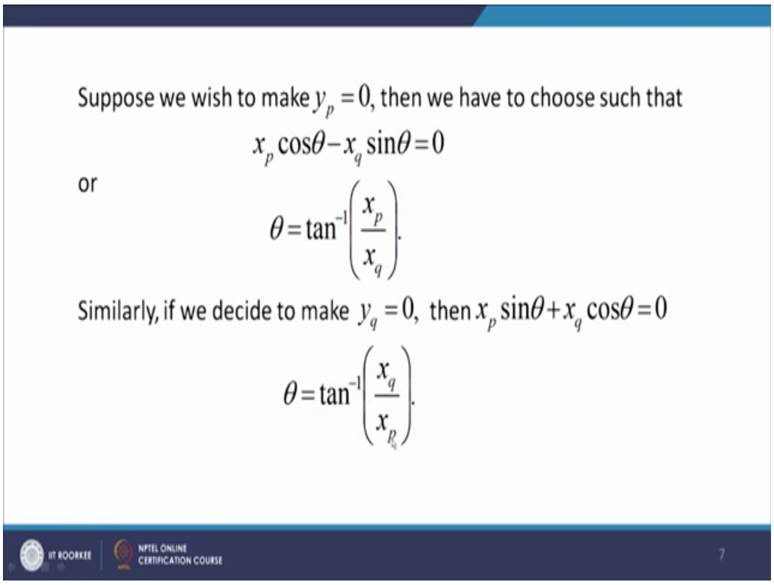

So, if you wish to make y p equal to 0 then x p cos theta minus x q sin theta is equal to 0. So, we have to take theta equal to tan inverse x p over x q. Similarly, if we decide to make by q equal to 0 then, we take then we have to put x p sin theta plus x q cos theta equal to 0. And which will give us theta equal to tan inverse x q over x p.

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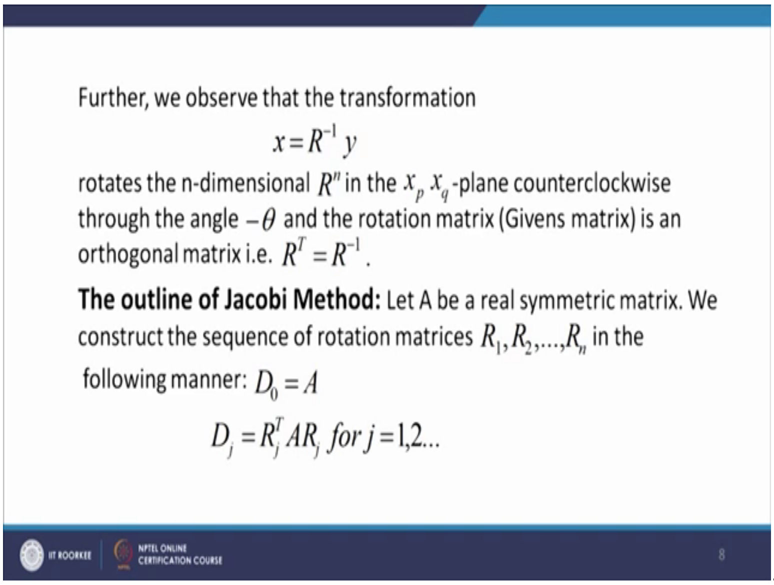

Now, first further we observe that, the transformation x is equal to R inverse y. X equal to R inverse y rotates the n dimensional space or to the power n in the x p, x q, x p, x q plane, counter wise clockwise through the angle minus theta and the rotation matrix that is givens matrix is an orthogonal matrix R transpose is equal to R inverse, you can see here if you find R transpose here. You will get the same as R inverse. So, R transpose is equal to R inverse or you can see that RR transpose. We notice that RR transpose is equal to identity matrix. Now, let us discuss the outline of the Jacobi method let A be a real symmetric matrix, we shall constant a sequence of rotation matrices R 1 R 2 R n in the following manner.

Let us take D naught equal to the matrix A the given matrix A and D j equal to R j transpose A R j for j equal to 1 2 3 and so on.

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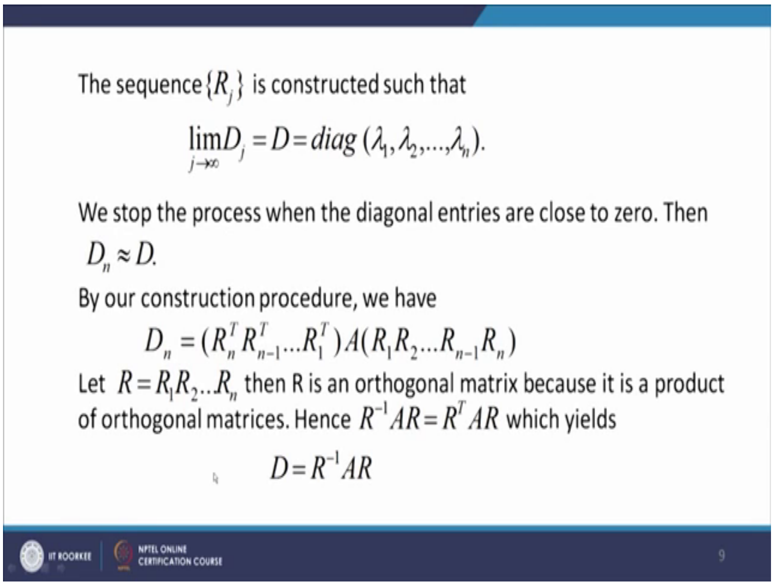

The sequence R j is constructed such that the limit of D j as j goes to infinity. The limit of D j as j j goes to infinity becomes a diagonal matrix. And as you know the on the diagonal the entries of the eigen values of a will be displayed. So, D is equal to diagonal matrix lambda 1 lambda 2 lambda n. We shall stop the process when the diagonal entries are off diagonal entries are close to 0.

So, we stop the process, when the off diagonal entries are close to 0 and then the, diagonal entries of the resulting matrix are become the eigen values of the given matrix A. And we say that D n is close to D. Now, by our construction procedure we can see that D as we have written D naught is equal to A. D 1 equal to R 1 transpose D naught R 1. D 2 equal to R 2 transpose D 1 R 2 and so on, D i plus 1 is equal to R i transpose D i R I.

So, we can say that D n is equal to D n is equal R n minus 1 transpose. D n minus 1 we have here sorry D n here is R n D naught is equal to A 1. D 1 is R 1 transpose D naught R 1 D 2 is equal to R 2 transpose D 1 R 2. So, D i plus 1 is R i D i is equal to or we can write D n. D n is equal to R n transpose D n minus 1 R n we can write. Now, let us we can put the values here.

We can put the value of D n minus 1 here then D n minus 2 and so on. And we can say that D n is equal to. So, D n is equal to R n R n minus 1 R n minus 2 and so on R 1 and we get R 1 transpose. So, R n or I can say we can write like this. Using the fact that A B transpose is equal to B transpose, A transpose, we can see here that B 2 is equal to R 2 transpose D 1. D 1 is R 1 transpose D naught is a R 1 ok.

So, if you put here value then D 2 will be equal to R 2 transpose R 1 transpose. Then D naught and then R 1 R 2 which can be written as R 1, R 2 transpose D naught R 1 R 2. So, in general D n can be written as R 1, R 2 R n transpose and D naught is equal to a ok. So, we have D dot R 1 R 2 R 1 R 2 R n. Now, if R is equal to R 1 R 2 R n then, we can say that D n is equal to R transpose A R. So, this is D n when we say that when is when the matrix is close to the D n is close to D, that is the diagonal matrix off diagonal elements are close to 0, then I can write D n approximately as D ok.

So, when D n is approximately the diagonal matrix D we have, D equal to R transpose A R. So, and R transpose is R inverse because, if A and B are 2 orthogonal matrices then, their product is also an orthogonal matrix we can easily prove that. If p and q are orthogonal matrices, that is p p p transpose is equal to identity matrix, and q transpose is equal to identity matrix then, p q is also an orthogonal matrix ok. So, we can prove this easily.

Let us let us let us show that p q into p q transpose p q into p q transpose is identity matrix. Let us show that p q into p q transpose is A identity matrix. I can write it as p q and then p q transpose is q transpose p transpose ok. So, matrix multiplication is associative. So, I can write it as p times q q transpose into p transpose, but q q transpose is identity matrix. So, I have p p transpose, but p p transpose is equal to identity. So, product of orthogonal matrices also orthogonal, here R is the product of orthogonal matrices R 1 R 2 R n and therefore, R is an orthogonal matrix. So, that is why we have written R transpose is equal to R inverse.

Now, or we can say A R is equal to, let us say A R is equal to R D. Now, R matrix has the columns of the matrix R are the eigen vectors of the matrix A in order in the same order in which eigen value is lambda 1, lambda 2 lambda n occur in the diagonal matrix we are going to prove that. So, let us say, let say the columns of the matrix R be x 1, x 2, x n each with n components.

So, R has columns x 1, x 2, x n then let us see what do we get from the equation A R equal to R D. So, A R will be equal to A R will be equal to A, A into x 1, x 2, x n.

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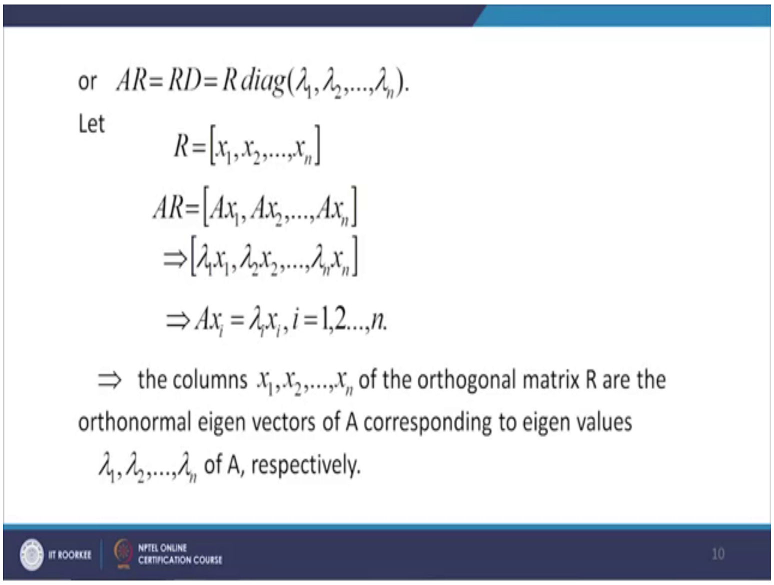

So, AR will be the matrix a multiplied by the matrix R and we shall have.

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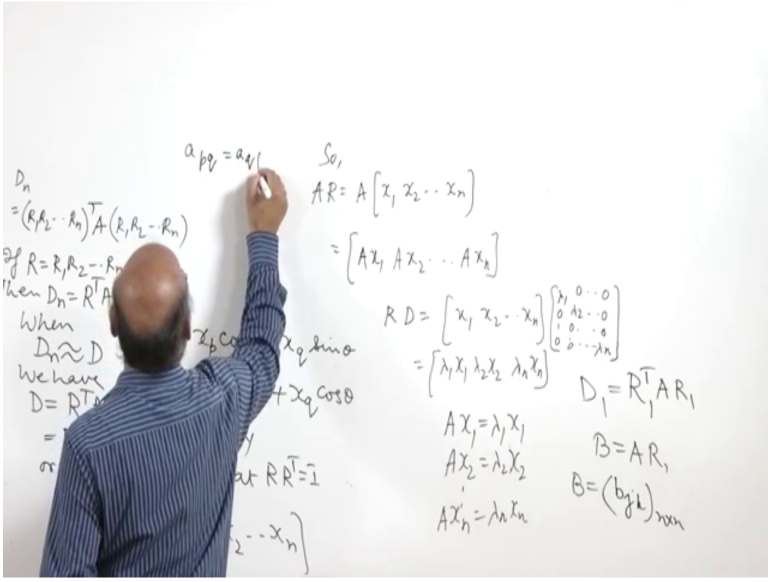

When the when the rows of the matrix A are multiplied by the first column of the matrix R we get the first column of the matrix A R so, we get A x 1. First column as A x 1 and then, when the all B rows of the matrix AR multiplied by the column x 2 ok. Second column we get the second column of the matrix A R so, we get A x 2 and so on. The nth column of A R will be equal to A x n ok.

Now, what is R D what is RD? R D is equal to x 1, x 2, x n and the diagonal matrix D is lambda 1 0 0 0 lambda 2 0 0 0 lambda n. So, when the first column is multiplied to the rows of the matrix. D what we get is lambda 1 x 1 lambda 2 x 2 this is the first column this is the second column and this is the nth column. So, what we get is. So, we get the following. So, now A R is equal to R D. So, what we get first column A x 1 is same as first column here.

So, A x 1 equal to lambda 1, x 1 second column A x 2 is equal to lambda 2 x 2 and so on. A x n is equal to lambda n x n. So, from here we can say that A x i is equal to lambda x i for all i equal to 1 2 3 and so on up to n and therefore, the columns x 1, x 2, x n of the orthogonal matrix R.

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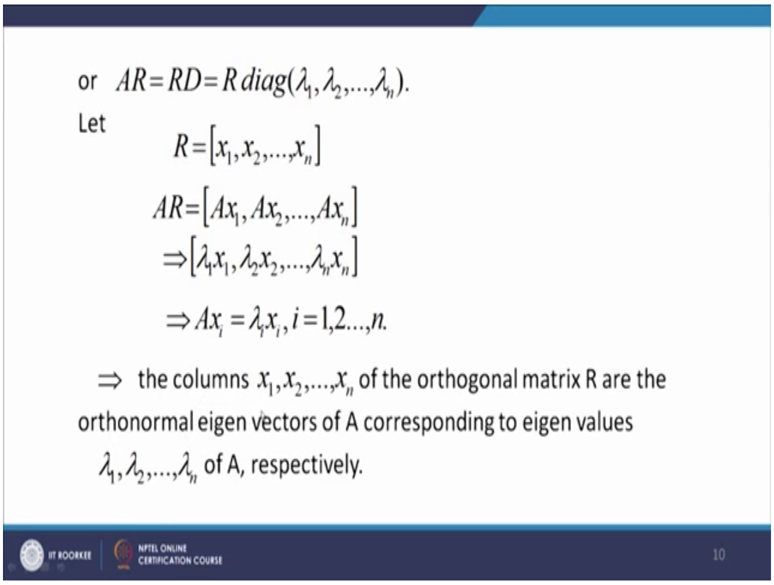

These x 1, x 2, x n these columns of the matrix R are the orthonormal eigen vectors of the matrix A corresponding to the eigen values lambda 1 lambda 2 lambda n of A respectively the general step of Jacobi's method.

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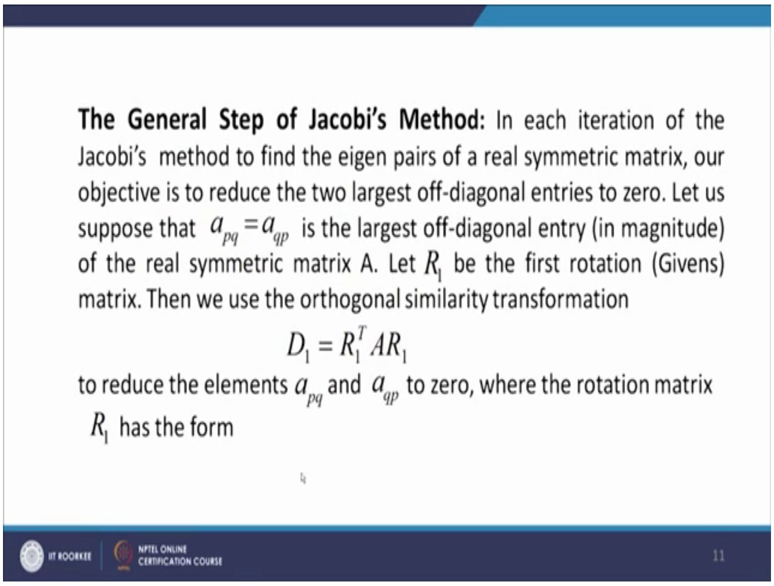

Now, in each iteration of the Jacobi's method our objective is to reduce I mean we want to find the eigen pairs of a real symmetric matrix by using the Jacobi's method. So, in each iterations of our objective is to reduce the 2 largest off diagonal entries to 0. Let us suppose that the p q th entry in the given matrix A is the is numerically the or numerically the largest off diagonal entry. And since, it is a symmetric matrix a p q will be equal to a q p. So, a p q equal to a q p is the largest off diagonal entry in magnitude of the real symmetric matrix A. Let R 1 be the first rotation matrix.

Then we use the orthogonal similarity transformation D 1 equal to R 1 transpose, A R 1 to reduce the elements a p q n q, a q p to 0 where the rotation matrix R 1 has this form.

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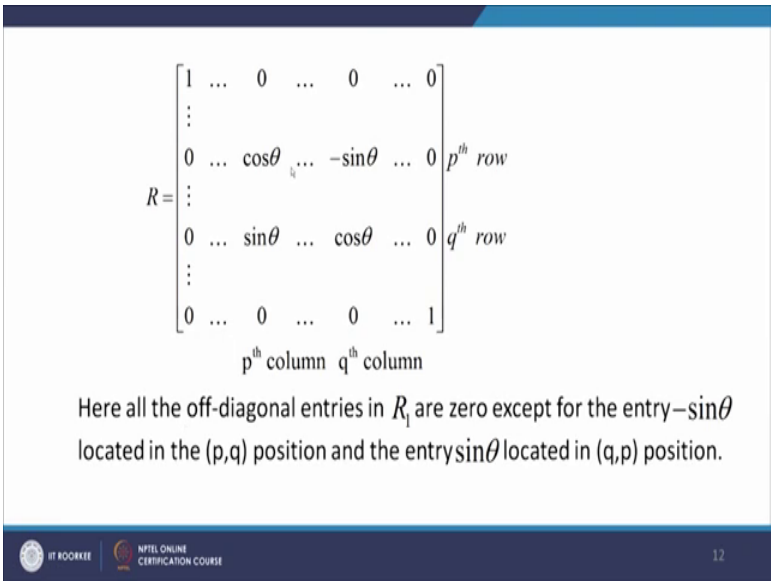

So, see in the given matrix A, we are assuming that in the in among the all off diagonal entries in the matrix A the p q th entry a p q is numerically the largest. And since, it is symmetric matrix if a p q is numerically the largest then a q p, a q p is equal to a p q. So, it is also the numerically largest. So, what we will do in the R th matrix in the p q th position in the p q th position we will take minus sin theta. In the q p th position that is q th row p th column we shall take sin theta.

And in the p p th position in the p p th position we take cos theta and q q th position also we take cos theta. All other entries on the diagonal are taken as one and remaining off diagonal entries are taken 0. So, corresponding to q th element a p q in the rotation matrix R, we take minus sin theta corresponding to the q p th element a q p in the q p th position in R we take sin theta. And then p, p th position, p th row, p th column, in the matrix R is taken as cos theta, q th row, q th element a q is also taken as cos theta. Remaining elements on the diagonal in RR taken as 1 all the remaining diagonal off diagonal elements are taken as 0.

So, this is how we construct the matrix R. And so, here all the off-diagonal entries in R 1 are 0 except for the entry minus sin theta located in the p q th position, and the entry sin theta are located in the q p th position. All the diagonal entries in R R 1 are equal to 1 except for the entry cos theta located in p p th position and q q th position, thus R 1 is a rotation matrix where theta is to be chosen in such a way that the entries d p q and d q p of D 1 located at the p q th and q p th position respectively in D 1 R 0.

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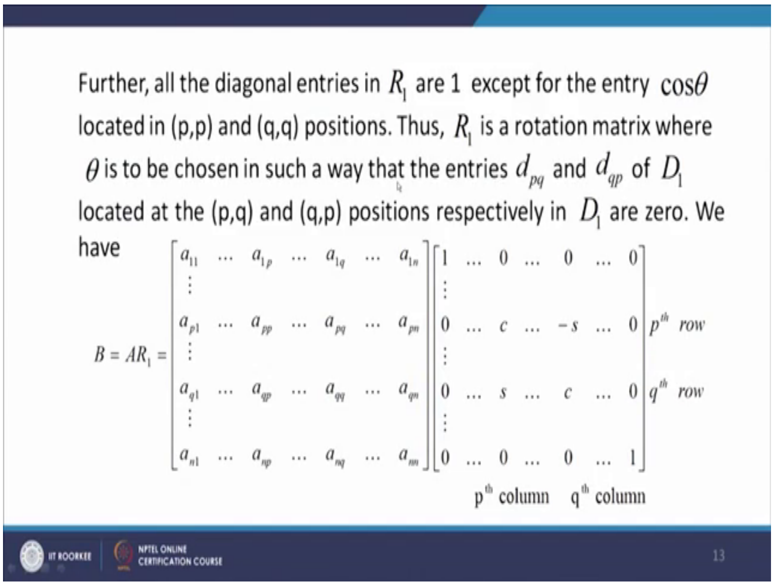

So, when you find the D 1 matrix ok. R 1 transpose D naught R the R you can say R 1 transpose A R 1 then, it should turn out that the element in the p q th position in D 1 is equal to 0. And the element in q p th position is 0 or you can say close to 0. Now, so let us see what happens, A is the given matrix A 1 1 A 1 at 1 2, A 1 p this is p th for the p th column. This is q th column a 1 q, A 1 and this is p th row of the matrix A a p 1, a p p, a p q, a p n this q th row a q 1, a q p, a q p, a q q, a q 1, and this is nth row of the matrix a and this is the matrix R.

So, first we are finding what happens when the matrix because, we have to find D 1 ok. D 1 is equal to D 1 is equal to R 1 transpose A R 1. So, first we are finding A R 1 first we are multiplying the matrix ay the matrix R 1, and in the matrix R 1 for simplicity for convenience. We have represented cos theta by c and sin theta by s.

So, you can see here in the p q th position we are taking minus s in the q p th position we are taking s, in the p p th position and q q th position we have taken cc. So, this is how we take write construct the matrix R and then multiply a by the matrix R. And let us see what happens then it will turn out that.

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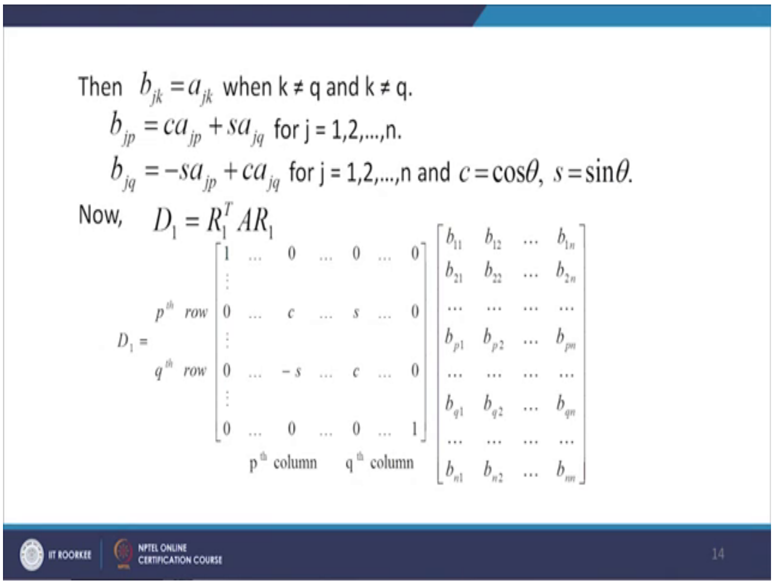

When k is not equal to q or k is not equal to q ok. K is not equal to q let us say see when you find the matrix v j B. Let us say, B matrix the matrix B the matrix B is equal to b j k. Let us say, the matrix B is equal to b j k n by n then, b j k will be equal to a j k whenever k is not equal to q and k is not equal to, k is not equal to p and k is not equal to q.

Because, only in the p th and q th when k equal to p and k equal to q, there will be changes there will be no change. So, b j p equal to c a j p plus s a j q. We get this when you multiply j th row of the matrix A. J th row of the matrix A by the p th column of R 1 we get b j p. So, when you so, b j p it turns out to be c a j p plus s a j q similarly, when you multiply j th row of the matrix a by the a q th column of r ok. We get minus s a j p plus c a j q for j equal to 1 2 3 and so on up to n. Here c is cos theta s a sin theta.

And then when you find D 1 R 1 transpose A R 1 this is my R 1 transpose. You can see here, the position of c and c remains unchanged because, they are on the diagonal while the position of minus a and s gets interchanged minus s comes from here, to here and s goes from here to here ok. So, because of transpose so, then when we multiply this R 1 transpose by the A R 1 A R 1 is my B matrix A R 1 is B matrix.

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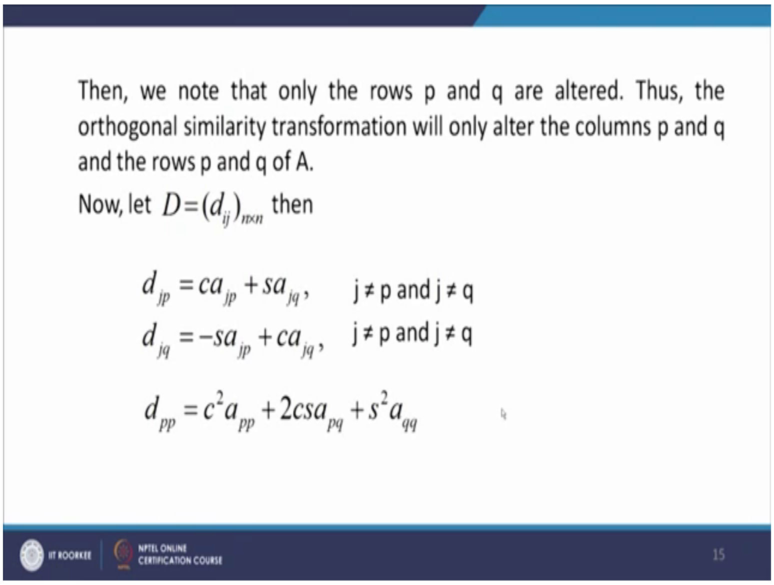

So, what I get is this we get the following. So, when we note that only the rows p and q are altered you see this is because, all other entries all other in all the other rows we have ones on the diagonal while other entries are 0. So, there will be no change except the except p and q th row p th row and q th row. So, only the p th and q th rows are altered does the orthogonal similarity transformation will only alter the columns p and q. When you find the matrix B columns are altered p and q, and then we find your R 1 transpose B then the rows p and q are altered.

So, D is equal to if we represented to d by matrix by d I j n by n then, it is easy to see that d j p d j p is the. When we multiplied j th row of r 1 transpose by p th column of b, we get c a j p plus s a j q. This will be the case when j is not equal to p and j is not equal to q. Similarly, d j q we can find when we multiply j th row of R 1 transpose by q th column of b. We get minus s a j p plus c a j q and d p p, when we multiply p th row of the matrix R 1 transpose by p th column of the matrix we. We will get c square a p p plus 2 c s, 2 c s a p q plus s square a q q.

Remember we here while finding the multiplication we have made use of the fact that a p q equal to a q p. So, when we made use of this fact that app a p q and a q p are same, d p p turns out to be this.

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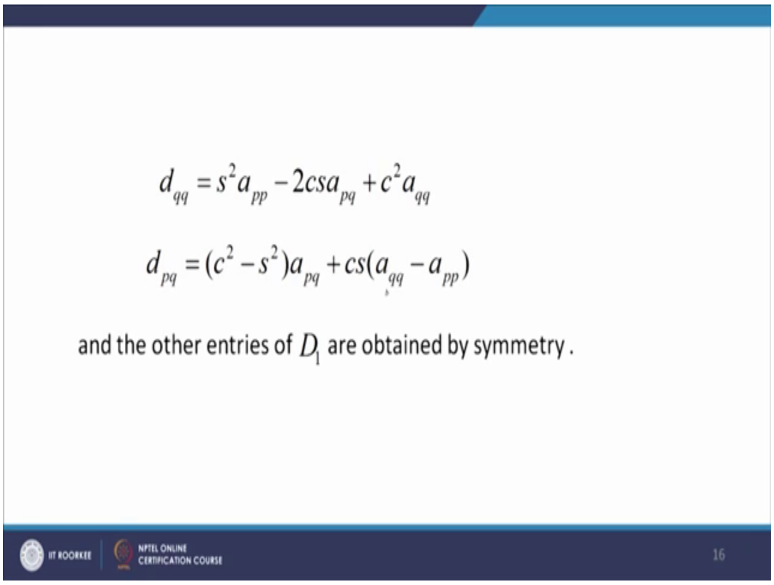

And similarly, we can find d q q d q q turns out to be s square app minus 2 c s a p q plus c square a q q. And d p q is equal to c square minus s square into a p q plus c s times a q q minus app. The other entries of D 1 are obtained by symmetry ok.

So, this is what we do, we will in the next lecture we will determine the formulas to determine the values of certain parameters. When we determine those parameters we will be able to determine the angle theta, which is to be chosen in such a way that a p q and a q p entries of the matrix A R 0. The largest numerically largest off diagonal entries of the matrix A R 0 so, that we will discuss. So, with this I would like to conclude my lecture.

Thank you very much for your attention.