Advanced Matrix Theory and Linear Algebra for Engineers

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Lecture No. # 39

Back To Linear Systems - Part 2

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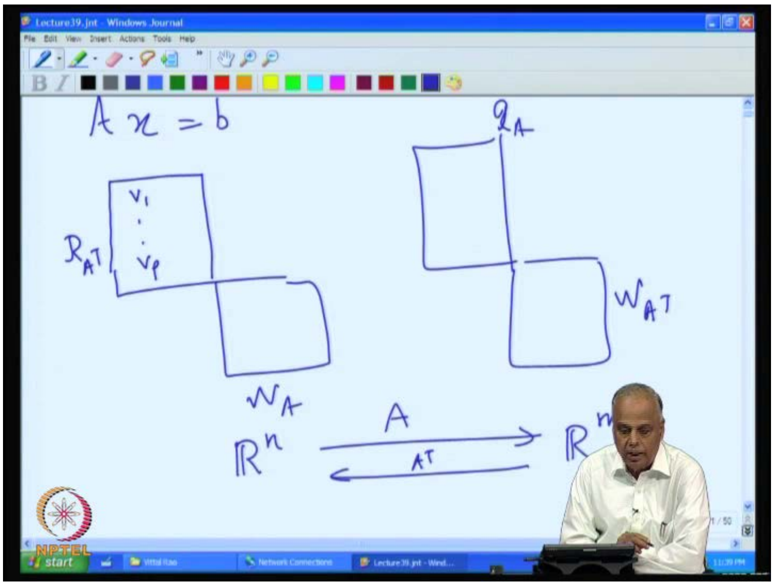

Let us recall, what we have been doing with the system of equations Ax equal to b. It was all based on the various bases that we chose for the four spaces. So, let us recall we had these four fundamental spaces - the range of A transpose, the null space of A, and the range of A and the null space of A transpose. These are subspaces on the R n side, and these are subspaces on the R m side; and, A is a transformation from R n to R m and A transpose is a transformation.

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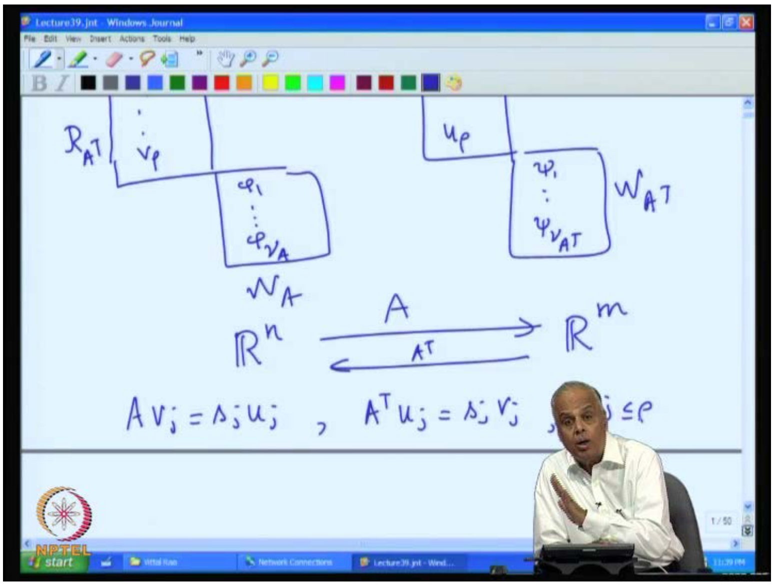

And, the bases that we chose aware like v 1, v 2, v rho is an orthonormal basis for the range of A transpose; phi 1, phi 2, phi nu A was an orthonormal basis for the null space of A; u 1, u 2, u rho was an orthonormal basis for the range of A; and, psi 1, psi 2, psi nu A transpose was an orthonormal basis for the null space of A transpose; and, the crucial relationship, where that A v j was s j u j, and A transpose u j was s j v j for 1 less than or equal to j less than or equal to rho. This was our fundamental choice of the orthonormal basis. Now, we were analyzing how to use this choice of the basis in the analysis of the system of equation.

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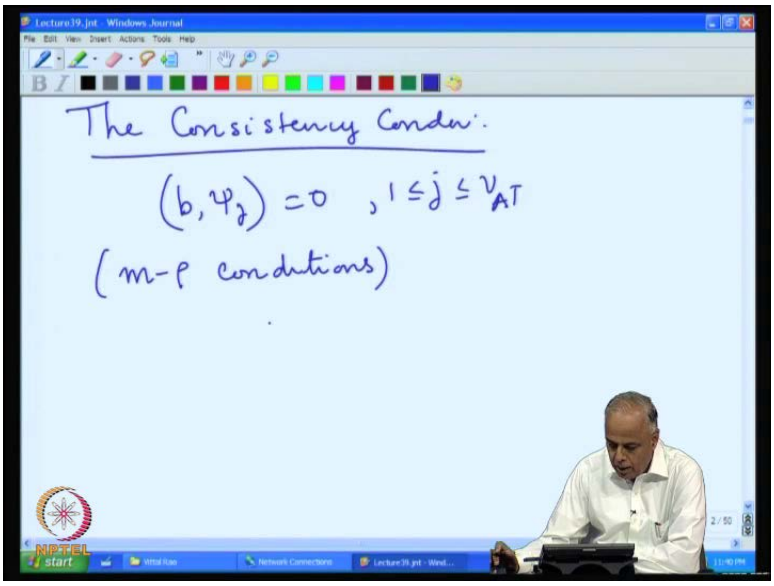

What we found was that the consistency condition for that b has to satisfy can now be written as b must be orthogonal to all the basis vectors of the null space of A transpose. And therefore, there are m minus rho conditions, because the nullity of A transpose is m minus the rank by the rank-nullity theorem. So, there are m minus rho conditions that b has to satisfy.

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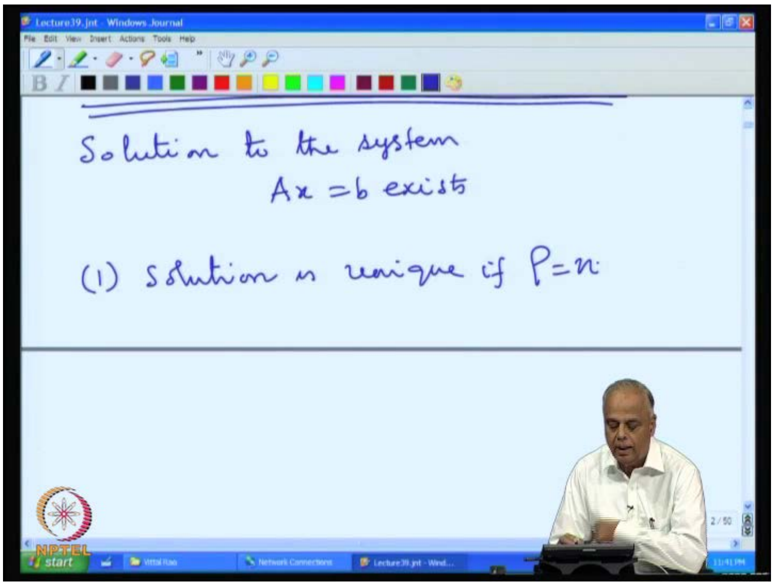

Then, we look at the case, when b satisfies these conditions, what are the results that we had. Solution exists – that is the first thing – solution to the system Ax equal to b exists. And, the solution is unique, if the rank is equal to n.

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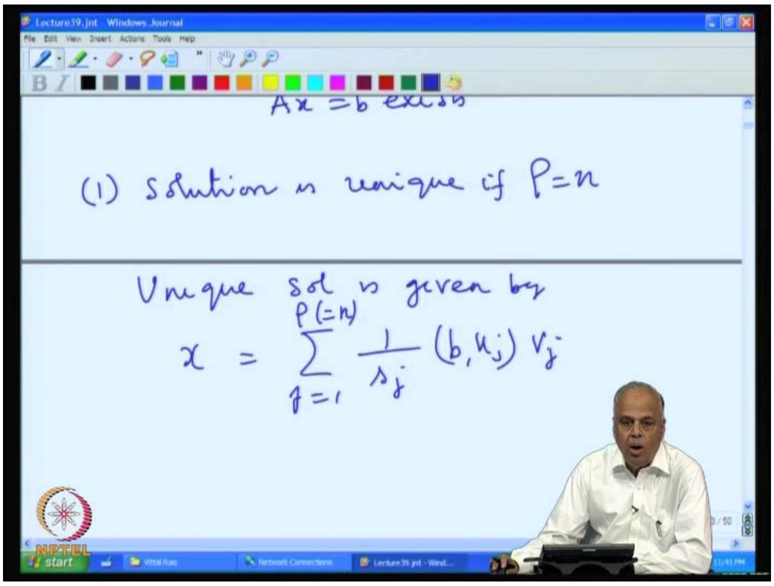

And, the unique solution in this case is given by x equal to summation j equal to1 to rho; where in this case, rho equal to n 1 by s j b u j v j; where s j is as we mentioned above, are the singular values of the matrix A.

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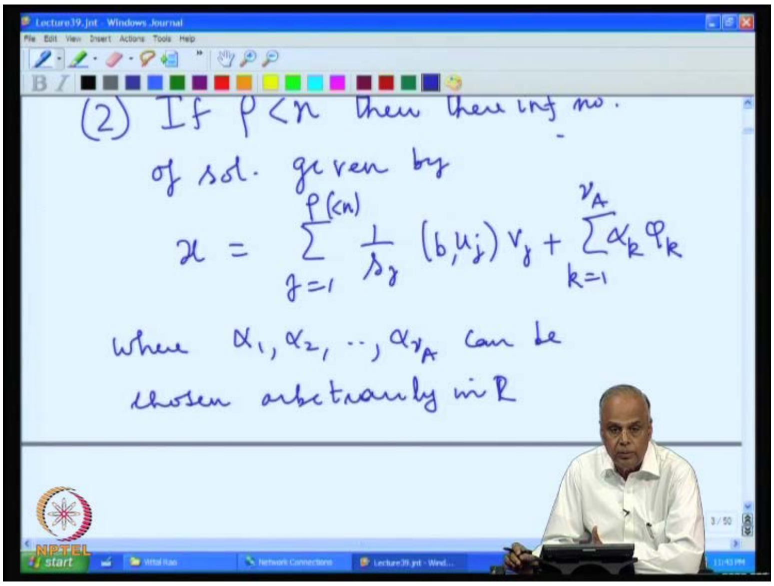

Then, if rho is less than n, the rank of the matrix is less than n, then there are infinite number of solutions given by – all of them can be written in the following form x equal to summation j equal to 1 to rho – now, rho is less than n – 1 by s j, b u j, v j. The b u js are the components of b in the direction u j plus summation k equal to 1 to nu A alpha k phi k; where alpha 1, alpha 2, alpha nu A can be chosen arbitrarily in R.

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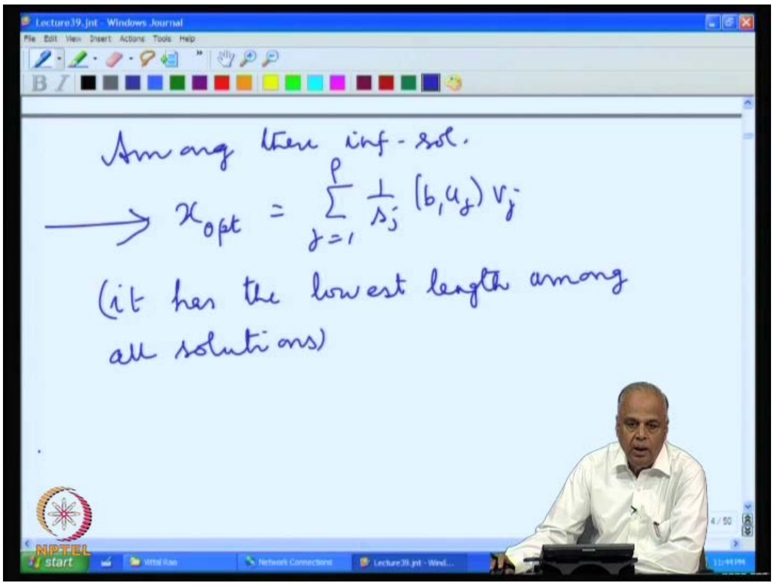

So, when rho was equal to m, we got unique solution; when rho is less than n, we get infinite number of solution. Then, among these infinite solutions, the solution x we call optimal, which is j equal to 1 to rho 1 by s j, b u j v j. This is obtained by taking all the alpha ks will be 0; all the arbitrarily constant alpha k to be 0. And, this is called the optimal solution; it has the lowest length among all solutions. Thus, when we have b satisfies the consistency condition, where solution exists, unique if rho equal to n, it is given by the expression x equal to this whole thing. And, when rho is less than n, we have infinite number of solutions; all of them are characterized as this and we have the optimal solution, which is given by this expression.

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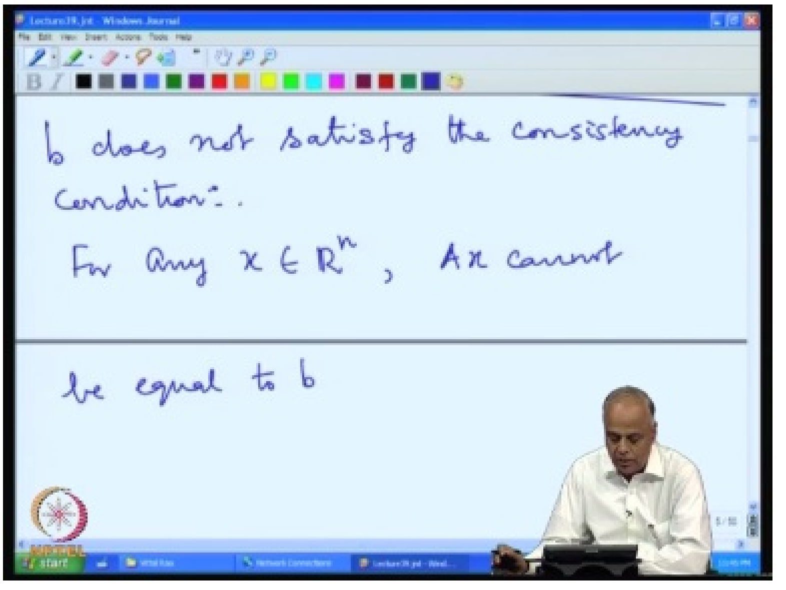

Now, having known what happens when the consistency conditions are satisfied, we can get our answers about the solutions purely in terms of the basis that we have chosen. So, the next, we looked at the case when b does not satisfy the consistency condition. This is the case that we have to deal with. And, we observed last time that what this mean is that, for any x in R n, Ax – it cannot be equal to b. What this means is Ax belongs to the range of A, but b does not satisfy the condition means b does not belong to the range of A. So, Ax and b can never match each other.

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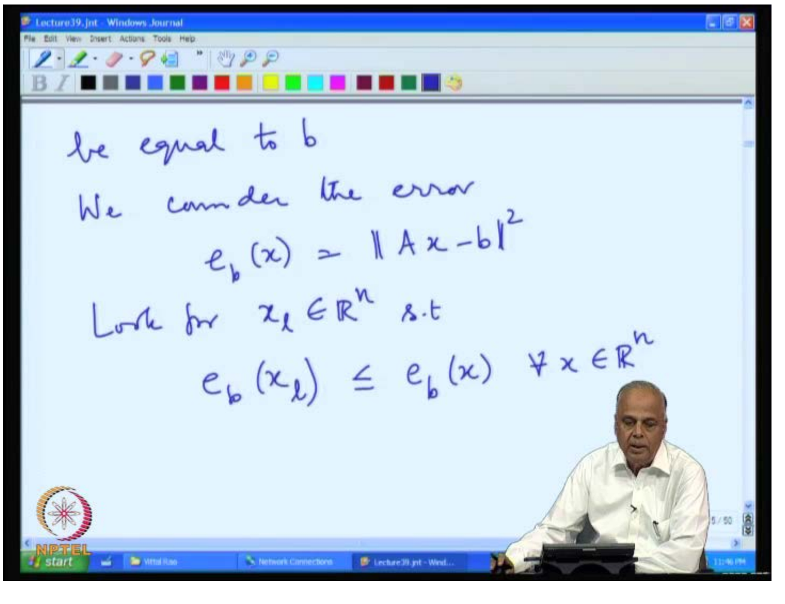

Therefore, we consider the error e b x as the length of the vector A x minus b. And, we look for a vector x l in R n such that this error is minimum. That is, if we take the error, we can even take this square error that will be less than or equal to the error obtained from any other vector; that is, x l get as close to the vector b as possible under the transformation A.

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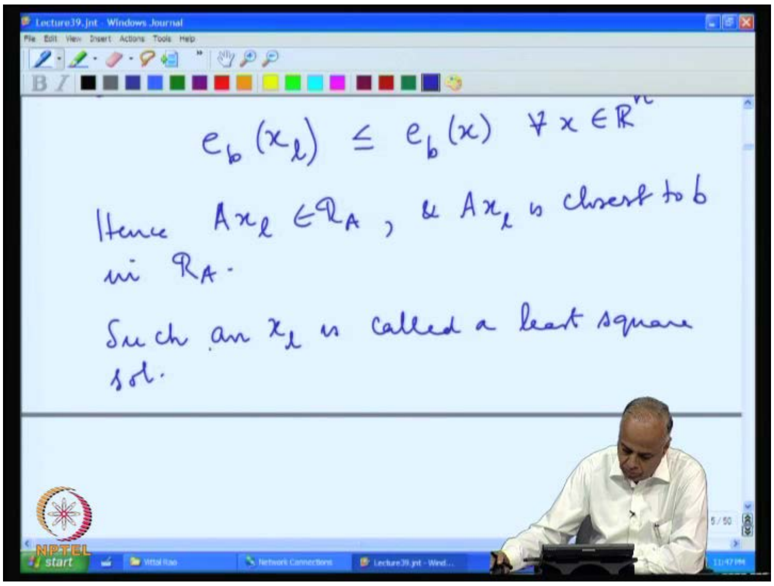

Hence, A x l belongs to the range of A first thing, because it is A of something. And, A x l is closest to b in the range of A. But, we know that the vector closest to the range of A is b r. So, any such vector x l, which you take as closest to b, such an x l is called a least square solution.

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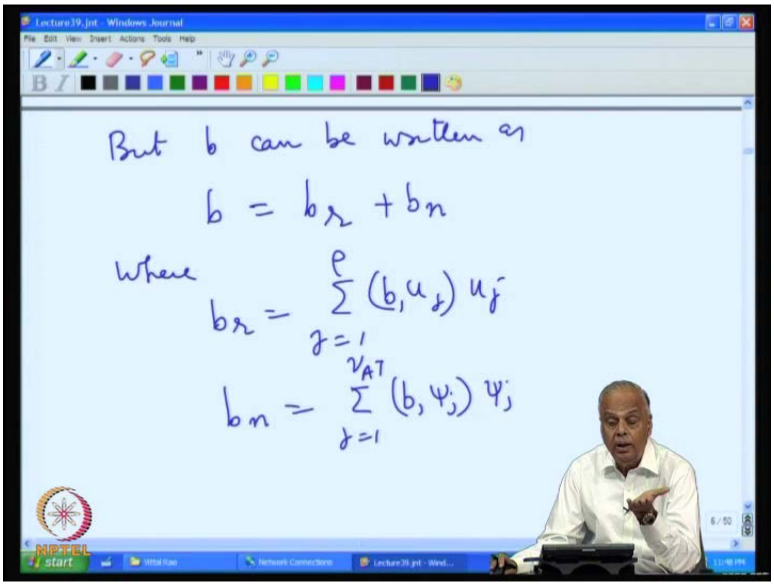

Now, how do we get this least square solution? We know that A x l closest to b and it is in the range of A. But, b can be written as b equal to b r plus b n, where b r is the projection of b on to the range of A. So, we write b r; that projection is j equal to 1 to rho b u j u j. That is the projection in the range of A. And, b n is the projection of b on the null space of A. Now, we know that the vector in the range of A, which is closest to b is given by b r, which we have seen by studying the orthogonal projections.

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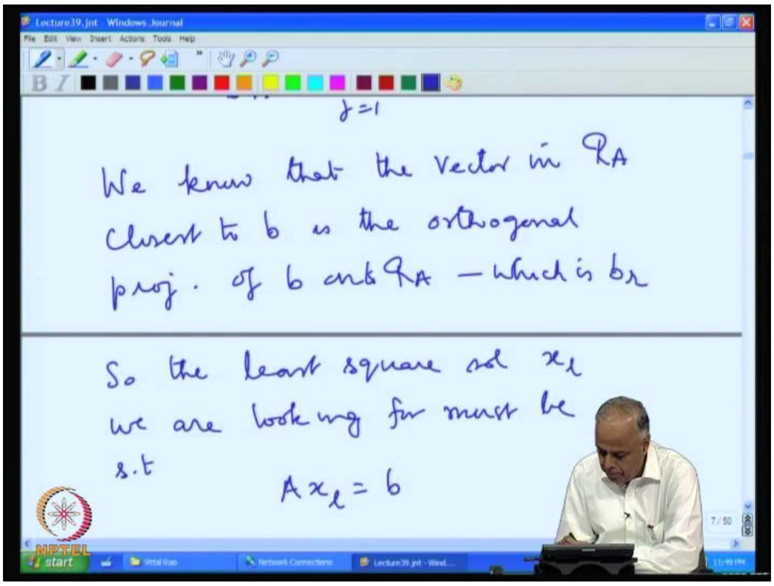

We know that the vector in the range of A closest to b is the orthogonal projection of b on to range of A and which is b r. Therefore, we are trying to make x l go to b r. So, the least square solution x l we are looking for must be such that A x l equal to b r.

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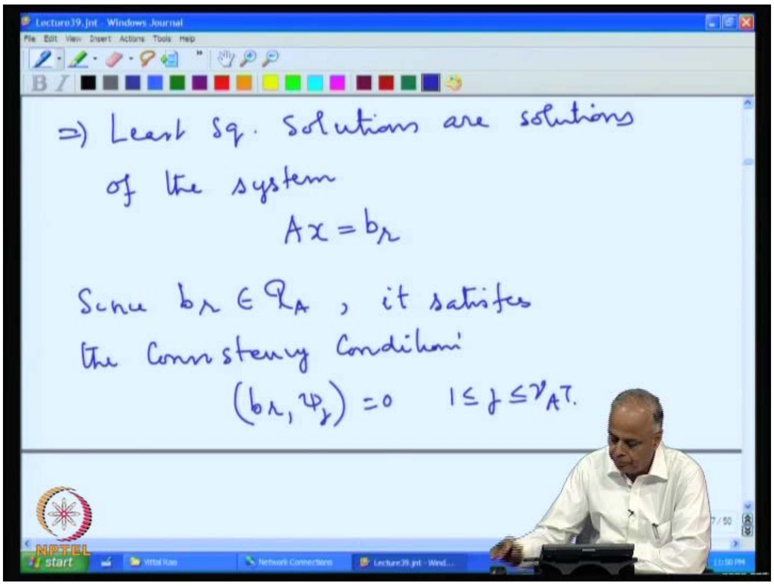

Therefore, least square solutions are the solutions of the system Ax equal to b r. Now, this system has a solution, because b r is in the range of A and it satisfies the consistency condition. Since b r is in the range of A, any vector in the range of A is perpendicular to all the vectors in the null space A transpose, because range of A and the null space of A transpose are orthogonal complements. So, it is satisfies the consistency conditions b r, psi j equal to 0; 1 less than or equal to j less than or equal to nu A transpose.

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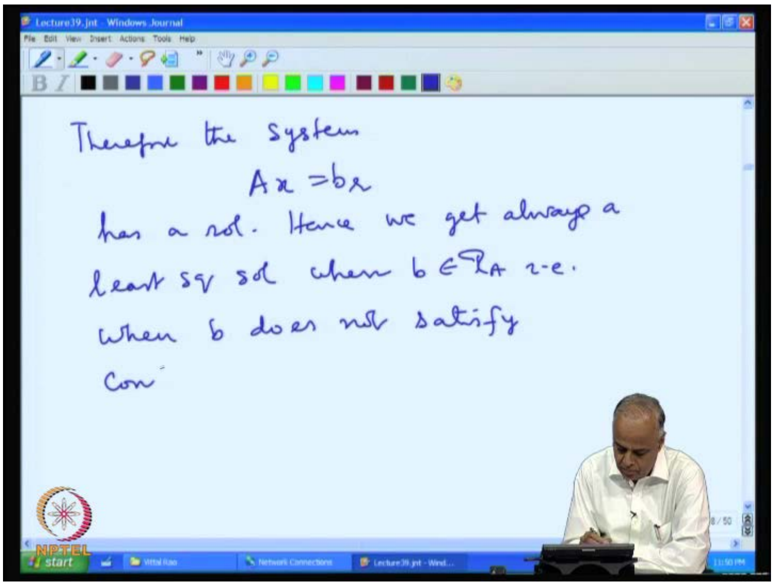

And therefore, the system Ax equal to b r has a solution. And hence, we get always a least square solution, when b does not belong to range of A; that is, when b does not satisfy consistency conditions. Now let us, analyze this least square solution. Now, Ax equal to b r; b r satisfies the consistency conditions. We have seen how to get the solutions when the consistency conditions are satisfied.

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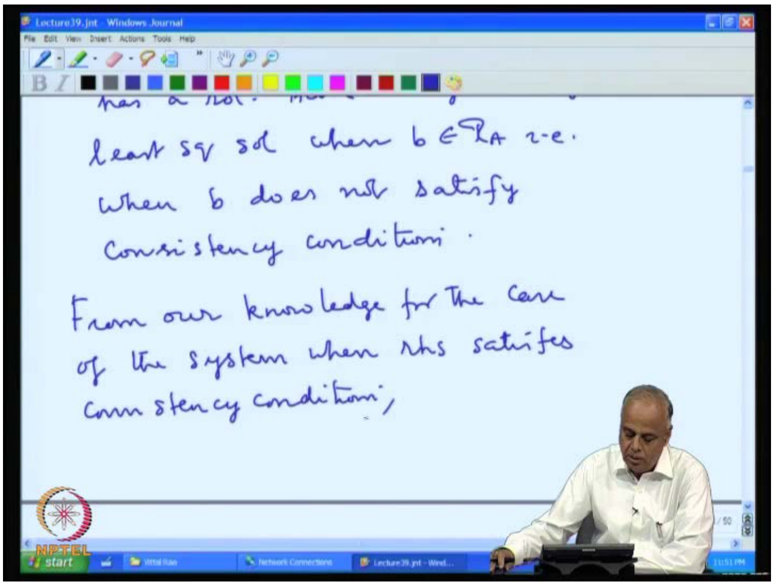

From our knowledge, which we mentioned in the beginning of the course – from our knowledge for the case of the system when rhs satisfies consistency condition.

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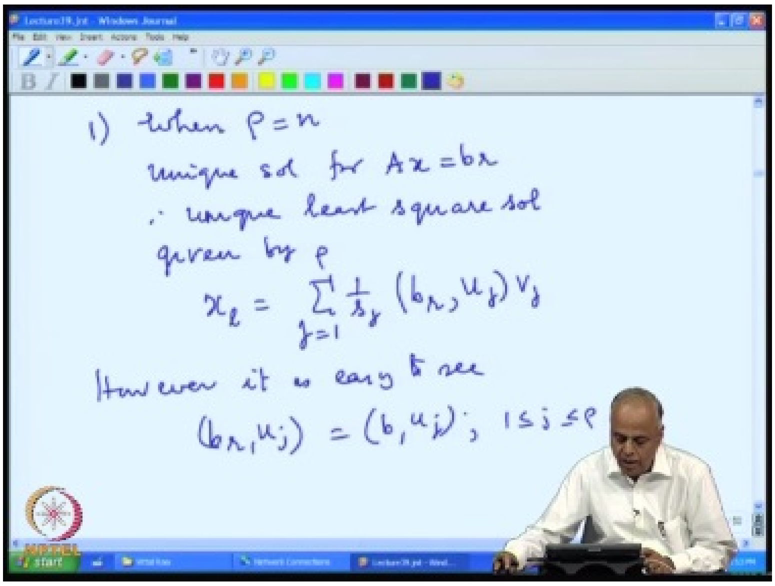

Look at that we had studied. We know that there were two cases: rho is equal to n and rho is less than n. When rho equal to n, unique solution for Ax equal to b r. Therefore, unique least square solution, because any solution for A x equal to b r is least square solution. Now, we have unique solution. Therefore, unique least square solution, and given by x l – from our work earlier with the case when consistency conditions were satisfied – 1 by s j; now, the right hand side is b r – so, it is b r u j into v j. However, it is easy to see that the b r u j is the same as b of u j. The projected vector and the original vector have the same component in the projected space. So, this is 1 less than or equal to j less than or equal to rho.

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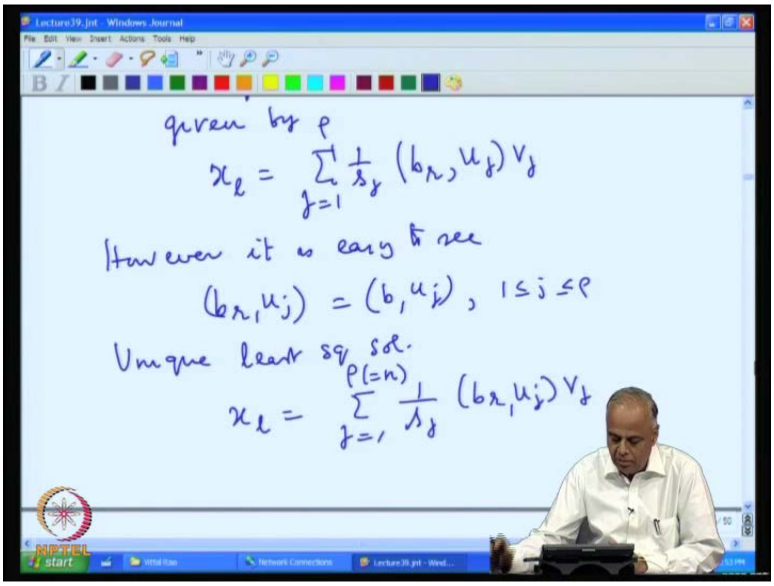

And therefore, the unique least square solution is given by x l equal to j equal to 1 to rho; in this case, rho equal to n – 1 by s j b r, u j v j. Thus, when the rank is n, consistency conditions are not satisfied, there is a unique least square solution; and, it is given by this expression (Refer slide Time: 15:34).

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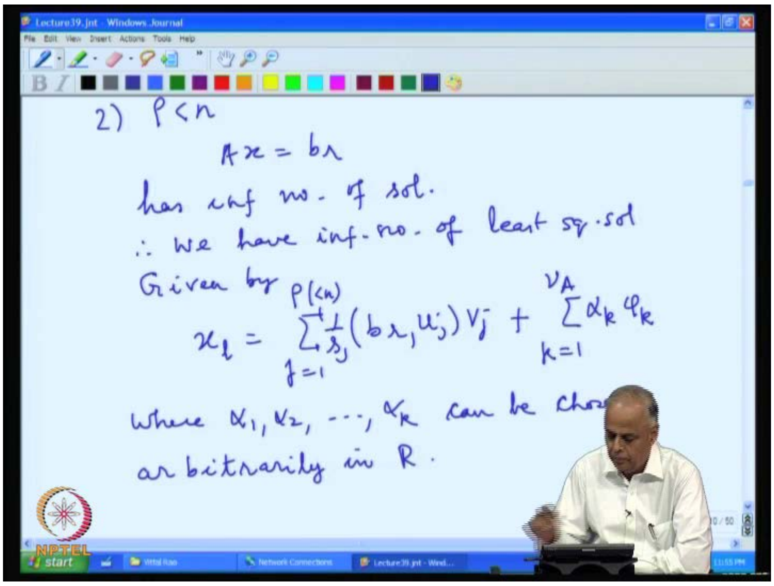

On the other hand, when rho is less than n, A x equal to b r has infinite number of solutions. But, any solution of this A x equal to b r, is called the least square solution. Therefore, we have infinite number of least square solutions. What are they? They are all given by – we know that the rho is less than n; all the solutions are given by in the case when a consistency condition is satisfied and because b r satisfies the consistency conditions, it is 1 by s j b r u j v j plus the arbitrariness comes from the fact that there is rank less than n. So, that will be the arbitrariness always come from the null space of A, where alpha 1, alpha 2, alpha k, can be chosen arbitrarily in R.

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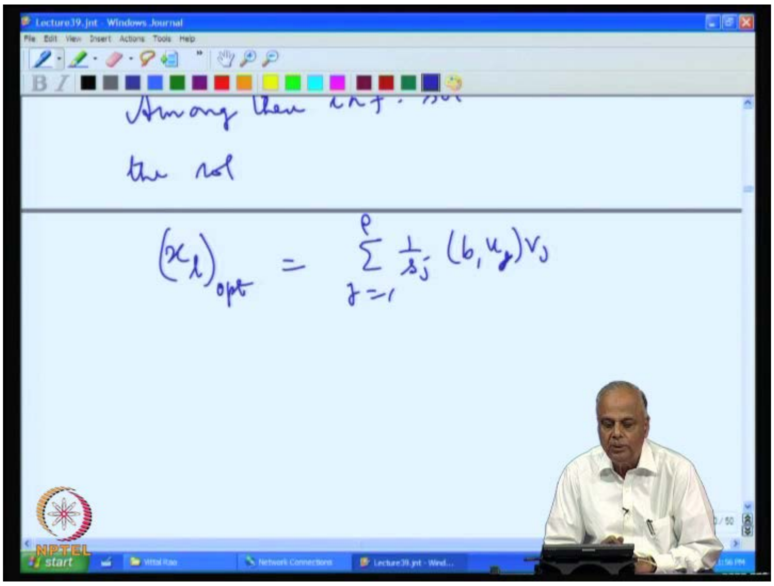

Now, again among these infinite solutions, the one consisting of only the range of A transpose part, that is, those involving v 1, v 2, v rho, that part uses a vector, which is a least square solution and which has the least length. And therefore, it is called the optimal solution. The solution, which we will call as least square optimal, which consists of only the unambiguous part – 1 by s j b u j v j. Notice that this b r u j can again be written as, this is (Refer Slide Time: 18:02) equal to b u j also as we absorbed above.

Therefore, in the case when rho is less than n, consistency condition is not satisfied, least square solution exists; there are infinite number of least square solutions given by this (Refer Slide Time: 18:19). And, among all these, there is one which has least length and that is unique and that is called the optimal least square solution. Therefore, in all the cases, whether the system satisfies the consistency condition or not, we always have the representation for the type of solution that we are looking for in each case in terms of the orthonormal basis that we have chosen. In the case when rho is equal to n, whether we are looking for regular solution in the case when b satisfies the conditions, all the least square solution when b does not, we always get a unique solution. This is what you notice that in the case rho equal to n. In the case of least square also, we have got unique solutions. And, we have seen last time that in the case when regular solution with consistency condition is satisfied, then also, we get least square solution. Similarly, in the case, rho is less than n, we got infinite number of solutions whether in the case of regular solutions or in the case of least square solutions. Therefore, we always look for optimal solutions.

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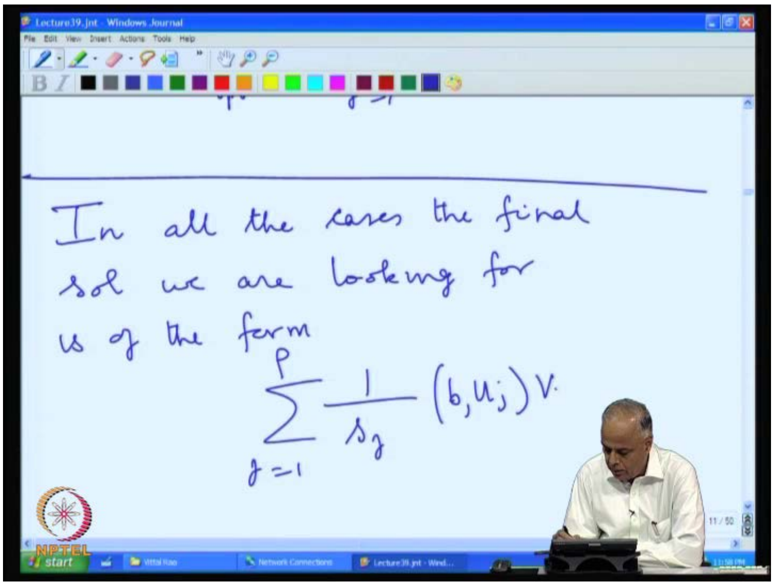

Let us now, summarize this. If you notice in all these cases, the final solution we are looking for is of the form summation j equal to 1 to rho 1 by s j b u j v j.

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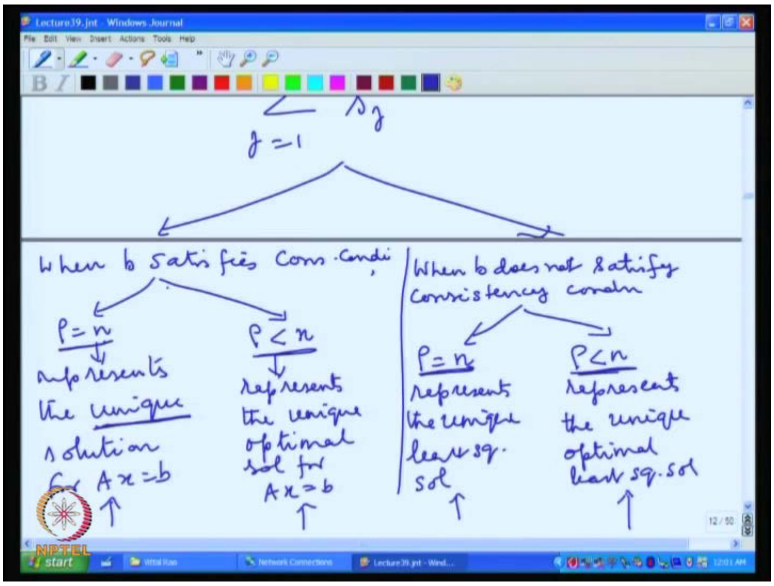

Now, what does the represent? This represents let us see; when b satisfies consistency condition and rho equal to n, that represents that expression there represents the unique… The uniqueness is because of the fact, rho is equal to n – unique solution for Ax equal to b. When b satisfies, when rho is less than n, that same expression represents the unique optimal solution of the system. And, when b does not satisfy consistency condition, the same expression again when rho equal to n represents now not the unique solution, but the unique least square solution. And, when rho is less than n, it represents the unique optimal least square solution.

Now, this you observe that whenever you had the case rho equal to n, whether in the case of consistency or inconsistency, that is, whether you are looking for exact solution or least square solution, you are always getting uniqueness. But, when rho is less than n, in both the cases, we got infinite number of solutions, but we got unique optimal solution. And, all these things that final solution, whether it is a unique exact solution, that is, in this case; or, the unique least square solution, that is, in this case (Refer Slide Time: 23:09); or, the unique optimal solution, that is, in this case; or, that is, unique optimal least square solution in this case, the representation of the solution is exactly this (Refer Slide Time: 23:22). It depends on rho now. So, it will be total vary, the summation index will vary in all these cases.

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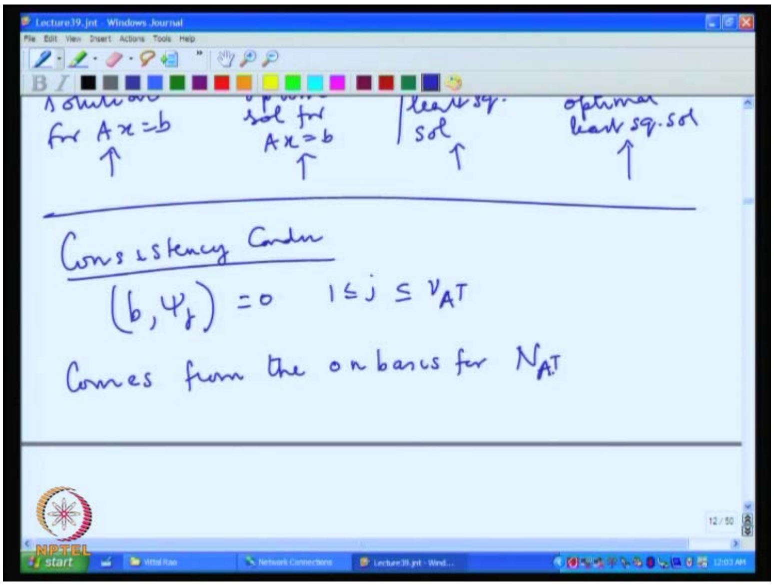

So, now, we have the answer for the solution of the system of equation in terms of the basis that we have chosen completely, whatever the case may be. We know the consistency condition, we know what happens when the consistency condition is satisfied. When the consistency condition is satisfied, we know when the solution is unique and when the solution is infinite. And, when the solution is unique, we know what the unique solution is. And, when the solution is infinite number, we know all of them and we know the representative solution, which is the optimal solution. When b does not satisfy the consistency condition, we know that least square solutions exist. And, when the least square solution is unique, we know that least square solution is infinite.

We know when the least square solution is unique, what is that solution. And, when least square solution is infinite, we know what are all those least square solutions and we know what is the unique representative optimal least square solutions. And, all these are obtained in terms of the basis we chose. The consistency condition comes from the bases for the null space of A. Remember, the consistency condition b psi j equal to 0 for 1 less than or equal to 0. So, it comes from 1 less than or equal to nu A transpose j less… comes from the b psi j, which are the orthonormal bases for the null space of A transpose. So, first a priori, the consistency conditions are determined for the bases that we have chosen for the null space of A transpose.

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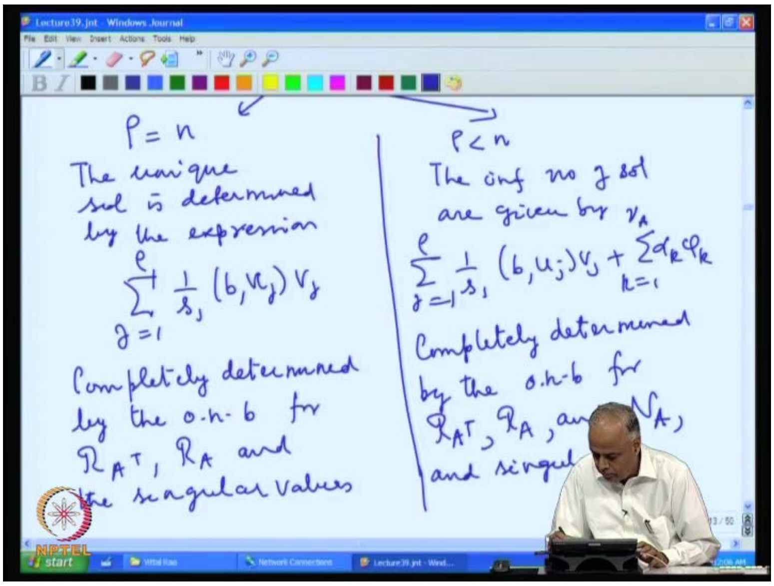

Then, the next is when consistency condition satisfied by b, when rho equal to n, we know that the unique solution is determined by the expression summation j equal to 1 to rho 1 by s j b u j v j. And, you see that this is completely determined by the v js and the u js and the s js. So, it is completely determined by the orthonormal basis for the range of A transpose, the range of A and the singular values. And, when I say the orthonormal basis, the orthonormal basis that we have chosen. The u js and the v js have been very specially chosen to be related within each other.

When rho is less n, the infinite number of solutions are given by again summation j equal to 1 to rho 1 by s j b u j v j plus summation k equal to 1 to nu A alpha k phi k. By varying alpha k over all possible real values, we got all the solutions of the system. And, this is completely determined again by the u js and the v js, which are the bases for the range of A transpose and the range of A, the s j, the singular values and the phi ks, which are the basis for the null space of A. So, by the orthonormal basis for the range of A transpose – these are v js; the range of A – these are the u js; and, the null space of A – these are the phi ks; and, the singular values.

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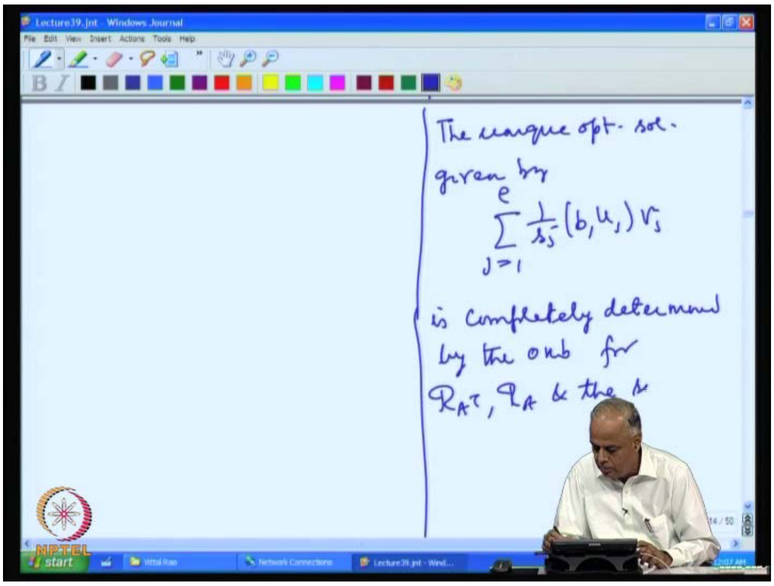

And, when the solutions are infinite in this case, the unique optimal solution given by summation j equal to 1 to rho 1 by s j b u j v j is completely determined by the orthonormal basis for the range of A transpose, the range of A and the singular values. At least now, we see that in the case when consistency conditions are satisfied, all our answers involve the basis of range of A, basis of range of A transpose and if necessary, the basis for the null space of A. The consistency conditions come from the basis for the null space of A transpose. So, all these four bases that we have chosen play a very crucial role in determining a complete analysis of the system of equation.

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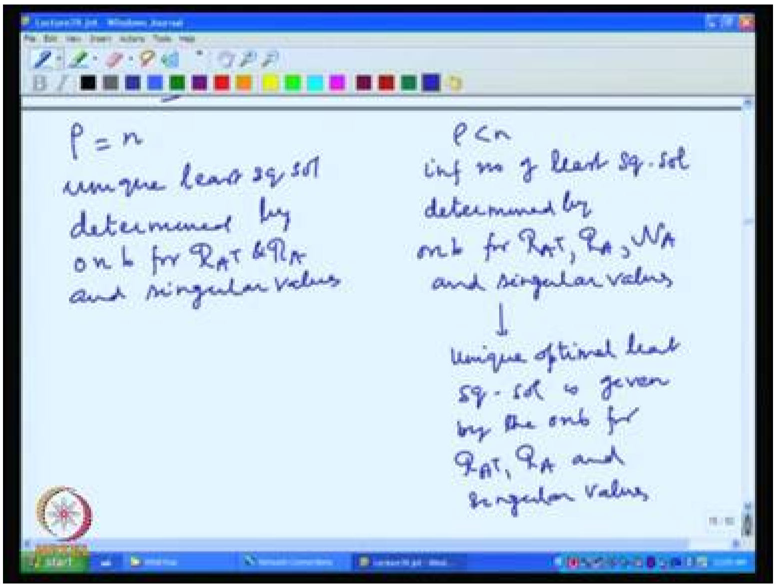

So, let us write again the case, when b does not satisfy consistency conditions, then – we will not write full expression again, we will simply mention – rho equal to n unique least square solution, again determined by the orthonormal basis for the range of A transpose and the range of A. When rho is less than n, infinite number of least square solutions determined by the orthonormal bases for the range of A transpose, the range of A, the null space of A and of course, the singular value; in all these cases, the singular values come in to the picture. And, in the case the infinite number of solutions, the unique optimal least square solution is given by the basis for range of A transpose, range of A and singular values.

So, the moral of the story is that the four orthonormal bases that we have chosen for the range of A transpose, for the null space of A, for the range of A and the null space of A transpose together with the singular values, together special way we have chosen these orthonormal bases completely answers all our questions about the solution of the system explicitly. We again repeat, the null space of A transpose appears in determining the consistency condition; the null space of A appears whether determining there are arbitrariness in the solution or not; and, the range of A and the range of A transpose always keep track of the main solution that we are looking for. These four bases are very important for us.

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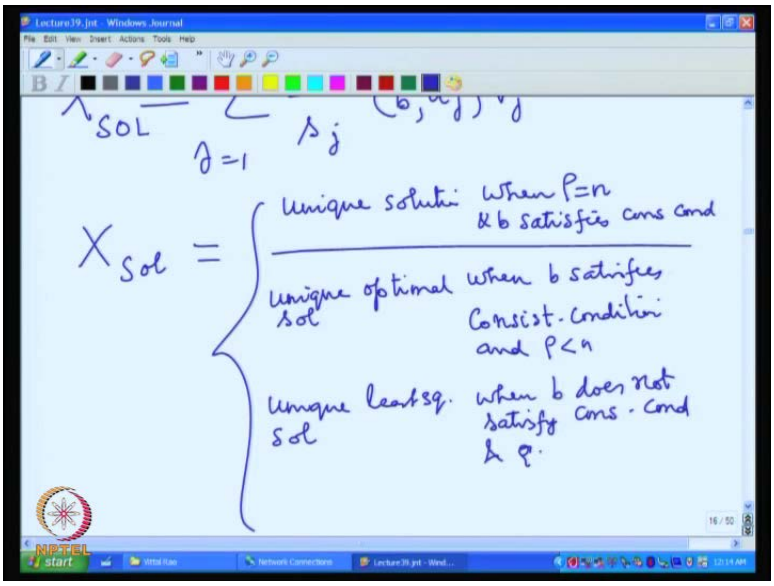

Now, let us get back to the structure of the solution and get to the notion of the so-called pseudo inverse of a matrix. So, let us get back. Let us look at the expression, summation j equal to 1 to rho 1 by s j b u j v j. If you look at all the varieties of cases we have studied, this part is common in all the solutions whether there is an exact unique solution, whether there is infinite number of unique solutions, whether it is the unique optimal solution, whether it is the unique least square solution, or whether the infinite number of least square solution or the unique optimal least square solution. In all these, this is an essential common part. And eventually, this is what we are extracting as the essence of all the solutions. So, recall that this is what we are looking for.

Always, our final answer for Ax equal to b is take x to be this; you cannot do anything better than this of all. If it is going to be exact solution, this will be an exact solution. There are many exact solutions, this will be the one, which have the least length. There is only one exact solution, this will be the unique exact solution. If there are only least square solutions and it is unique, this will be least square solution. If there are many least square solutions, then this will be the unique optimal least square solution. So, this is in a sense the essential answer for the system of equations Ax equal to b.

Now, let us write this expression, therefore, we will call it as (Refer Slide Time: 34:32) X sol. This is solver of the system – X sol. I will again repeat, X sol is unique solution, when the uniqueness comes from rho equal to n and solution comes from and b satisfies consistency condition. So, when b satisfies consistency condition and rho equal to n, this X sol will be the unique solution. It will be the unique optimal solution when b satisfies consistency condition and rho less than n. And, this will be unique least square solution when b does not satisfy consistency condition and rho equal to n.

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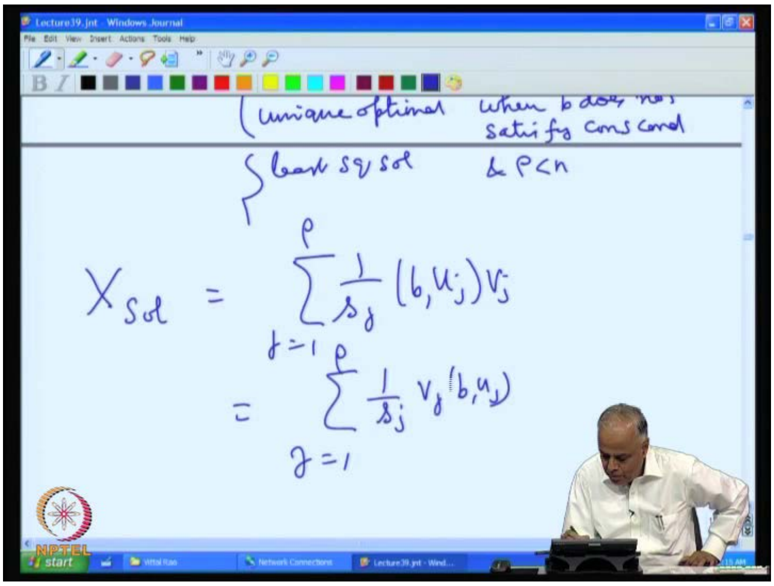

And finally, this will be the unique optimal least square solution, when b does not satisfy consistency condition and rho less than n. So, the same expression represents different things and the different cases. Therefore, that expression captures the essence of the solution in all the cases. So, we will again look at X sol, which is summation 1 by s j b u j v j; and, we will now write it in a special form, which is equal to summation j equal to 1 to rho 1 by s j v j b u j; we will write the number b u j to the right of the vector.

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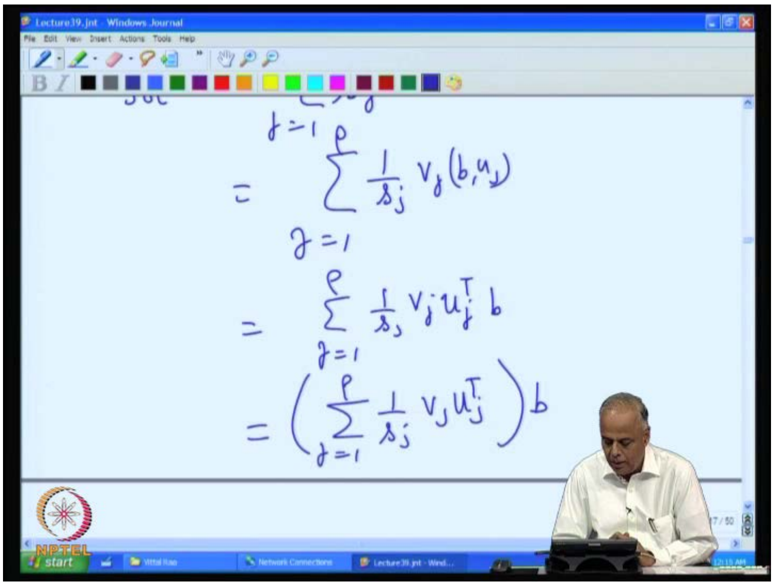

And, now, that can be written as j equal to 1 to rho 1 by s j v j; the inner product can be written as u i transpose b. And therefore, we can write this as summation j equal to 1 to rho 1 by s j v j u j transpose acting on b; each term is multiplying b. So, we can combine using the distributive law for matrix multiplication. This can be written as the sum matrix multiplying.

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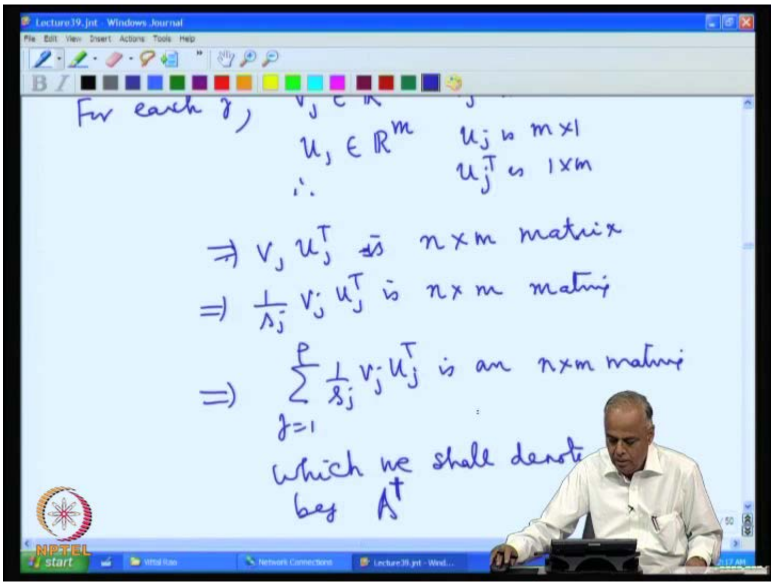

Now, look at what is inside. For each j, v j belongs to R n, because v j was the basis for the range of A transpose. Therefore, they are living on the R n side. Therefore, v j is n by 1. And, u j belongs to R m. And, therefore, u j is m by 1. And therefore, u j transpose is 1 by m. And therefore, v j u j transpose will be an n by 1 matrix times 1 by m matrix. So, is n by m matrix. v j u j transpose is an n by m matrix. And, that says if I multiply it by a scalar, that will also be an n by m matrix. Therefore, this v j u j transpose times 1 by s j is always an n by m matrix. Now, this sum involves rho terms; each term is an n by m matrix. So, if I add rho of the n by m matrix, again I will get an n by m matrix. So, that says if I add all these (( )) is an n by m matrix, which we shall denote by A dagger and call this as the pseudo inverse of A.

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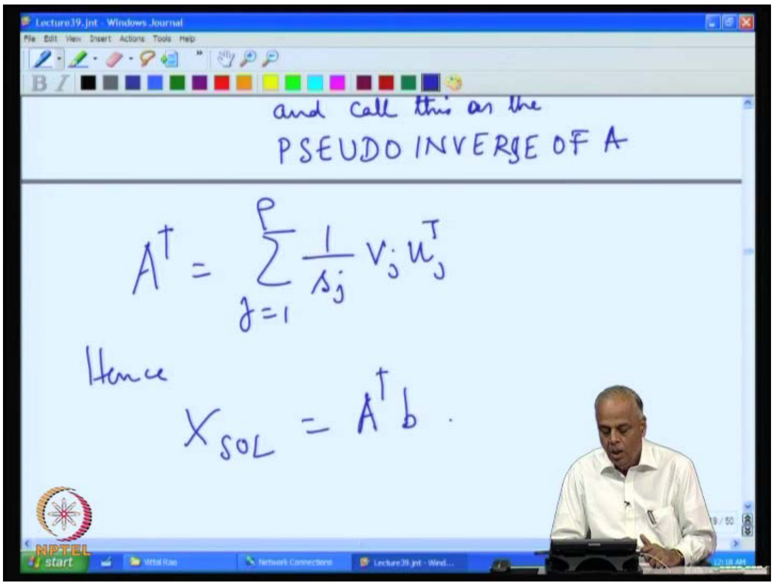

So, we have A dagger is equal to summation j equal to 1 to rho 1 by s j v j u j transpose. And hence, this entire X sol can be written as X sol equal to A dagger b.

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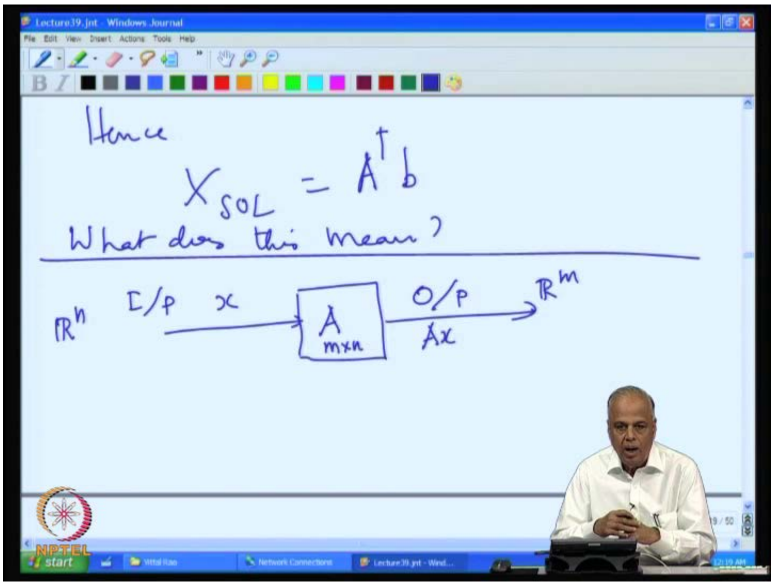

So, what does all that mean in terms… Let us look at the sol thing from a different perspective. We have a system… Let us put what does this mean. We have a system we are looking at it from the system point of view, the system matrixes A and the inputs for the system are all from R n. And, when you put the x in input and there is an output, there all in R m, and the output is given by A times x; and, A is an m by n matrix. So, this is how the matrix can be viewed as. So, the matrix A is a black box, is an input output system; in that system, the whole system operation or action is controlled by this m by n matrix A. And, the inputs that are accepted into the system are vectors from R n; and, the outputs that are going to come out are vectors in R m. So, when we input a vector x, the output is simply the vector x, is pre multiplied by the matrix A. Since, A is m by n and x is n by 1, the output A x is going to be in R m.

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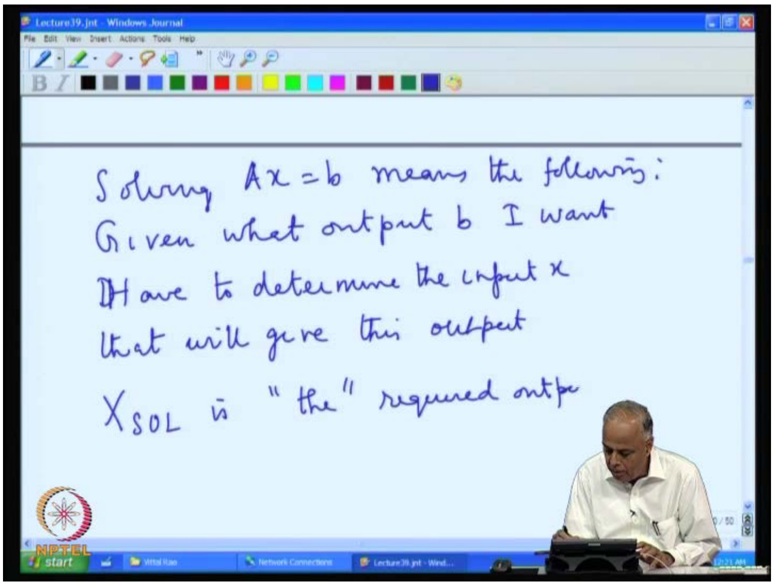

Now, whenever we know the system, it means we know the matrix A. And therefore, whenever the input is known, we can calculate the output. The question we are asking, what is meant by solving Ax equal to b means the following. Here I do not know x; I am given what output b I want; then, I have to determine the input x that will give this output.

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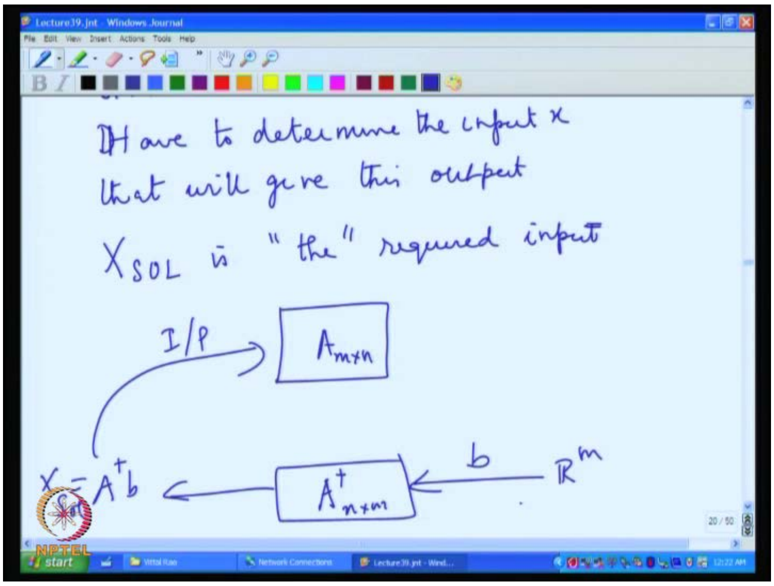

Now, what does X sol say? X sol is the required output. What do we mean by the required input? So, what does that mean? So, again, you have this system A; and then, from b, I have a new system A dagger, which is an n by m matrix. So, when b is given in R m, I input into this new system, I get a dagger b, which is the X sol; and now, I take this as the input; will I get x, will I get b, the required output? I will get b if it is possible; or, if it is not possible to get any input, which gets b, then X sol will give that input, which will take you closest to b. And, among all such possible things, X sol will have the least length. So, it will get closest to b.

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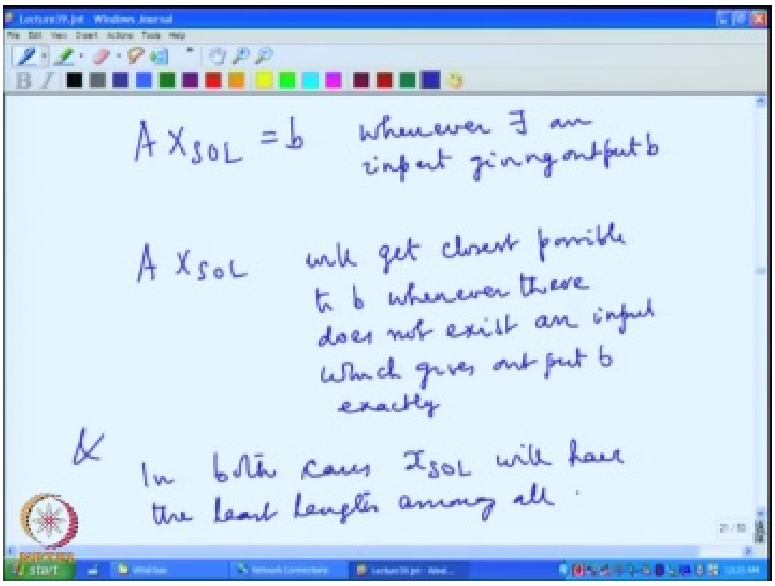

So, A X sol will be equal to b whenever there exists an input giving output b. And, A X sol will get closest possible to b whenever there does not exist an input, which gives output b exactly. And, in both the cases, X sol will have the least length among all such required inputs. So, it is like a control. I want to control the output b by controlling the input. I want to control the system in such a way that I get the output b. I want to see how do I control by controlling the input. I would say that you can control by putting the input X sol; and, that is the best you can do. What do we mean by saying that is the best we can do? When you put the input X sol if the output b you were looking for is a genuine output for the system, then X sol will definitely give that output b. There may be many inputs, which may give the same genuine output b you are looking for. But, among all that, X sol will have the least length.

If by chance, you are looking for an output for which it is not a genuine output, that is, there is no input for the system, which is going to produce that output, then X sol will be that input, which will produce an output, which is as close to be as possible. No other input can get anywhere closer to that output b. So, X sol is the best approximation for the answer that you are looking for. And, if there are many such solute inputs, which can take you same as close to be possible, then the X sol we have given is the one that have the optimal length and still gives the closest to b. So, that is what is meant by the inverse system or the pseudo inverse system. So, if the system is A, A dagger is the pseudo inverse system and it tells you how to control the output. That is the way this has to be interpreted in terms of the systems.

Now, let us again look at these things in various ways. When we have the concept of a square matrix, you have the concept of an inverse. But, in this context, we have also introduced the notion of a pseudo inverse for any type of matrix m by n. In particular, if I take m equal to n, I have a square matrix, and therefore, I can talk about its pseudo inverse. What is the connection between the pseudo inverse and the inverse for a square matrix?

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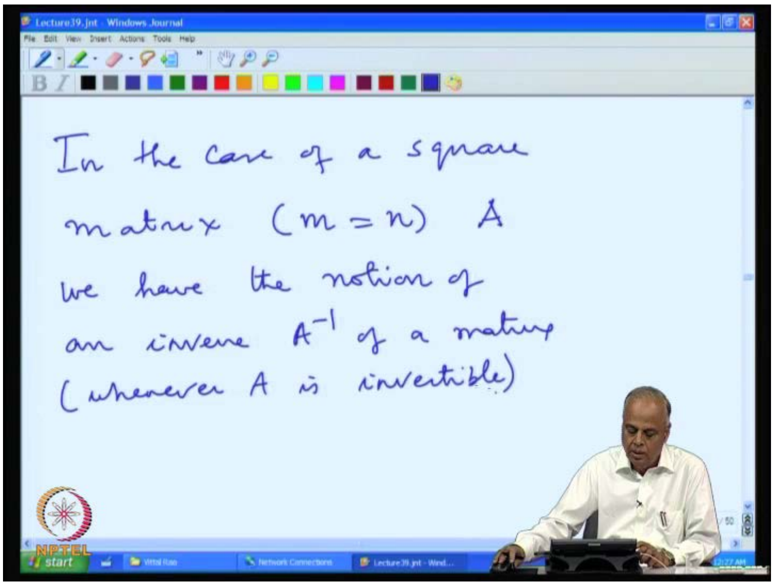

Let us in the case of a square matrix m equal to n. So, we have a square matrix A. We have the notion of an inverse of a matrix whenever A is invertible. Now, we have also a notion of pseudo inverse A dagger. If A is invertible square matrix, what is the connection between A inverse and A dagger? Because we have got too many varieties of inverses, we have to make sure what we are talking about. So, let us analyze this question.

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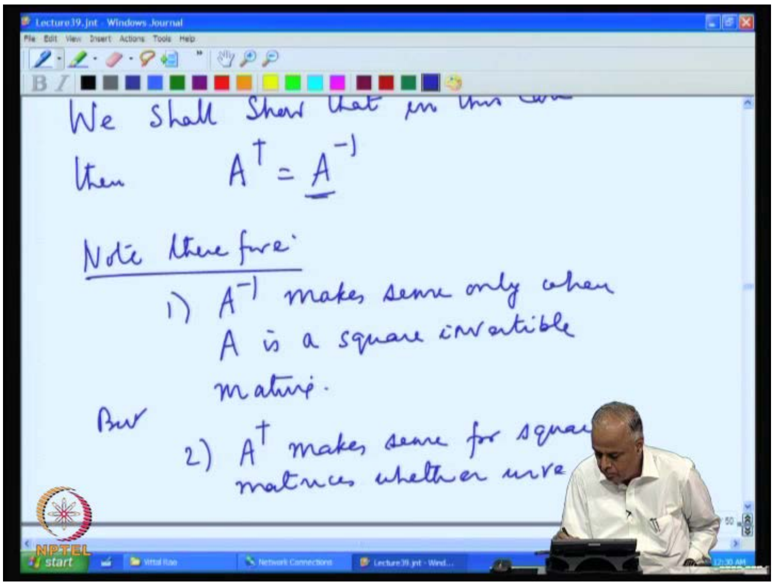

We shall show that in this case, that is, when A is an invertible matrix, square matrix, then A dagger is the same as A inverse. The pseudo inverse add the original concept of inverse will tally. Therefore, there is no confusion; whether you talk about that inverse or this inverse, both are same. Now, what happens is, this notion of inverse – the notion we have got is A inverse make sense only for square matrices when they are invertible. However, the notion of A dagger make sense for square matrices even when they are not invertible and even for rectangular matrices. And, when square matrixes, also when they are invertible also, it makes sense. And, whenever they are invertible, it carries on to the same notion as the old invertible, an inverse. Therefore, A dagger is a generalized notion of the inverse. Note therefore, the following points. The notion A inverse makes sense only when A is a square invertible matrix. But, A dagger makes sense for square matrices whether invertible or not and also for nonsquare matrices.

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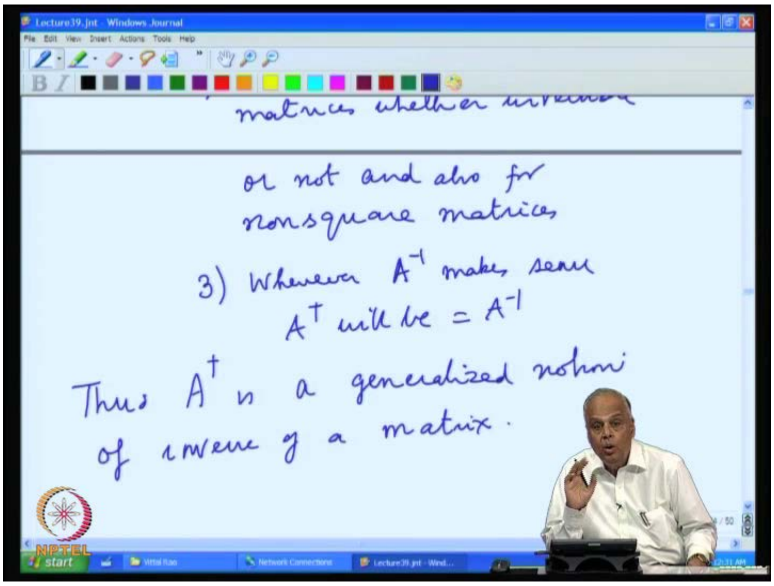

And, three, whenever A inverse makes sense, A dagger will be equal to A inverse. Thus, A dagger is a generalized notion of inverse of a matrix. So, A dagger is a far reaching generalization of the notion of inverse. We can talk about the A dagger, whether the matrix is square or not, whether the matrix is square and invertible or not, whether the matrix is rectangular or not. So, makes in all these cases. And, whenever the classical notion of inverse makes sense, the two will coincide. So, we shall now establish the fact that A dagger is equal to A inverse whenever A inverse makes sense. For this, we will begin with some preliminaries.

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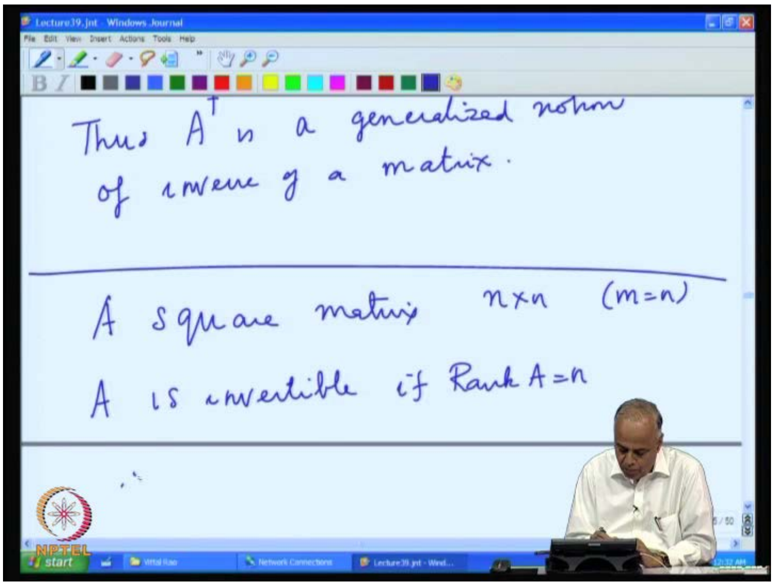

Let us take; A as a square matrix say n by n. So, here m equal to n, we can think of. It is a rectangular matrix, but where the number of rows is equal to number of columns. So, when A is invertible? A is invertible if rank of A is equal to n. Therefore, rho is equal to n.

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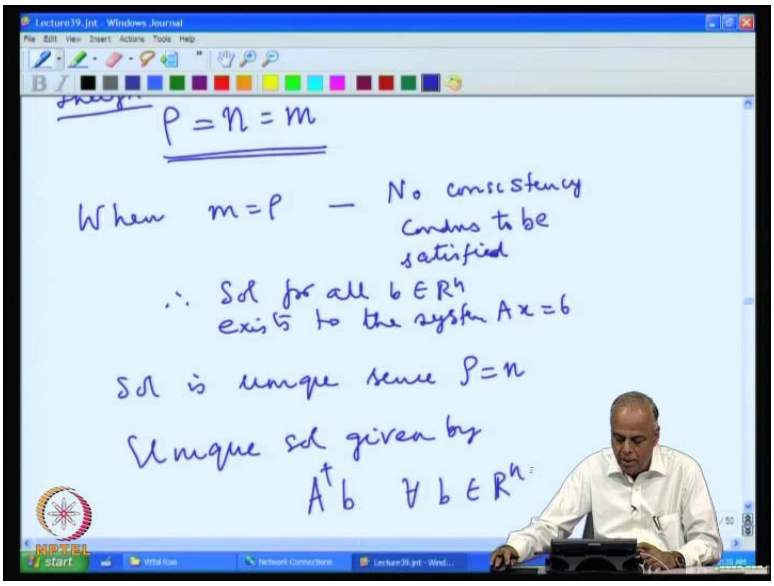

So, what is the situation that we have? n is equal to m and rho equal to n. Therefore, we have a matrix, where the number of rows is equal to the number of columns and all of them are equal to the rho. So, with this in mind, let us look at, what is the situation that we have. Because rho equal to m, we will have how many consistency conditions are required for the system A x equal to b, we need m minus rho conditions. And then, we shall see that, because m minus rho is 0 in this case, there is not going to be any consistency condition. So, the first thing that we get will be A x equal to b, will have a solution for all b in R m. And, because rho equal to m, we will be in the case that the solution is unique. Therefore, when rho equal to n equal to m, we have a unique solution for the system, for all b in R m. And, that unique solution will be written as A dagger b.

We will again go through this carefully. When m equal to rho, we have no consistency conditions to be satisfied. Therefore, solution for all b exists to the system Ax equal to b. Solution is unique since rho equal to n. Null space consists of only the zero vector. Unique solution given by A dagger b for every b in R m.

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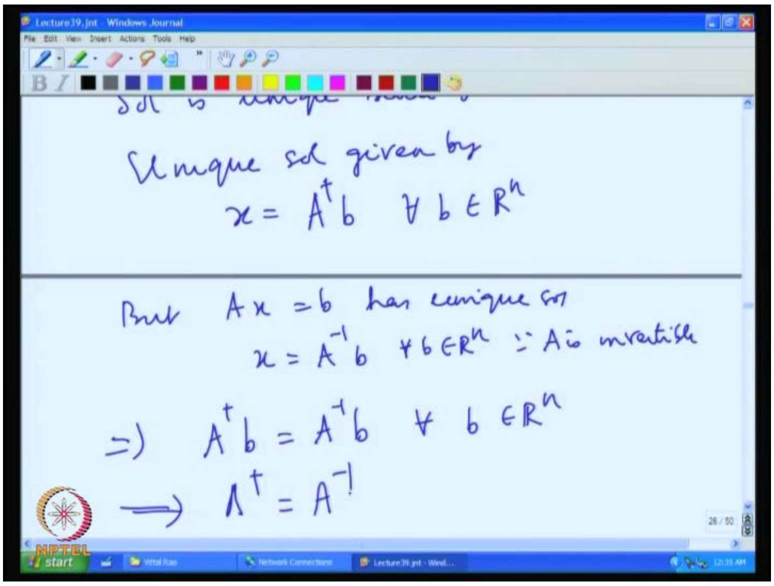

But, A x equal to b has unique solution x equal to A inverse b for every b in R n, because A is invertible. Now, on the one hand, we have the unique solution given by the A dagger b; on the other hand, we have given by A inverse b. We have A dagger b is equal to A inverse b for every b in R n. Since the action of this matrix is the same on all the vectors, we get A dagger equal to A inverse. So, whenever the A inverse exists, we have that it is the same as the pseudo inverse. But, the pseudo inverse makes sense even in the most general cases.

In the next lecture, we will go back to the basic fundamental questions that we raised in our first two lectures and takes talk whether we have the answers to all these questions. Before we do that, we will again review the solution concepts that we have got in another view point.