Dynamical Systems and Control

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Lecture - 08
Solution of Linear Systems - II

Hello friend, welcome to this lecture and in this lecture we will continue our discussion of finding n linearly independent solution of x dash = Ax. And we have seen that we can find out n linearly independent solution provided that we can find n linearly independent eigenvectors of the coefficient matrix A.

And we have one result in this regard that if you have n distinct eigenvalues of the coefficient matrix A then we can find out n linearly independent eigenvectors and with the help of this eigenpairs can as we can find out the nn linearly independent solution of x dash = Ax, and we have seen one example. Now let us; in that example we have considered that if we have say lambda 1, lambda 2, lambda 3 all are distinct eigenvalues then we have find out the eigenvectors and we have written down the general solution of x dash = Ax.

Now we move forward and let us assume that, that if we have say complex eigenvalues then how we can handle the situation.

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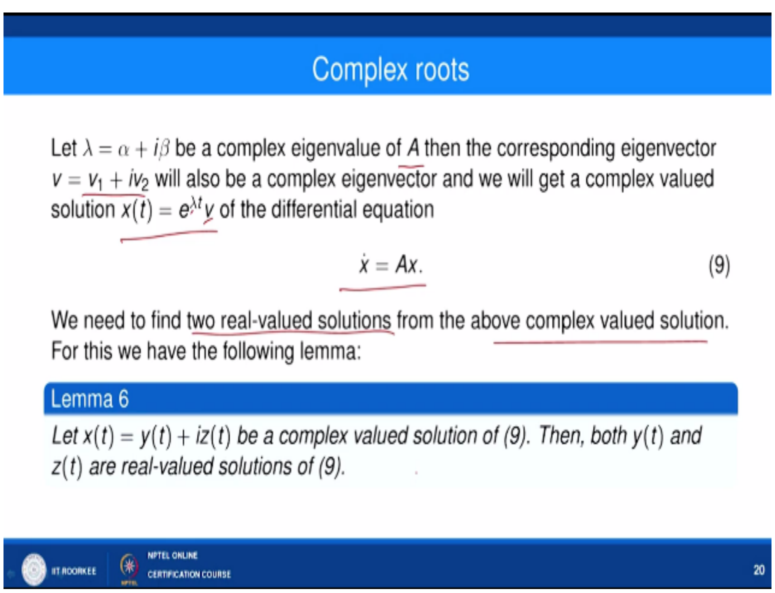

So here let us consider that let lambda = alpha I beta be a complex eigenvalue of A, right. So if we have a complex eigenvalue it may happen then the corresponding eigenvector let us assume that it is given as v as v+1 I v2 will also be a complex eigenvector. And in this case we will get a complex-valued solution that is x(t)=e to the power lambda t of the differential equation x dash = Ax.

But since all these problems are coming from real world problem so we are interested in real-valued solutions also. So from a complex-valued solution we need to find out the real-valued solution. So our idea is to how we can form; how we can find out to real value solution from the above complex-valued solution that is x(t)= e to the power lambda tv where lambda is alpha + I beta and v is v+I v 2.

So from a complex solution how we can find out to real value solution that is what we wanted to discuss here. So in this we have the following Lemma that let x(t)= y(t)+iz t be a complex-valued solution of x dash =Ax. Then both the real part and the complex part are real- valued solution of 9. So it means that we have x(t)=e to the power lambda t v as a solution and since it is a complex solution I can write down this y(t)+I z(t) where y(t) is real part of this x(t) and z(t) is imagery part of this.

Then this Lemma says that this y(t) and z(t) be two real-valued solution of 9 and we can also prove that they are linearly independent solutions. So first let us prove that these are real value solution for which we simply say that since x(t) is a solution of this so it will satisfy here.

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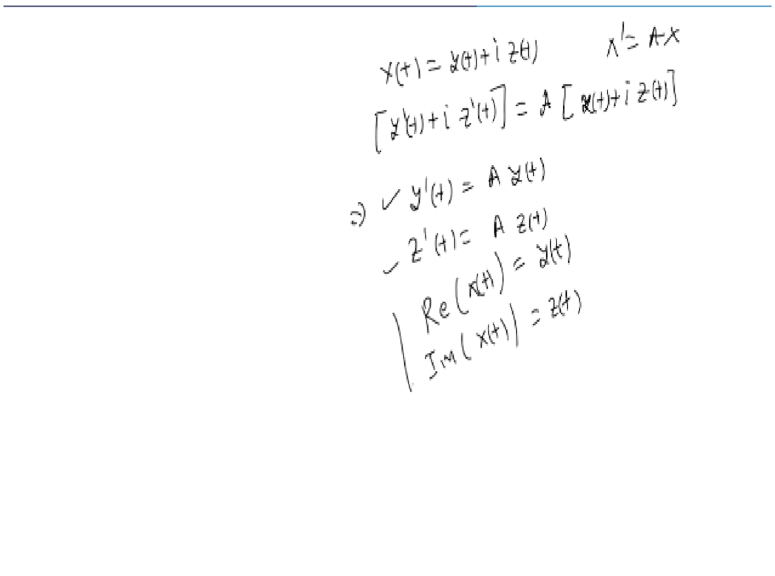

So let us write down x(t) =y(t) + i z(t) and x(t) is a solution of x dash =Ax so it means that y dash t + i z dash t must be equal to A times y(t) + I z(t). And if simplify the real image we separate the real imagery part we have y dash t =A of y(t) and z dash t =A of z(t). So it means that y(t) and z(t) separately satisfy the system x dash =Ax, hence we can say that the real part real of x of t and imagery of x of t that is here it is y(t) and hear this is it z(t) will also be a solutions of x dash =Ax. Okay.

So once we have this Lemma then we can see that if you have a complex solution then from that complex solution we can obtain two real-valued solution of x dash =Ax.

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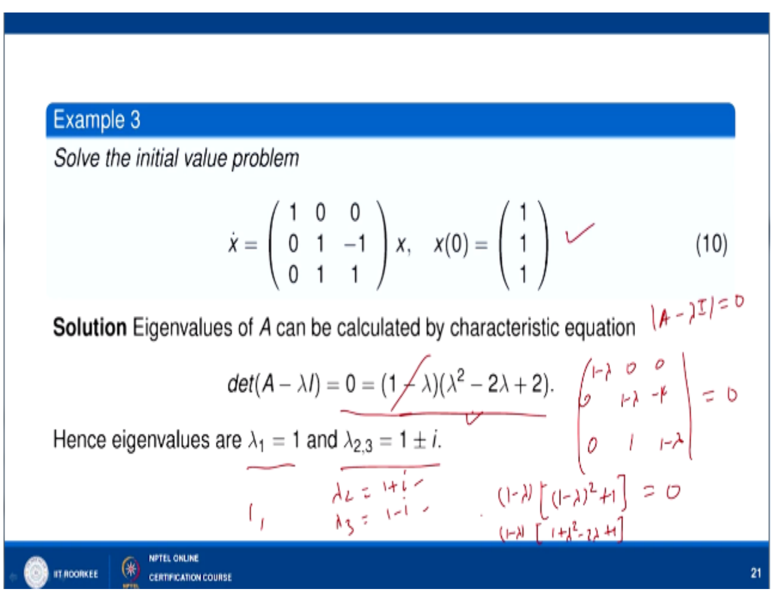

So now latest consider the following example. So solve the initial value problem x dash =Ax where A is given as 100; 0 1-1; 011 and x(0) is 11 1. So here let us focus on finding the particular example rather than finding the general solution. Of course you can find out the particular solution after fixing the constant C1 C2 C3. So let us see how we can find out a particular solution of the system x dash = Ax where x(0) is given as 11 1.

So first thing is we need to find out the eigenvalues and eigenvectors. So let us calculate the eigenvalues here. So you find out characteristic equation, characteristic equation A - lambda I = determinant of this is =0 so 1 - lambda 00; 0 1 -lambda -1 0 1 1 - lambda =0 so determinant of this is =0 so 1 - lambda times you can write down 1- lambda whole square + 1 and that is all it is =0 and you can simplify 1- lambda * 1 + lambda square - 2 lambda + 1.

And you can write down this has 1- lambda * lambda squared - 2 lambda + 2 =0. So this is your characteristic equation. And here you can find out one root from this and that is lambda 1 =1. And you can find out the second root from this that is lambda square - 2 lambda + 2 =0. And when you solve this we can get a complex root that is 1 +-I.

So let us denote this as lambda 2 as 1+ Psi and Lambda 3s 1-Psi. So; because they comes in a conjugate pair basically, so now let us find out the eigenvectors. Now it is also obvious here that 1 1+i and 1-i are all distinct eigenvalues. So here one thing is very much sure that we will get linearly independent eigenvectors. So first let us find out the eigenvectors.

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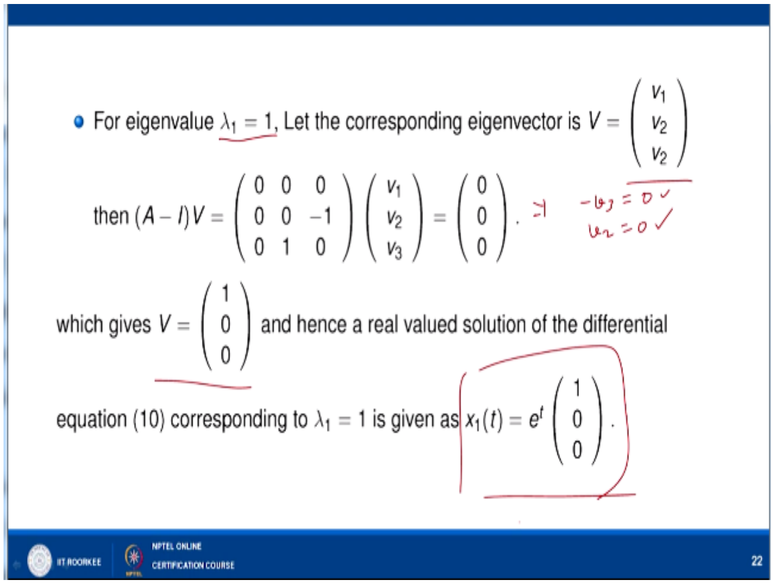

So for eigenvalue lambda 1=1. Let assume the corresponding eigenvector is given by V1 V2 V3 and we can find V1 V2 V3 solving this A - Lambda 1 I v = 0. So lambda 1=1, so we have 0 0 0; 0 1 -1; 0 10 * V1 V2 V3 = 0 0 0. So if you simplify this is = what that -V3 =0, V2 =0. So it means that he here the V1 V2 V3 then V2 and V3 must be 0 so the only possibility left out is that V1 is arbitrary, so V1 is arbitrary so let us take V1 as value 1.

So we can say that V=100 is an eigenvector corresponding to lambda 1 =1. The corresponding solution is given by x1(t)=e to the power lambda 1 t that is e to the power t*1 0 0. So here one real value solution is obtained by this.

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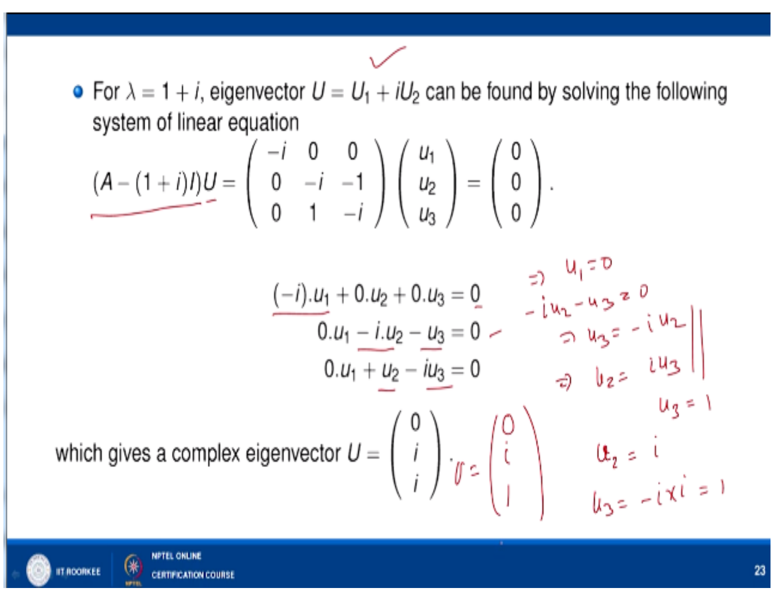

Now second real valued solution we want to find out but here we have a complex root that is lambda = 1 + Psi. Now eigenvector responding, eigenvector we are writing u1+I of u2 and the eigenvector we can find out like A - lambda I * equal to this thing; =0. So A-1 + Iu=0 is -I 0 0; 0 -I -1; 0 1 -I; u1 u2 u3 = 0 0 0. So this is what - I u1 =0; -I root 2 - root 3 =0; u2 -Iu3=0. So first equation gives you that u1=0. And second equation gives you that -u2 -u3 = 0.

So you can say that u3 is equal to -I u2 and second and third last equation is that u2=-I u3. So you have u3=-I u2 and u2=I u3. So let us take u3 as a 1 and you can find out u2 as I hear and you can check whether it is satisfying here. So u3 you are getting as - I * I so that is 1 here, so it means that it is satisfying. So it means that your u is coming out to be u1 is 0, u2 is I and u3 is 1. So 0 I 1 is your eigenvector that is u. So u is written as 0 I 1 as a eigenvector.

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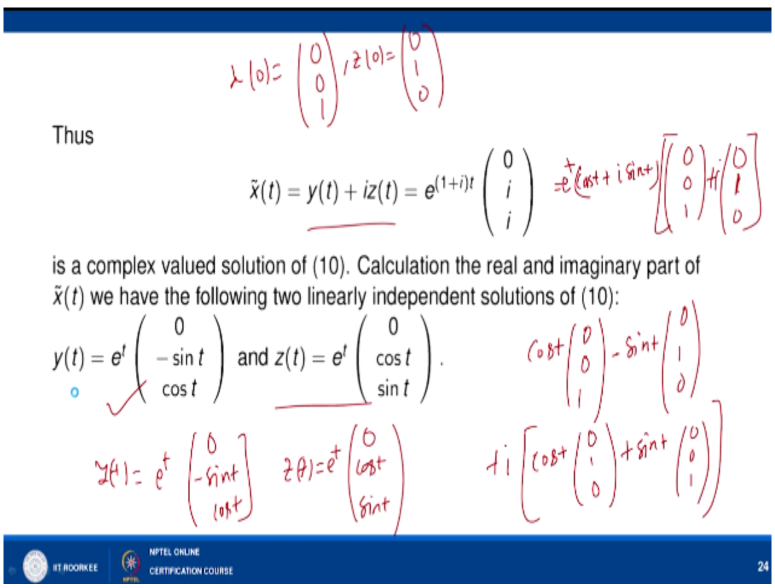

So you can find out the solution complex eigenvector complex solution has y(t) + z(t)=e to the power the power 1 + I t * z0 I and 1 here. So now we can simplify this. We can simplify as writing here as this is nothing but cos of t + I sin t and here we have say 0; say 0 0 1 + 0 say I and this is 0 and from this week and find out the real and imaginary part. We can I write down here this as I times here it is 1 only.

Then we can write down the solution real part you can simply say that it is you can simplified is cos of t * 001 right, here you will get -sin of t 0 10; this is the real part and imaginary part will be cos of t 01 0 and imagery part is also here it is sin t 001, so it is imagery part. So your y(t) is given as you can write down sorry e to the power t is missing here. So your y(t) is coming out to be e to the power t.

And it is what let me write it here 0 -sin and cos of t so real part is given as e to the power t* 0 -sin t cos t that is written here and z(t) you can find out about t times, what is your solution 0 as cos of t and it is sin t. So your immediate solution is given by z(t) * e to the power t 0 cos t sin t. And you can easily verify that these are linearly independent solutions for that you can simply take t=0 if you take t=0 your y(0) will be what y(0)=0 0 and 1 and z(0) is coming out to be 0 1 and 0 and you can check that these are linearly independent vectors in 3.

So it means that y(t) and z(t) are also linearly independent solution of x dash=Ax. And if you look at here we have not solve for lambda = 1-I because if you solve for lambda = 1-I you will also get the same real and imaginary part. So here in the case of complex root we take one root say among the pairs you take one say one element let us say lambda = 1 + I and with this we try to find out complex solution and then we can find out to real solution given by y(t) and 2(t). So now we have three linearly independent solutions given to us.

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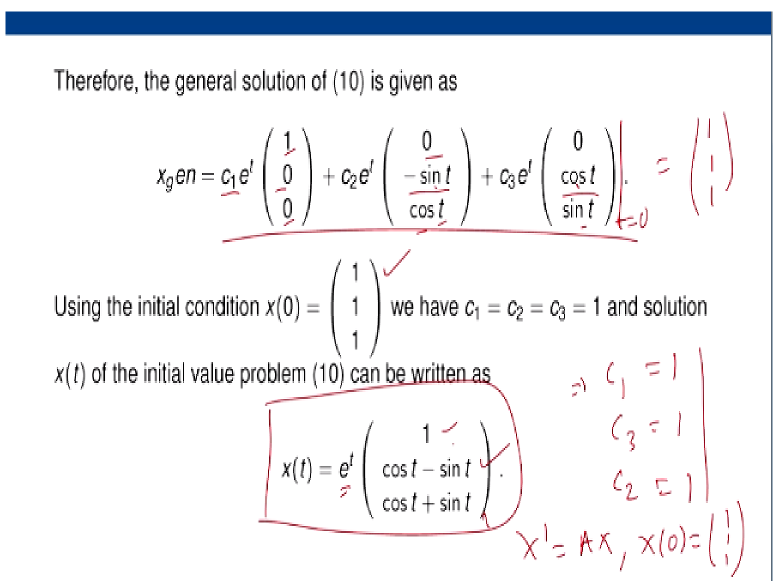

So we can write down a general solution has C1 e to the power t * 1 0 0 + C2 e to the power t * 0 -sin t cos t + C3e to the power t0 cos t sin t., so this is your general solution we can write it like this. But we want to find out a particular solution it means that we want to find out a solution with satisfy the initial condition given at 0 as x(0) as 111. So now with the help of this we can fix our C1 and C2 and C3.

So we can fix that let us say that x(0)means that t=0 we have it is 111.So if you look at look at this here to C1 and here we have 0 so C1=1 and if look at the second thing here we have 0 so we cannot get anything here also we will get 0. Here we will get C3=1. And if you look at the third entry that it is zero here it is you will get C2 and here also you will get. So C2 is also got. So it means that here C1 C2 C3 all are one and you can find; we can find out the particular solution by putting C1 C2 C3 as 111.

And we can write down the solution has e to the power t is common in everyone, so you can write down e to the power t out so it is 1, so 1 at first place. Second place it is -sin t + cos t, so cos t - sin t and in third place we have cos t + sin t, so we have this think. So this is the solution of x dash = Ax with the initial condition that x(0) is 1 1 1. And we can check that it is actually a solution of this.

Now moving on; so it means that we have discussed the case, when we the roots are distinct. First case was distinct and real and second case when distinct but complex. And we have; we have seen one, one example in each case. Now let us move in the case when we have equal roots. Because in case of distinct root we have the guarantee that the corresponding eigenvectors are linearly independent eigenvectors and we can find out the solutions. But in case of equal roots that theorem will not be applicable.

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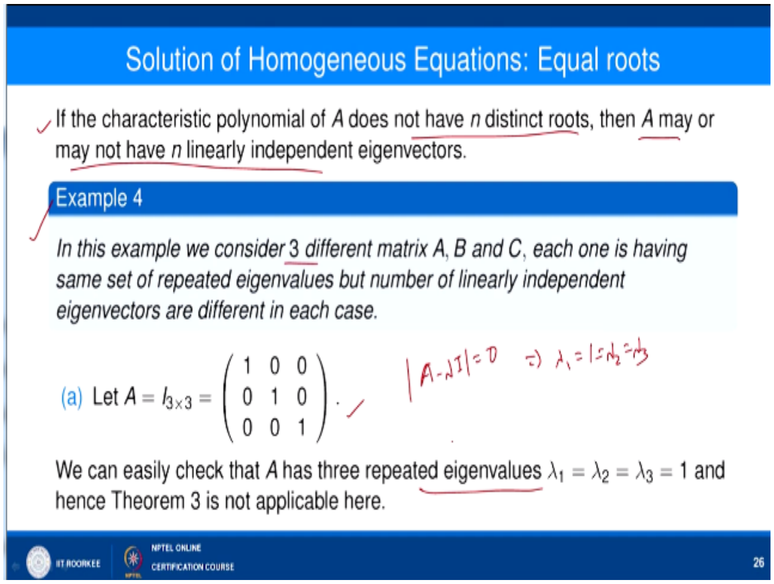

And we can say that, we can consider the thing that if the characteristic polynomial of a does not have n distinct roots, then A may or may not have n linearly independent eigenvectors. So that we can understand from this following example, so in this example we considered three different matrices say A B and C.

Each one is having same set of eigenvalues of course repeated eigenvalues. But number of linearly independent eigenvectors are different in each case. So let us say with this example we try to see that in case when we have repeated eigenvalues then there is no guarantee at all that it has n linearly independent eigenvectors or not. So let us consider the following example, so first example let us take A as identity matrix say of 3*3. So we have 1 0 0; 0 1 0; 0 0 1.

And we can see that A has three repeated eigenvalues we can check that eigenvalues of this we have to write down A - lambda I=0 so this implies that lambda 1 = 1 = lambda 2 = lambda 3, so we; here we have three repeated eigenvalues that is each is equal to 1. And hence the previous theorem that distinct eigenvalues will give you linearly independent eigenvectors will not be applicable here.

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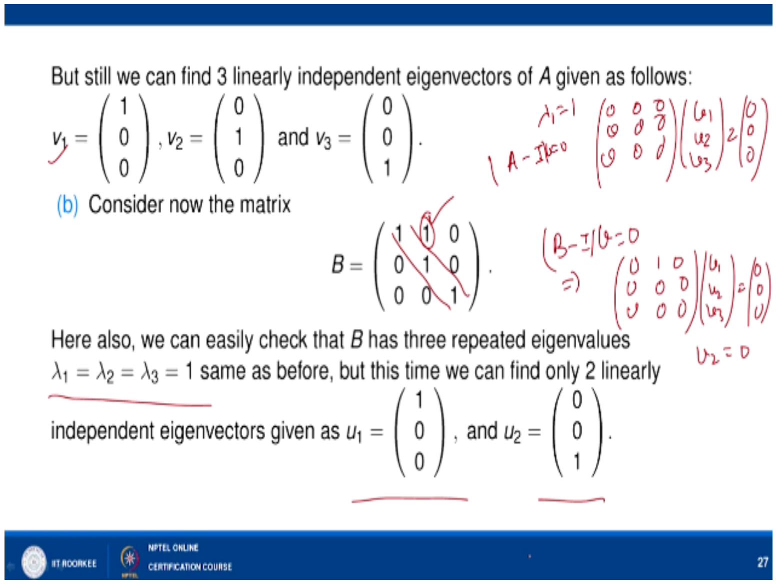

But still in this case we can find out three linearly independent eigenvectors of A given as follows. How we can say, you take lambda 1 = 1 and look at hear A- say I-I; A-I = v=0 and here we have 0 0 0; 0 0 0; 0 0 0 and V1 V2 V3. So here we can have 3 linearly independent solution and it is given as V1 as 1 0 0; V2 as 0 1 0 and V3 as 0 0 1. So in this case do we have repeated eigenvalues, but still we have linearly Independent eigenvectors.

And we want three linearly independent eigenvectors and we are getting three linearly independent eigenvectors. Now consider the second matrix. Here in diagonal entries it is all one month, only one tree is a non-zero because in our previous case only the diagonal entries are non-zero rest are all 0. But here we have just perturbed our identity matrix and we simply took one non-zero value at this 1 2 place. So B is just a perturb case of A.

So here also we can check this since it is say upper triangular matrix we can easily check that the eigenvalues and nothing but diagonal entries. So 1 1 1 is also repeated eigenvalues in this case also. In this case also we have repeated eigenvalues and each one is = 1. But here we have only two linearly independent solutions, how we can check; we can simply say B- I, this is V=0. Let us solve this and here we have 0 1 0; 0 0 0; 0 0 0 and here we have V1 V2 V3 = 0. So here we simply say that your V2 has to be 0.

So this means that condition is V2 has to be 0 so we have two free variable that is V1 and V3 and we can simplify we can find out to linearly independent solution as 1 0 0 and 0 0 1. So here we have three repeated eigenvalues but in this case rather than having three linearly independent eigenvectors we have only two linearly independent eigenvectors. So here we simply perturbed our identity matrix by at only one position but the result is very say disappointing that here we are not able to get three linearly independent eigenvectors.

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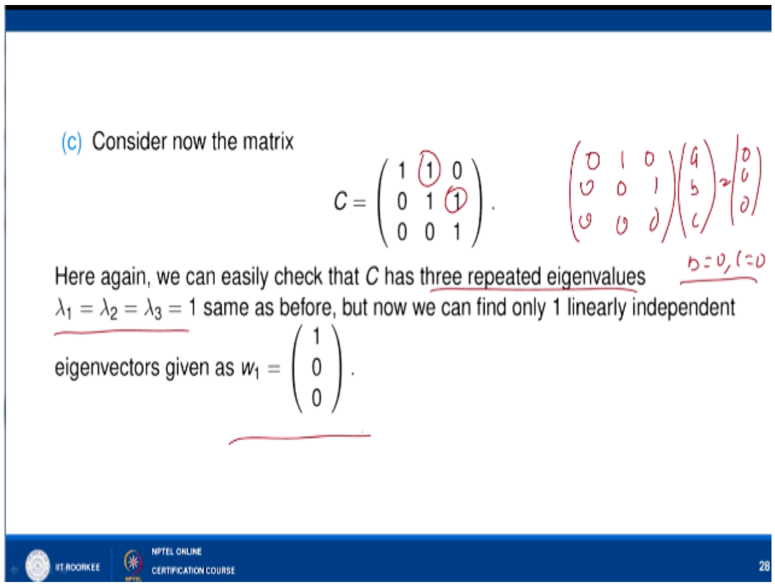

Now consider one more case, here we again perturbed our matrix B. And in place of having; in only one place we have changed in B but now here we change the value at two different places that is in diagonal as well as the above diagonal entries are also known non-zero. So here again since it is upper triangular matrix eigenvalues are nothing but the diagonal entries that is 1 1 1, so here again three repeated eigenvalues given as 1 1 1.

But here if you want to find out say eigenvector corresponding to lambda 1 then we have 0 1 0; 0 0 1; 0 0 0 and here we want to find out ABC as 0 0 0, so here we condition is a B=0 and C=0. So here, B=0 and C=0 is fixed so only one eigenvector is possible that is 1 0 0. So we have seen that three cases basically. In both the cases having same set of repeated eigenvalues that is 1 1 1. All the three are 3*3 matrices.

But in one case, we have three linearly independent eigenvectors in another case we have to linearly independent eigenvectors and in the last case we have only one linearly independent eigenvectors. So here we cannot how we check that we are dealing with which kind of a matrix that is A B or C. So in this case when we have repeated eigenvalues it is quite difficult to find out n linearly independent eigenvectors.

Of course we are not denying that in some cases we may have n linearly independent eigenvectors. But it is quite difficult to check whether our matrix given matrix falling in that particular category.

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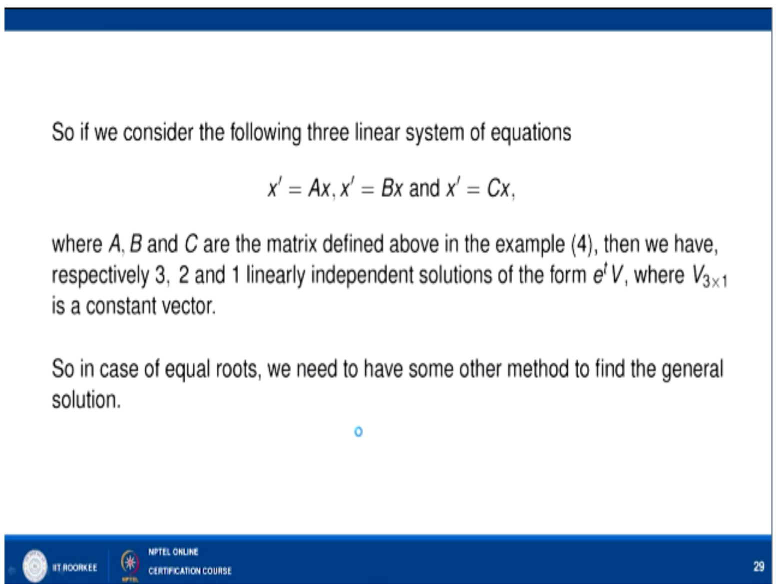

So if we consider the following three linear system of equation that is x dash = Ax; x dash = Bx and x dash = Cx, where A B and C are the matrix define in the example 4. Then in this case we have only 3 in the first case, 2 in second case and 1 linearly independent solution of the form e to the power t v, where v is 3 cross 1 is a constant vector. So in first case we have 3 linear independent solution, second case we have 2 linearly independent solution and third case we have one linearly independent solution.

So this means that only in first case we can write down the general solution but in second and third case we may not write down the general solution in this particular situation. So we need to find out the remaining linearly independent solution of this system x dash = Bx and x dash = Cx.

Why? Because we have already have the guarantee that the solution space of x dash = Bx and x dash = Cx has a dimension and so there exist n linearly independent solution.

The only thing is that in this coming situation we may not be able to find out in this particular form that is e to the power t*v. So now in case of equal roots we need to have some other method to find out the general method, general solution.

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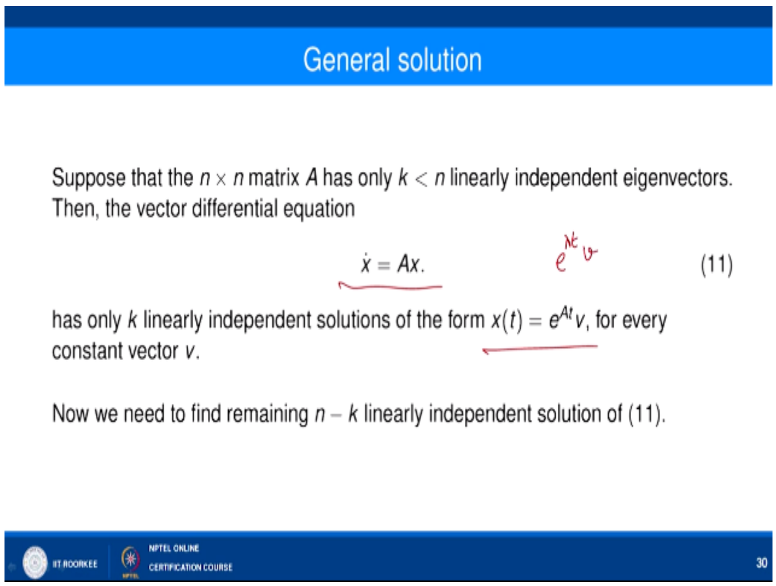

Suppose that n*n matrix A has only k which is <= < n linearly independent eigenvectors. So it means that we have system x dash = Ax. But we are able to find out only k linearly independent eigenvectors; we not able to find out the n linearly independent eigenvectors. So it means that some of them must be repeated eigenvalues. So in this case we have only k linearly independent solution given in this in the form of e to the power in the form of e to the power lambda t and v.

So now; for a given constant vector v. Now we need to find out remaining n-k linearly independent solution of this system x dash = Ax.

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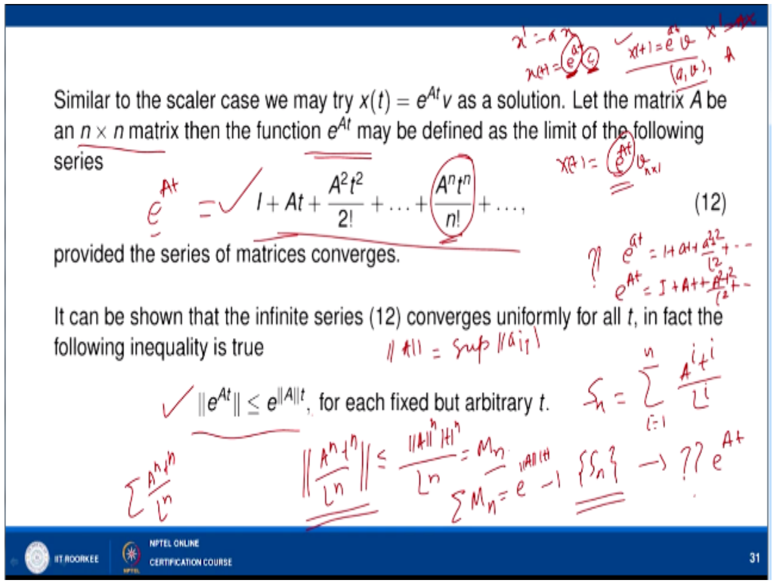

So here we get the idea from the scalar case as we have pointed out that we have x dash = Ax then the solution is given as x(t) = e to the power At * constant. So in first case what we try to do here, we take that this C in place of constant value, if we replace this C/A vector it means that x(t) = e to the power At * that v then under what condition this will serve as a solution. And B have shown that this will act as a solution provided that this A, v is an eigenpair of A.

So corresponding to system x dash = Ax, this x(t)= e to the power At v will work has a solution provided that this Av is an eigenpair of matrix A. And we have seen that this will this work very fine provided that A has n distinct eigenvalues. Now and in case of repeated eigenvalues we may not have n linearly independent solution. Then our ideas now look at this component, so here we have two component a C and this.

Here in the first case we have generalized the C as eigenvector; C as a vector and we try to work with this kind of a function has a solution. Now in place of this now let us generalize this, in place of these let us try to see what happened; can we define this kind of a solution where x(t) = e to the power At v, where v is n+1 and e to the power At is something similar to e to the power At in scalar case.

Now here how we define this matrix and whether this defined or not, so we may try first of all that x(t) = e to the power At v as a solution and let the matrix AV is n*n matrix and then the function e to the power At maybe define as a limit of the following series. Because here we are taking the motivation from the scalar case and we know that in the scalar case our solution e to the power At is nothing but 1+At+A square t square/factorial 2 and so on. So in case of when this A, small a is replaced by matrix then can we have e to the power At as I+At+A square t square/factorial 2 and so on.

Can we have similar kind of structure? And if it is, if you have this solution; if we have this kind of structure will it; will it work as a solution of the system x dash = Ax. So here we may define this as a limit of this infinite series provided that this infinite series converge. So here first thing we need to work whether this infinite series converge or not. So to show that this infinite series converge or not we use the criteria that is quasi method of criteria, so for that let us consider this Sn as summation I=1 to say n this Ai ti upon factorial i, this as a sequence of matrices.

So here this Sn is a sequence of matrices. And if you recall we have discussed the under what condition this Sn will converge to a limit. So here we say that this will converge to a limit which we call this has e to the power At. So first thing we want to show whether this will converge or not.

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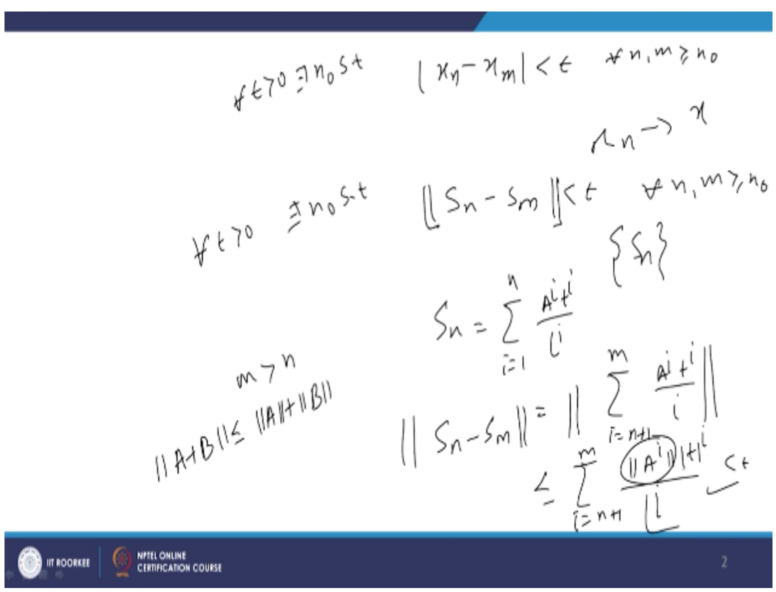

For that, if you look at that we have one result in case of scalar thing that Xn is a sequence where Xn is a sequence of partial sum it will converge provided that it satisfies the quasi criteria that for every epsilon > 0 they exist n0 such that modulus of Xn-Xm is < epsilon for every n, m > or not. If you have this kind of criteria, then the Xn will converge and converge to some limit let us call this Sn. So we are using the similar kind of result for Sn here so Sn-Sm < epsilon for every epsilon > 0, we need to find out n0 such, this is true for every n, m > or = n0.

So only thing is that here the modulus is replaced by norm here. Is that okay? So here we want to check that whether this Sn-Sm, Sn this sequence Sn of matrices forms a quasi; satisfy a quasi criteria or not. So where Sn is define as summation, let us I=1 to n, A to power i t to the power i upon factorial i. This will satisfy the quasi criteria or not.

So if you look at the Sn-Sm let us assume that m is bigger than n. Since m is not equal to n so let us assume one way. So norm of this is equal to norm of summation, here it is i=n+1 to m, A to the power i + i/ factorial i. Now we want to show that this is <= 1. So here we can use any of the norm and we can simply say that here we using this norm of A+B is <= norm of A+norm of B. So we are using this summation, so we can write down this as summation I=n+1 to m and norm of Ai ti factorial i, factorial i.

So now this norm of Ai is some number because norm is a function from space to real line. So all these are; so it basically reduces to your sequence of real numbers and we simply say that if this can be made arbitrary small then we are done then Sn is a quasi; satisfy the quasi criteria and we are done. But if you look at this modulus of norm of 3i this can be further less than this.

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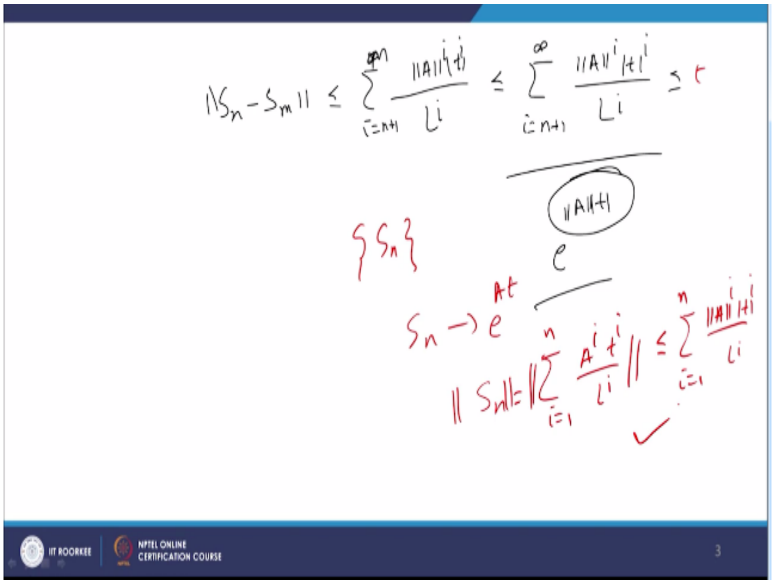

So we can write it here. Say norm of Sn-Sm <= summation i=n+1 to m. Here we have norm of Ai ti and factorial i. So here let me take this modulus of tn. And this can be further, I can write it i=n+1 to infinity norm of Ai modulus of t i upon factorial i. Now if you look at this norm of A is a particular number then this is nothing but the tail of e to the power norm of A, right. So this is the tail of this series e to the power norm of A modulus t.

We already know that for fixed t this is a convergence series, this is for the <= e power; okay so this is tail of, let me write it here, okay. So here if it is a tail of this convergence series then as tending to infinity this is tending to 0. So it means that as n is very large this can be make arbitrary small, right. So here we are using this fact that Sn is a sequence of matrices which satisfy the quasi criteria.

And hence we can say that Sn will converge to some limit and we call this limit as e to the power At. And not only this we want to show that here if you look at here Sn is basically what, Sn is summation I=1 to n A power i t to the power i upon factorial i. And we can say that this is norm of this is <= the norm here. I t is you can write it here summation I=1 to n norm of Ai modulus of ti upon factorial i, right.

So norm of Sn is less than equal to this; so we can say that this will require later. So first thing we have proved that Sn is a sequence which satisfy the quasi criteria and hence this infinite series will converge, right. And if this infinite series converge we can define this value as e to the power At. So we define this as e to the power At. So it means at least this term that e to the power At make sense. Now we can also show that this infinite series converges uniformly for all t. So for that we can apply the, (() (34:39).

And if you look at in this series, what is your nth term, nth term is An tn upon factorial n and which basically say that norm of An t to the power n upon factorial n is <= norm of A power n modulus of t to the power n upon factorial can call this as Mn. We already know that this summation Mn is a convergence series in fact it is a part of it is nothing but e to the power; this is; this can be written as e to the power norm of A or less t – 1.

So we can see that this is, this is a convergence series and hence this, hence we can say that this term let us, so it means that this summation An tn upon factorial n is also a convergence series, not only convergence series it is a uniformly convergence series. So we can say that using (()) (35:45) we can say that the convergence of this infinite series to e to the power At is uniform.

And we can also prove the following identity that norm of e to the power At is <= e to the power norm of At. Here I have taken this particular example norm of A is, you can take the supermom of Aij basically, modulus of Aij, I have taken this. We can take any kind of; and we can prove that norm of e to the power At is <= e to the power norm of A*t. So we have shown that this infinite series is converge and converges uniformly to e to the power At. And limit we are defining as e to the power At.

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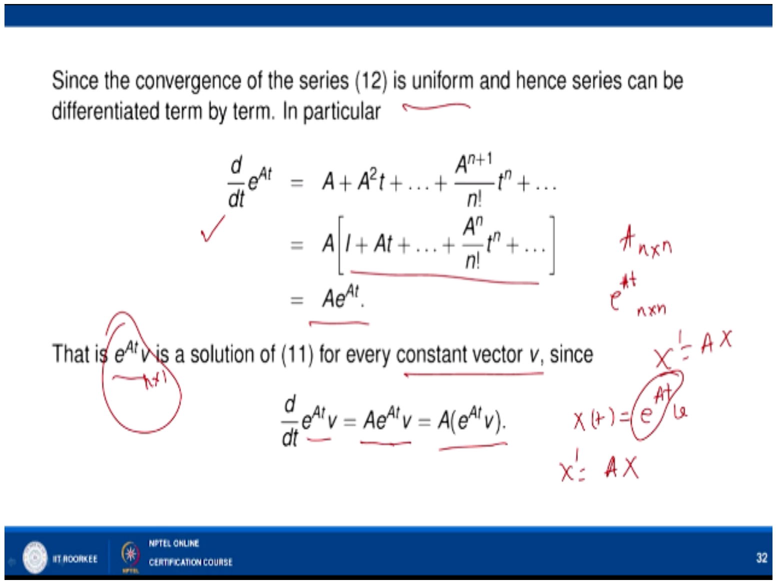

Now since the convergence of this series is uniform and hence we can differentiate this series term by term and we can evaluate d/dt of e to the power At, and it is nothing but A+A square t + so on. And if we take out this A common then it is A*I+At and so on and this is nothing but e to the power At. So we can write down d/dt of A to the power At is A e to the power At. So here Ae At is n*n and e to the power At is also n*n matrices.

So, let us relate this with our system of linear equation x dash = Ax. And we say that e to the At*v where v is n cross 1 then this will be n cross 1 function then we want to show; we claim that e to the power At v is a solution of x dash = Ax for every constant v. So d/dt of e to the power At v = At power At, v is just a constant we can take it out and it is nothing but A*e to the power At v. So it means t hat x(t)=e to the power At*v is a solution of x dash = Ax, right.

So it means that if we can calculate e to the power At for a given matrix A then we can write down the solution as e to the power At*v. But this problem of calculating e to the power At is a very difficult problem, because first of all e to the power At is an infinite series as we have pointed out here. It is a very; it is an infinite series and calculating these infinite sum is a quite difficult.

So here we have to find out certain tools by which we can simplify our procedure to sum this infinite series in a finite term. So here, here we want to make note that if somehow we are able to find out e to the power At then our system our solution given to with the help of e to the power At v.

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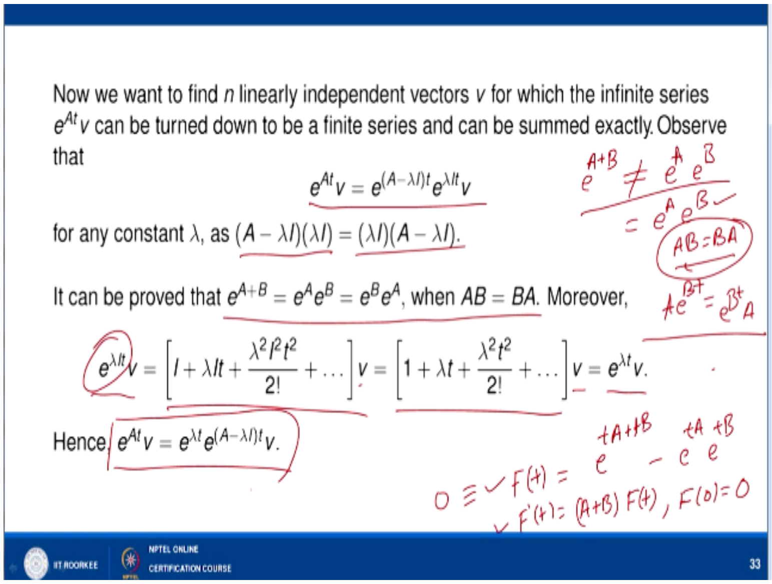

So now we want to find out n linearly independent vectors v for which this infinite series e to the power At will; can be turn down to finite series. So everything will depend on this vector v. It means that if we have to find out vector v in a way such that e to the power At v is reduced to finite series. So we need to find out this v says for every vector v it is a solution. So in particular if we put some condition on v then also this e to the power At v will be a solution of this.

And we want to choose factor v in a way such that At v is a finite series or it can be reduce in just one few term say 1 term, 2 term and so on. So here we look at e to the power At v as e to the power A-lambda it * e to the power lambda it. So here one thing note down here, that here it may not be true that e to the power A+B need not be true for e to the power A * e to the power B. This is very say common in terms of scalar series but in case of matrix series it may not be true. But it is true in a particular case. When AB=BA, right.

So in case of AB=BA this e to the power A+B can be written as e to the A * e to the power B. So this you can try. But in this case when AB=BA you can easily prove that e to the power A+B = e to the power A * A to the power B, I can take, I can give you one hint that let me e to the power tA + tB – e to the power tA*e to the power tB, you take this as a function say F(t) and so that this F dash (t)=(A+B) F(t). Try to show that this F(t) is a solution of F dash(t)=(A+B) * F(t).

Here, please observe here e to the power tA if we find out the derivative of this it is Ae*e to the power tA. Pleas observe here that you cannot write this as e to the power tA*A, right. It is not true at all. It may not be true. So here F dash(t)=(A+B) * F(t), and you can check that F(0) is simply 0. So if you can show that, that F(t) is a solution of this with initial condition F(0)=0 then F(t) has to be ideally equal to 0. And hence the result, it is true for all t, so in case of particular 240=1 also and we are done.

So here, so far I am not used this condition but while proving that F(t) is a solution of this initial value problem you have to use this AB=BA in fact we have to show; we can show that if A and B then A*B to the power Bt will also commute. So here assuming that you try at least ones, okay. So using this condition that if AB=BA then e to the power A+B=e to the power A * e to the power B, I can write this e to the power At v as e to the power A - lambda I t* e to the power lambda I t here, A-lambda * lambda I = lambda I*A-lambda I.

So here we are using this condition. Now I can; so e to the power At v can be written as e to the power A-lambda I t * e to the power lambda I tv. And e to the power lambda I tv can be written as; we can write down this for e to the power lambda I t as I + lambda I t+1 and if you take out this I square is same as I and so on, we can take out this I and we can write down this as, this is e to the power lambda t I*v is nothing but v, so we can write down this as e to the power lambda I tv is nothing but e to the power lambda tv.

So it means that you can write down e to the power I tv = e to the power lambda t*A-lambda I tv. So here we have this simplification that if we are trying to choose v in a way such that this can be truncated in a finite number of times. So this we will continue in next lecture. Here I will stop. We will discuss this, how to choose v in a next class. Thank you very much.