Multivariable Calculus
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Lecture - 28
Applications of Multiple Integrals

Hello friends. So, this lecture is the last lecture from the integral calculus. And in this lecture, I will introduce few applications of multiple integral. So, what I will do? First, I will introduce the applications of double integral, and then I will extend them in case of triple integral.

So, let us start with double integral.

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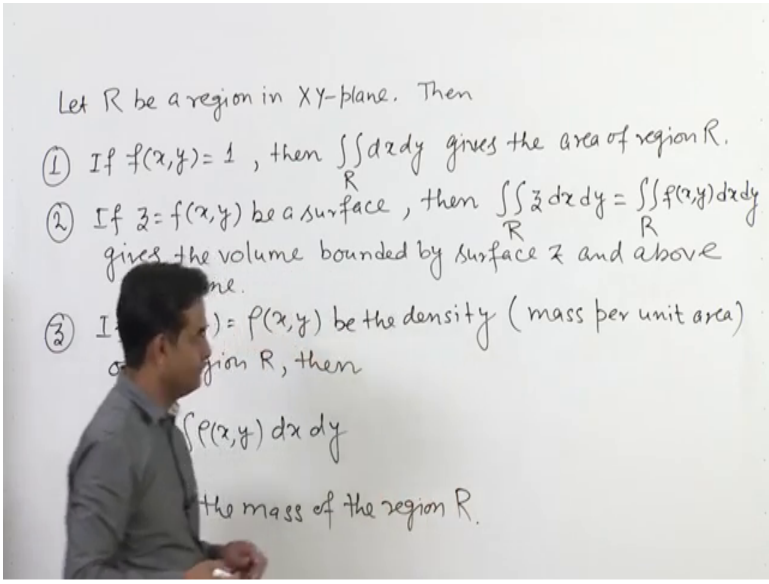

So, let R be a region in xy plane, then number 1. If f of xy equals to 1 then the double integral over the region R dx dy gives the area of region R. Number 2, if z equals to f of xy be a surface, then double integral over region R z of dx dy.

So, what we can do? Z is a function of x and y so, we can replace the z with that function, f of x y dx dy gives the volume of the region bounded by the surface z, and above xy plane. Number 3, if f of xy equals to rho of xy be the density ; density means mass per unit area of the region R, then double integral over R f of xy or rho of xy dx dy gives the not total mass of region R ok.

So, if f of x y equals to rho of xy that is the density function, then x bar equals to 1 upon M; where M is the mass of the region R.

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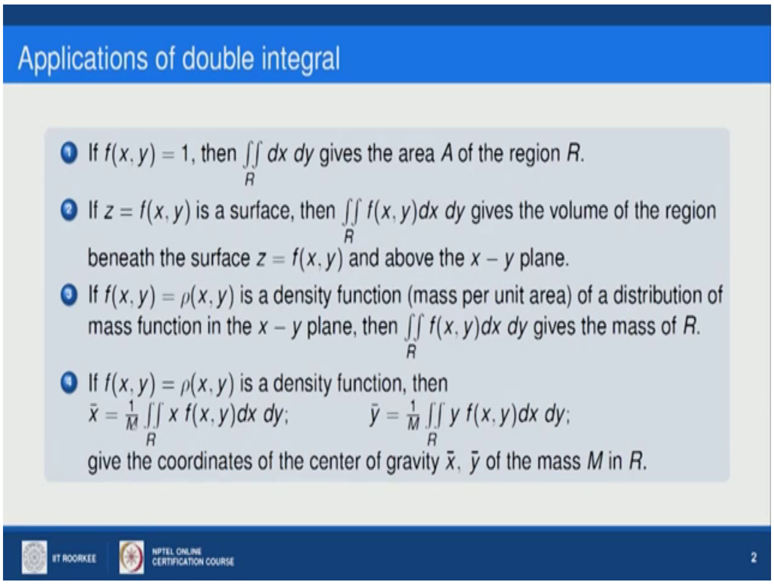

Double integral over R x times f of x y dx dy and y bar equals to 1 upon M double integral over region R y times f of x y dx dy gives the coordinates of the center of gravity that is x bar y bar of the mass M in R. If f of x y is rho, x y is a density function, then I x equals to double integral over the region R y square f of x y dx dy, and I y equals to double integral over R x square f of x y dx dy, give the moment of inertia of the mass in R about the x axis.

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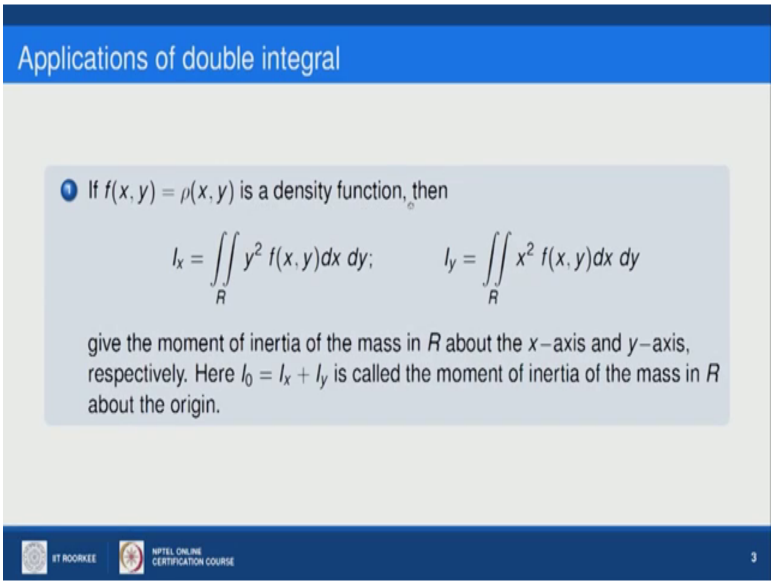

So, this is about x axis, and about the y axis this one. If I 0 is I x plus I y, then we say that moment of inertia of the mass R about the origin.

If I want to find out, the moment of inertia about a line let us say x equals to a, then it will become I x over R.

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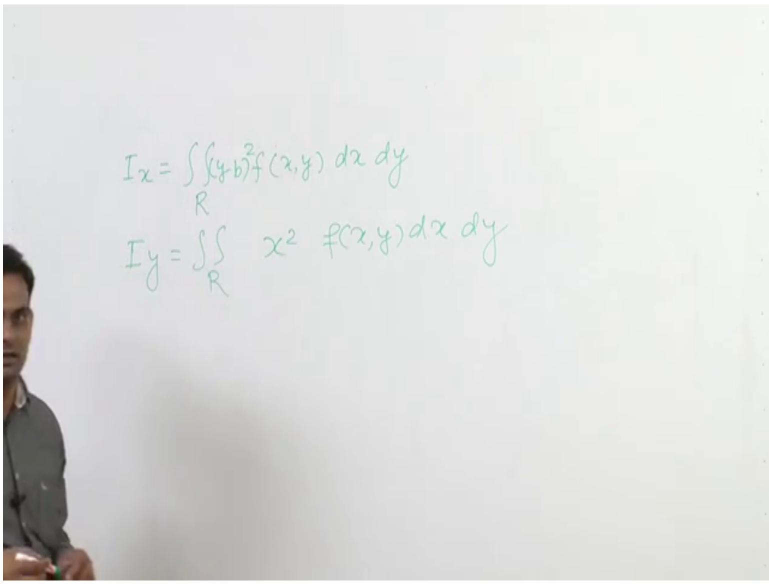

So, it will remain y square f of x y dx dy and then I y will become R x minus a whole square f of x y dx dy. So, here you can see that this is the moment of inertia about the line x equals to a. If I want to find out the moment of inertia about a line y equals to b.

So, there will be no change in I y; however, this term will become y minus b whole square.

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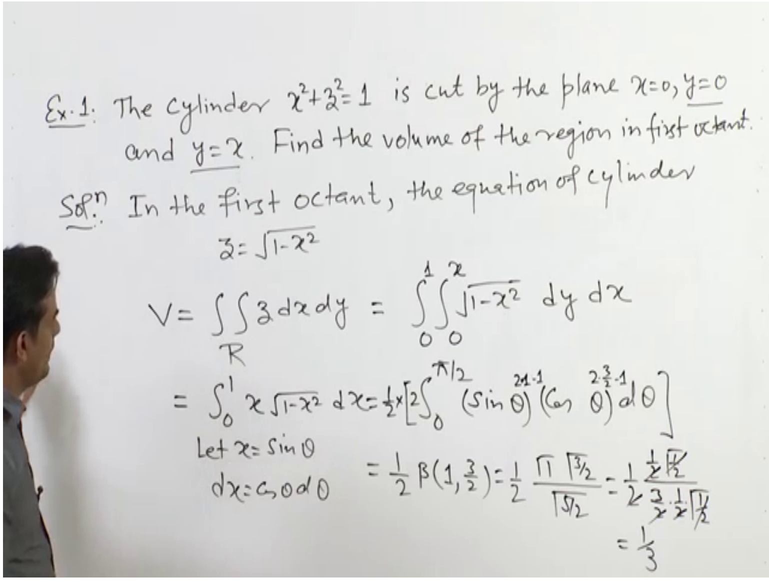

Now, we will take some example based on these formulas with just I have introduced to you. So, the example one is the cylinder x square plus z square equals to 1 is cut by the plane x equals to 0 y equals to 0 and y equals to x. So, this is the cylinder so, this will be the along y axis a cylinder of radius 1, and it is cut by the these 3 planes, x equals to 0 y equals to 0 and y equals to x. Find the volume of the region in first octant.

So, let us find the answer of this question. So, it is given that, we have to find a volume of the region only in first octant. So, in the first octant, the equation of the cylinder can be written as z equals to square root 1 minus x square ok. Now volume is given by the formula z dx dy over a region R. So, the projection of this region this particular region in the xy plane will be something like this. So, z will be square root 1 minus x square. So, y will be 0 to x y equals to x and y equals to 0 and in the xy plane where z equals to 0 x will go from 0 to 1, and then dx dy

So, this will become so, only thing I made a dy dx because these are the limits for x. So, it will become 0 to 1, and then I will integrate this with respect to y. So, y will come so, it will become x square root 1 minus x square dx. So, let x equals to sin theta, then dx will become cos theta d theta. So, this way integral will convert 0 to pi by 2, because when x is one, theta will become pi by 2 x is sin theta, this will be 1 minus sin square theta cos square theta square root will be cos theta and cos theta. So, sin theta cos square theta d theta ok.

So now I will show you that here for solving this we can use, the definition of beta and gamma functions, this I can write 1 by 2 into 2 times this one. Now this is sin theta so, this I can write sin theta raised to power 2 into 1 by 2 minus 1 2 into 1 minus 1. And this I can write cos theta square.

So, this I can write cos theta raised to power 2 3 by 2 minus 1. So, by the formula that beta of xy equals to 2 times integral 0 to pi by 2, sin theta raised to power 2 x minus 1 into cos theta raised to power 2 minus 1 d theta. The same thing we can see here x is 1 y is 3 by 2. So, this will become 1 by 2 beta of 1 3 by 2. This will become 1 by 2 gamma 1, gamma 3 by 2 upon gamma M plus 1. So, gamma pi by 2, this will be 1 by 2 1 by 2 gamma 1 by 2 upon 3 by 2 1 by 2 gamma 1 by 2 ; so, these 2 things will be canceled out. So, answer is 1 by 3.

So, in this way we can do this particular example. Another example I am taking here from the center of gravity. So, find the coordinates of the center of gravity of a plate.

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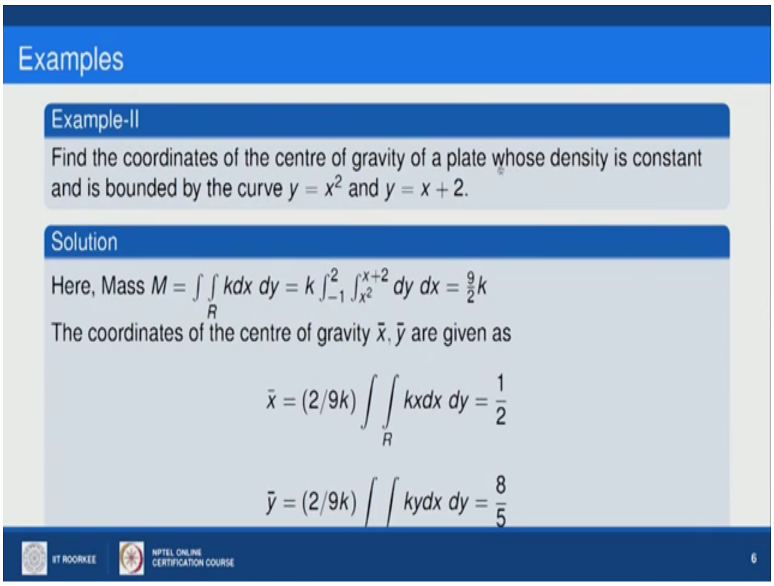

So, please note here it is a plate.

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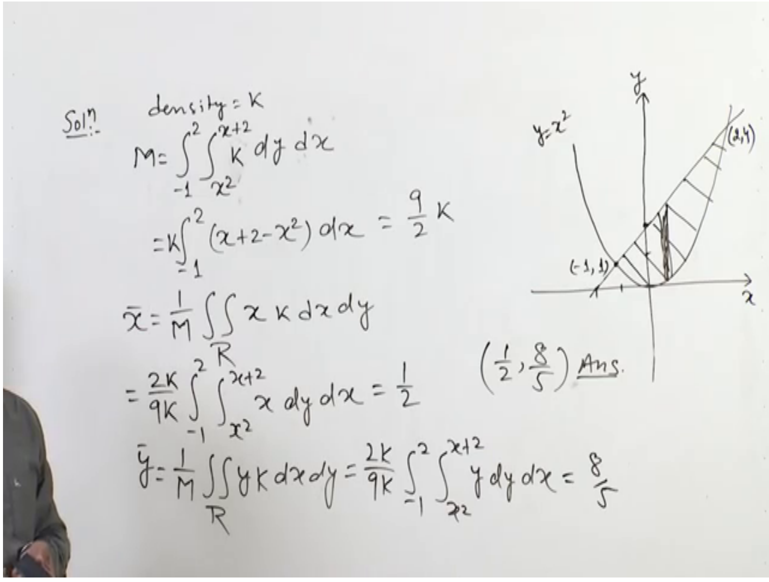

So, 2-dimensional region, whose density is constant, let us say k and is bounded by the curve y equals to x square and y equals to x plus 2. So, the plate is like this x y. So, y equals to x square, and the other one is y equals to x plus 2. So, it will cut 1 2, 1 2. So, the region is this one. So, we have to find out the moment of inertia of this plate when the density is constant. So, it will intersect at this point; which will be minus 1 and 1 and this point will become 2 and 4.

So, first of all for finding the coordinates or moment of inertia we need to calculate mass. So, mass will be let us say M, M will become k, let us say density is k which is a constant dy dx. Here y lower limit is x square if I take a this kind of strip, vertical strip, and the upper limit is x plus 2. While the limit for x is minus 1 to 2. So, this will become minus 1 to 2 k I will take out. So, y so, it will become x plus 2 minus x square dx. And this will come out as 9 by 2 times k.

Now, we need to find out the coordinates of center of gravity. So, x bar is given as 1 upon M integral over region R. R is this plate, and then x times density density is k dx dy. So, this equals to M is 9 by 2 k. So, 2 upon 9 k one k is here. So, this k I can take out, again limit will be minus 1 to 2 x square 2 x plus 2, and then x times dy dx, and this comes out to be 1 by 2 after solving this.

Similarly, y bar is given by 1 by M integral over region R y k dx dy. So, this will be again 2 k upon 9 k minus 1 to 2 x square 2 x plus 2 y dy dx. And after solving this it will come 8 upon 5. So, hence answer is 1 by 2 and 8 upon 5 ok.

So, we have taken two examples one for volume, another one for center of gravity. Now, we will talk about the application of triple integral ok; so, in case of triple integral first of all more mass. So, the mass of a region omega so now, omega is a volumetric region 3-dimensional region, that is given by the triple integral over the region omega rho xy dx dy dz.

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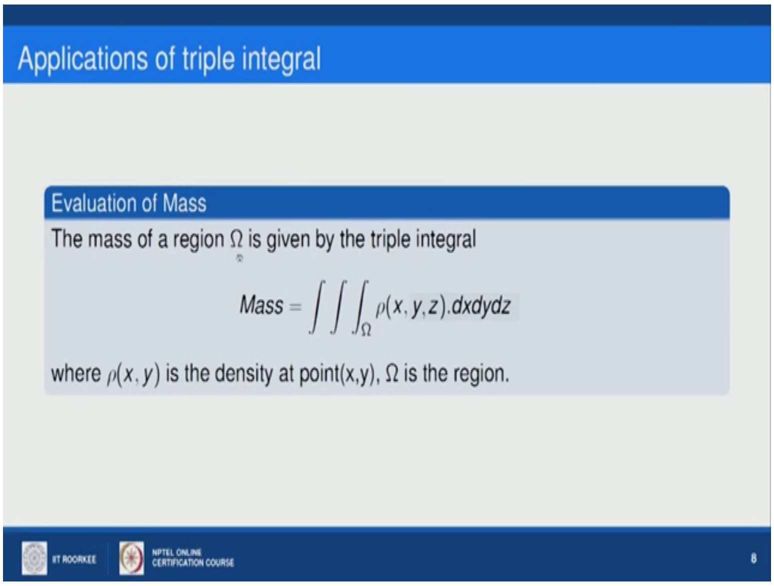

Where rho xy is the density at a point x y and omega is the region. So, for the application of triple integral let us take this example.

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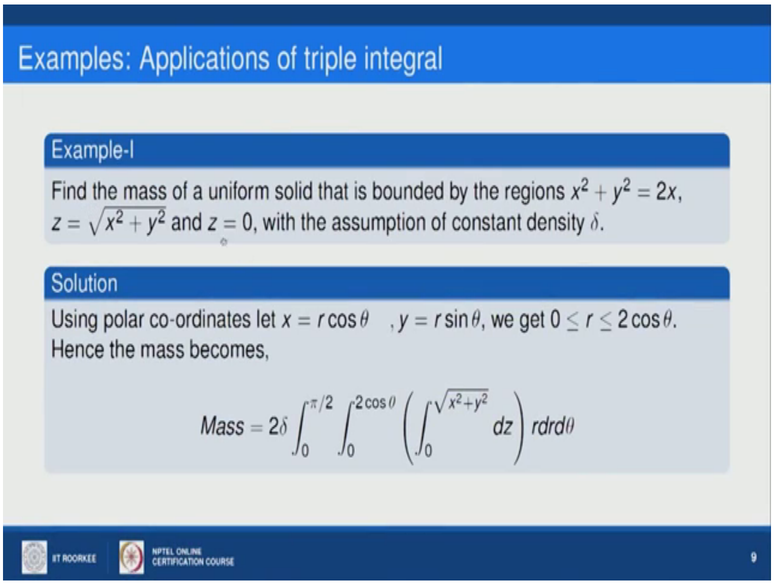

So, find the mass of a uniform solid that is bounded by the region x square plus y square equals to 2 x, z equals to square root x square plus y square, and z equals to 0 with the assumption of constant density ok. So, let us try to find out the way how to solve this example.

So, here I need to find out mass.

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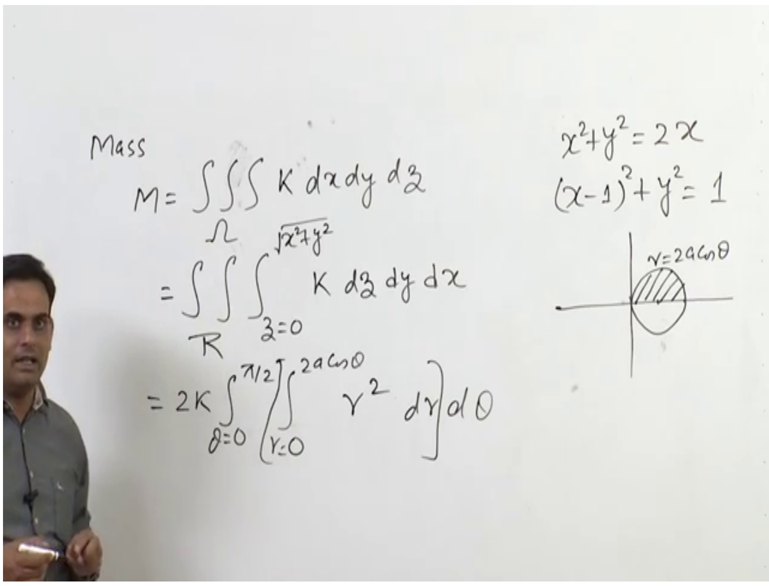

So, mass will be M equals to the region omega triple integral over region omega, density let us say that is the constant. So, I will take k and then dx dy dz. Now, region is something like that. Z is given equals to 0 to square root of x square plus y square, and then a region R; which is given as x square plus y square equals to 2 x and then k times dx dy dz; So, that I will take inside. So, dz dy dx, now the region is given as x square plus y square equals to 2 x. So, it means x minus 1 whole square plus y square equals to 1. So, it will be a circle having center at 1 and 0 and having radius 1. So, it will be a circle like this.

So, whatever mass we will be having above a x axis the same mass I will be having below. So, what I can do? I can write is 2 k. So, k I have taken out, and now I will concentrate on this region only. So, if I change it in polar coordinate, this curve will be R equals to 2 a cos theta. So, my (Refer Time: 22:29) will start from R equals to 0 and it will end on the curve R equals to 2 a cos theta and theta will move from 0 to pi by 2.

So, it is 2 k 0 to pi by 2 and then 0 to 2 a cos theta. So, these are limit for R these are the limits for theta, and then square root x square plus y square will become R and dy dx or dx dy will be R dr d theta. So, basically it will become r square dr d theta. So, first solve this one, and then you can solve for theta.

So, this is the way of handling this particular example. Now, another application of triple integral in terms of center of gravity or center of mass sometime because both are the same thing. So, center of gravity of a solid is given by the coordinates x bar y bar z bar, where x bar is given as 1 upon mass.

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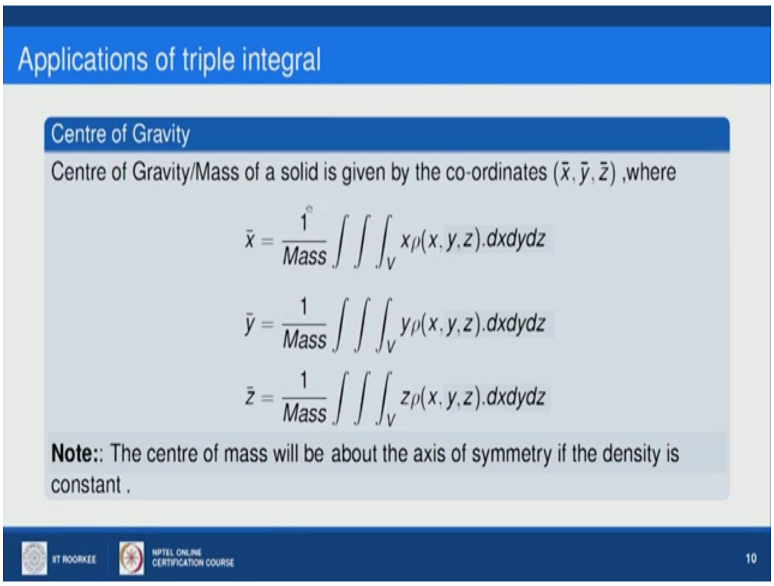

Triple integral over the region v, x times rho xy dx dy dz, here rho xy is the density. Similarly, for y bar just replace this x by y, an in-z bar just replace this y by z ok. So, in this way we can calculate x bar y bar z bar, and please note here we need to calculate mass; For calculating the center of gravity or center of mass.

One more remark, I would like to mention here, the center of mass will be about the axis of symmetry, if the density is constant. So, this is the example find the z coordinates of the center of gravity of the hemisphere of radius a ok.

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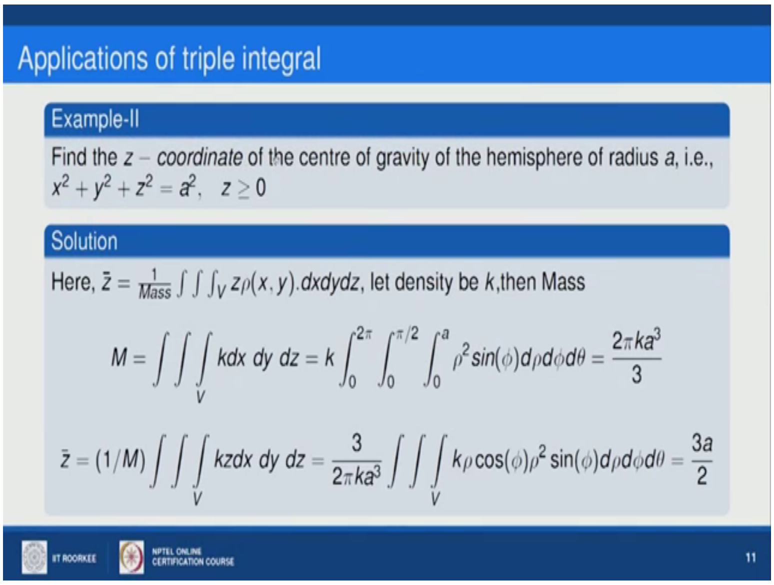

So, it is hemisphere so, above xy plane and give of radius a and I need to find out z coordinate of this. So, let us try to do it so, x square plus y square plus z square equals to a square, and z is greater than equals to 0.

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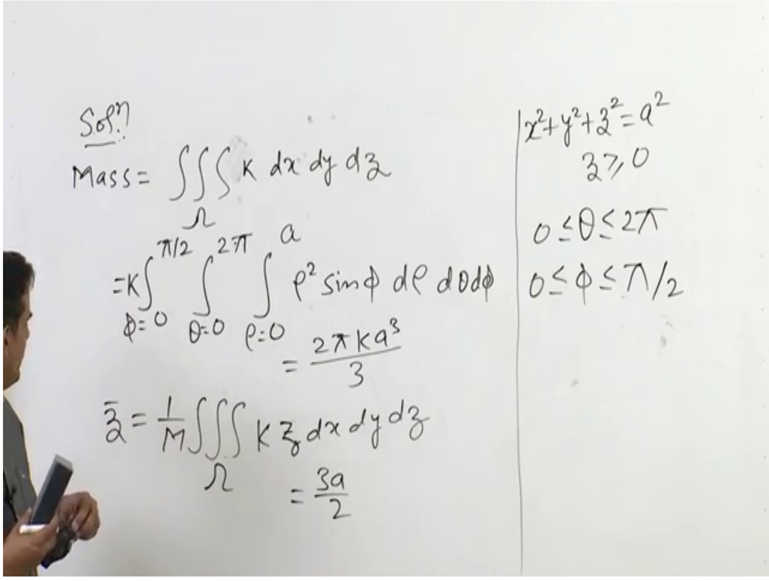

So, basically first of all we need to calculate the mass, and then we will calculate the z coordinate of the center of gravity. So, the idea is mass will be so, density is constant here dx dy dz. Now, if I use the Cartesian coordinates here, the things will become quite complicated; because z will become 0 to square root of a square minus x square minus y square and so on.

So, let us find out there the simple way and we will change it into a spherical coordinates. So, spherical coordinates will be x equals to rho sin phi, and then rho cost phi. And after that d x d y d z will be rho square sin phi d rho d phi d theta. We have seen prove this in some previous lecture in change of variables lecture.

So, hence and the limit will be theta will go for a circle if fair 0 to 2 pi, and for the complete if fair phi will go 0 to pi, but since it is a hemisphere. So now, my phi will go from 0 to pi by 2. So, mass will become 0 to pi by 2 k times, then I am having 0 to 2 pi that is 4 phi this is 4 theta, and then R will because radius is a. So, 0 to a dx dy; So, rho square sin phi d rho d phi d theta.

So, after solving it sorry d theta d phi, because we have written integral in this order; So, after solving it we got it 2 pi k a cube upon 3. Moreover, we need to calculate z coordinates of the center of gravity. So, it is given by 1 over M integral over the region R v, omega, sorry I have taken omega k times z into dx dy. So, again change everything in polar coordinates sorry spherical coordinates and then solve it. So, it will come out as 3 a by 2.

Similarly, you can calculate the x bar and y bar also in x bar here we will be having x in y bar, here we will be having y. Now, another application of triple integral is moment of inertia.

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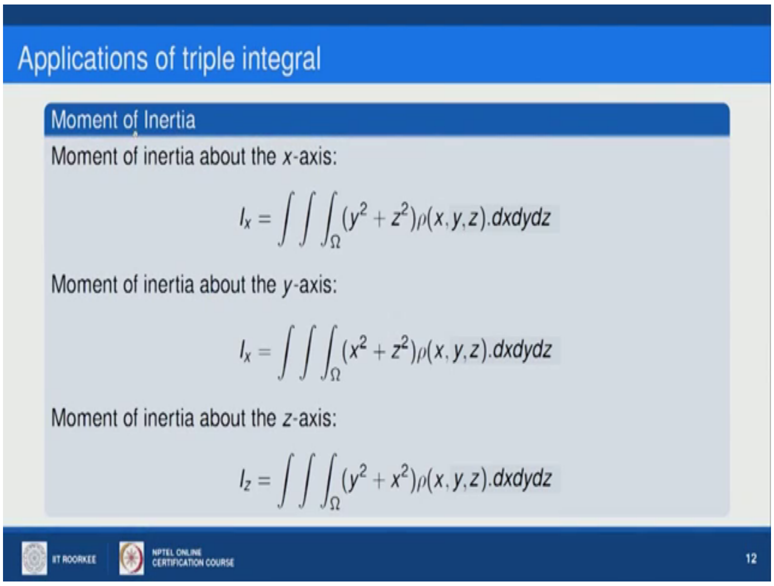

So, the moment of inertia about the x axis is given by Ix equals to triple integral over the region omega, y square plus z square into density into dx dy dz.

Similarly, moment of inertia about y axis is I y triple integral over omega x square plus z square into density function dx dy dz. Similarly, the moment of inertia about the z axis is given by I z triple integral over omega, y square plus x square into density function dx dy dz.

So, for example, if we are having this one, calculate the moment of inertia of a cylinder define y x square plus y square equals to 1, z is greater than equals to 0, and z is less than equals to 2, when it is rotated around x axis, y axis and z axis

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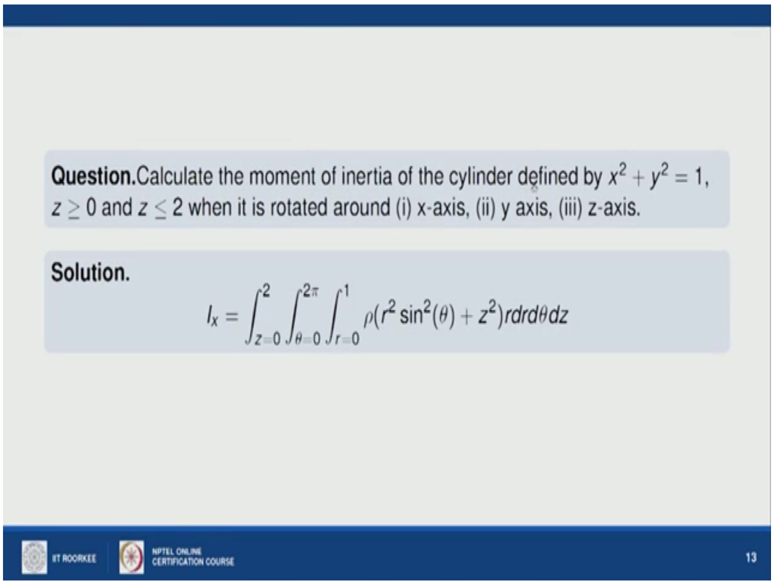

So, moment of inertia about x axis I x is given by; so, z is given 0 to 2. Now x square plus y square equals to 1. So, this is the cylinder of radius 1 along z axis. So, on the xy plane the projection of this cylinder will be circle of unit radius. So, we can change it in 2 polar coordinates so, theta will move from 0 to 2 pi and r will move from 0 to 1 rho r square sin square theta plus z square r dr d theta dz.

So, we are using here cylindrical coordinates. Rho square sin square theta for y and z will be a such density and dx dy dz will become rdr d theta dz. So, by solving this we will calculate Ix, if I need to calculate I y, only change will be this sin square theta will become cos square theta. Because, it will be x square plus z square so, x will be R cos theta and if I need to find out I z will be r square sin square theta plus r square cos square theta. So, r square I will take common. So, it will become rho r square r times d r d theta dz. So, in this way we will solve this particular example.

So, in the brief summary of this lecture, I have introduced the applications of double integral, and triple integral, the applications were basically finding the area finding the volume, finding the mass of a region, finding the coordinates of center of gravity and then finally, finding the moment of inertia about a given line. So, with this I will end this lecture.

And, thank you very much.