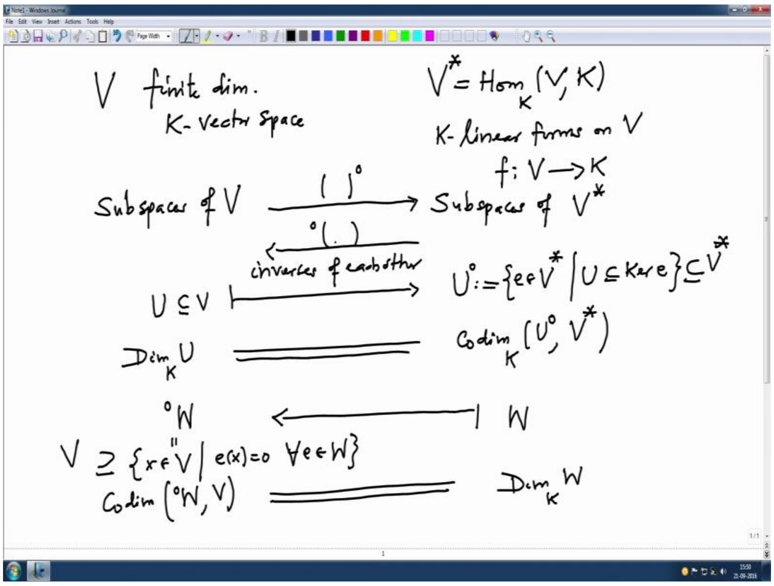
Linear Algebra

Prof. Dilip P Patil
Department of Mathematics
Indian Institute of Science, Bangalore

Lecture – 34
Quotient spaces

Today lecture, I will continue for a few minutes about the dual spaces. Last time we did dual spaces, I will continue for few minutes and then we will go to quotient spaces.

(Refer Slide Time: 00:46)



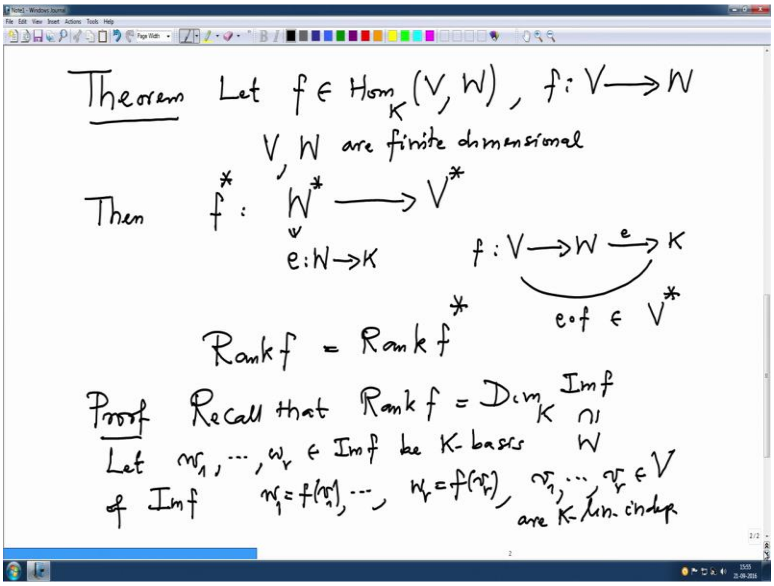
I will summarize that we did in the last lecture about dual spaces. So, I will summarize only in finite dimensional case. So, V finite dimensional K vector space and V star then the dual of V this is Hom K V K. The element are linear forms on V K- linear forms on V. So, there are linear maps on V to K.

And what we have seen is; we have seen in this case is to study the subspaces of V and subspaces of V star on the other hand. We gave a map from this direction and also gave a map in this direction. This map was denoted by the first map is any subspace U of V is map to the right circle on U. That is by definition all those linear forms which vanish on U; that means, U is contained in the kernel of e. This is clearly subspace of V star. And also we have proved that the relation between in there if dimension U has dimension U then here that dimension is nothing but co-dimension of this.

And we have seen this map is injective. Actually we have proved is the inverse in the case of finite dimension the inverse is take any subspace W and map it to W left circle W. That is by definition all those vectors x in V such that every linear form in W vanish on this x; e x is 0 for all e in W. This is clearly a subspace of V. And again the same; dimension of W here that is equal to co-dimension of codim this in V. Actually these maps are e inverses of each other, this map is a denoted by somebody left circle this map is denoted by somebody right circle then these are inverses of each other. That means, these composition is identity and these composition is also identity.

So, in the particular case of these we had seen earlier namely the n minus 1 dimensional subspaces of V which are called hyper planes the correspondent to one dimensional space of V star they are precisely generated by one linear form one non zero linear form. So, this is in general to and last lecture we have seen this how do we prove this their bijective.

(Refer Slide Time: 04:52)



So, this lecture I will continue. Just one theorem I want to prove with the following. This is the main interest of this few minutes discussion, this is the theorem. If I have a linear map f; let f belong to Hom K V, W, it is a linear move from V to W. And let me assume it is not always necessary to assume V and W finite dimensional, but let us assume V W are finite dimensional.

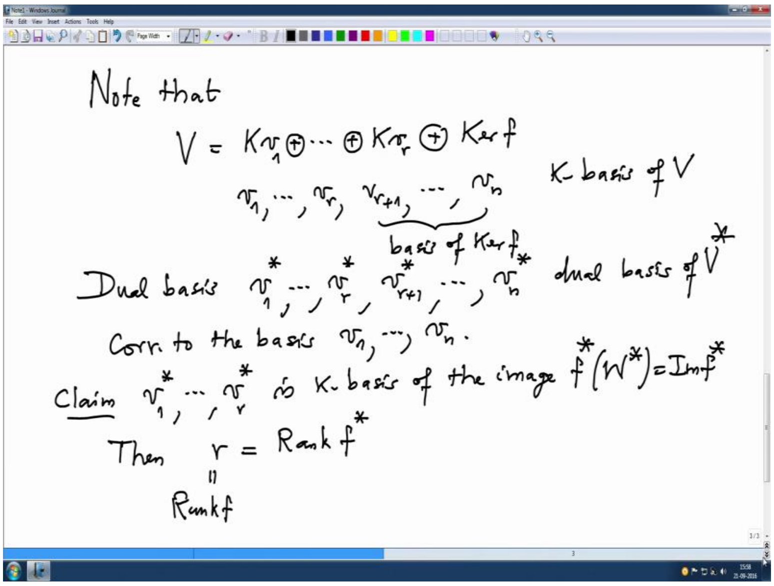
Then I have a dual map corresponding to f that I am denoting by f star, and f star is the map from the other direction W star to V star. This is f is map from V to W. So, dual map of f either map in the other direction on the dual space and by definition it is. That means, they have to define what it is on the linear forms. So, if I have e here that means, e is a linear form on w that means, e is a map K linear map from W to K. And I want to get by using this and f linear form on V. And what is best we can do? We can take f that is from V to W and followed by this e; this is e. So, this map is nothing but f compose e which is now indeed in element in V star. And because e linear f is linear e compose it f is also linear. So, we get a map from the dual spaces of W to dual space of V; the arrow are changing, the directions of the arrow are changing.

Now the assertion I want to prove is rank of f equal to rank of f star. So, let us prove this. Proof: recall that rank of f, rank of f is by definition the dimension of image space image of f. So, I will try to imitate what we did in the proof of rank theorem. So, choose a basis here. So, basis here, so let W 1 this is a subspace of W. W 1 etcetera, etcetera, W r in image f be a basis be a K basis of image of f. So, these are elements in image. Therefore, certainly I can write it as W 1 equal to f of v 1 etcetera, etcetera, W r equal to f of v r; where v 1 to v r are elements are vectors in v.

And because W 1 to W r is a basis we have noted that this V 1 to V r are actually linearly independent, because if the linear combination is 0 apply if to that and then you will get linear combination between f of V 1 to f of V r, but f of V 1 to f of V r are linearly independent because it is a basis of the image, therefore each coefficient is 0.

So, therefore, we have proved that this v 1 to v r are K linear linearly independent. So, we can complete it to a basis. And actually the compliment is nothing but the kernel.

(Refer Slide Time: 09:19)

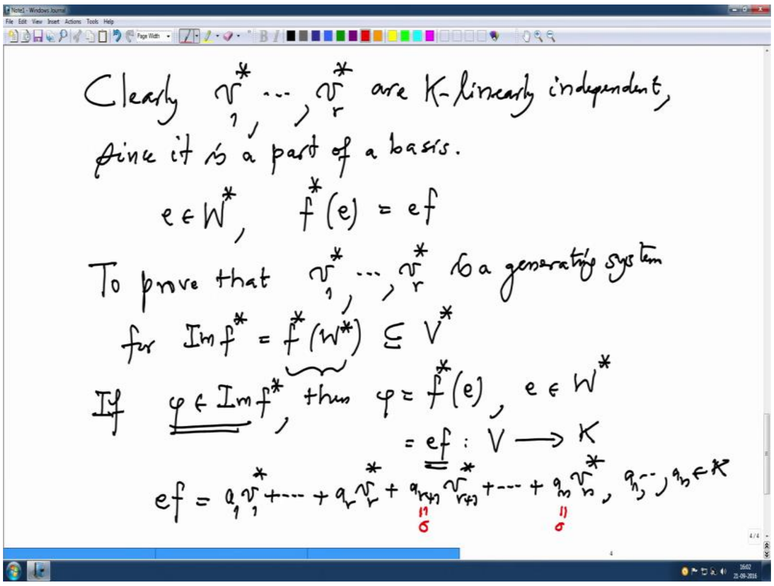


So, note that V is a direction of the subspace generated by v 1 to v r along with the kernel f. Let us call the basis of kernel also you extent this. I could add some also take v 1 to v r and v r plus 1 to v n. This is a basis of kernel of f. So, altogether is K basis of V. Now I have a basis of V so I will take the dual basis corresponding to these bases of V star. So, dual basis that is v 1 star etcetera, etcetera, v r star v r plus 1 star etcetera, etcetera, v n star a dual basis of V star corresponding to the basis v 1 to v n. And we have seen that this is a basis of V star.

Now I want to claim that this part v 1 star v r star is a K basis of the image f star of W star, which is image f star; that is a claim. Once I proved, this claim we have done, then what I will get is the number of then that is r is also rank of f star, but these r was a rank of f. So, that will finish the proof if I prove the claim. So, I have to prove it is a basis. So, I have to prove two things: they are linearly independent and they generate.

So, linearly independent is obvious because something bigger then that is a linearly independents, so smaller set is always independent.

(Refer Slide Time: 12:35)

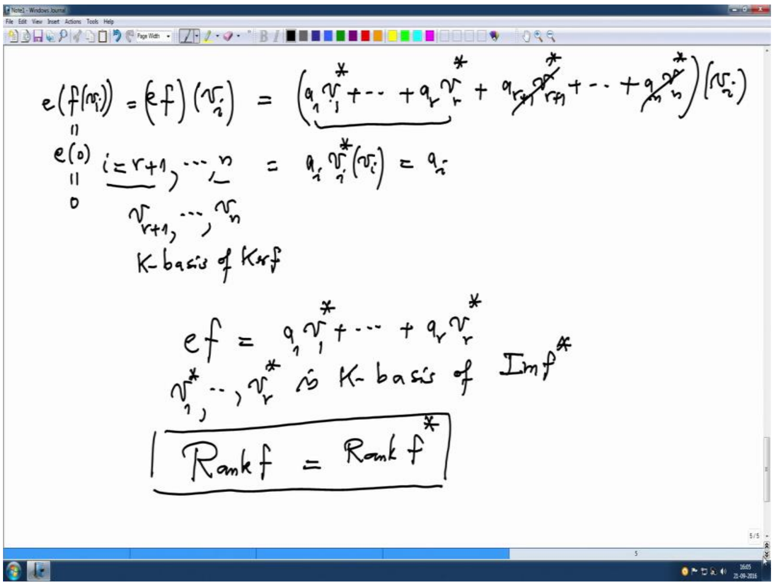


Clearly, v 1 star v r star are K linearly independent. Since it is a part of a basis; basis is a linearly independent and therefore part is linear independent. Now what do I do prove, I have to prove that given its; first let us understand. So, if e in W star then I know f star of e is nothing but e compose f. And we want to prove now, what is you want to prove that we want to prove that this is a generating source so we are proving to prove that v 1 star v r star; I made little.

So, this one is a no nothing, is generating system for the image of f star which is f star of W star which is subspace here V star. That means I want to prove that if I have anybody here in the image it should be a combination of V 1 star up to V r star. Take anybody in the image, but just now I said anybody in the image will look like this. So, if some phi belongs to the image f star, then phi must be f star of some linear form e, e is the linear form in W star. And this I know it e f and I want to write this phi has linear combination of V 1 star to V r star. That means, I want to right this e f as elements of the linear combination of the V 1 star to V r star, but e f is linear form on V.

So, definitely e f has because V 1 star etcetera, etcetera, V n star is a basis of the dual space of V star this definitely as a linear combination like this a r v r star plus a r plus 1 v r plus 1 star etcetera a n v n star or sum a 1 to a n scalars I am interested in proving that this guys are 0. All these coefficient after a r plus 1 etcetera up to a n r 0 I want to prove that. And how do you prove that? I am going to you evaluate this both the sides on the vectors V r plus 1 to vector V n.

(Refer Slide Time: 16:28)

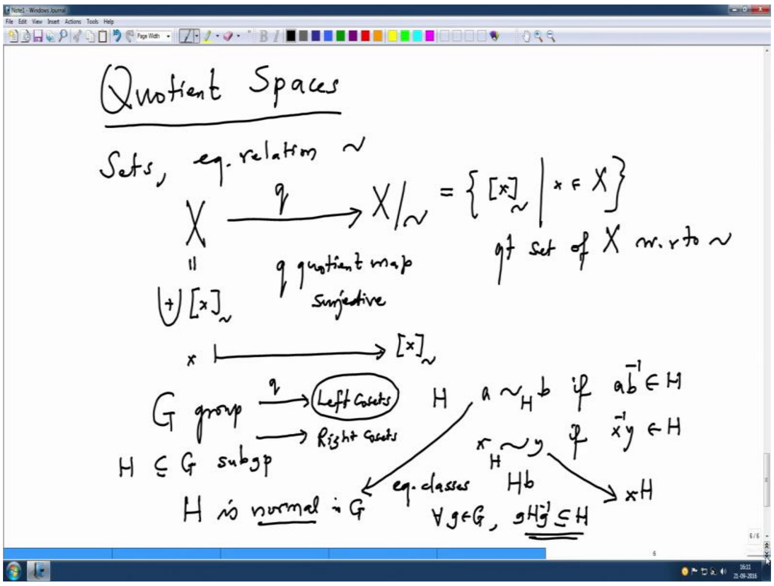


So, we are looking at e of e f evaluated on v i and i is from r plus 1 to n. Note that v r 1 etcetera v n that was the K basis of kernel of f. So, this is by definition e of f of v i, but of v i 0. So, this is e of 0, so this side is 0. On the other side it is a 1 v 1 star etcetera, etcetera plus a r v r star plus a r plus 1 v r plus 1 star plus plus plus plus a n v n star and this you want to evaluate on v i.

And what will remain? Obviously, i is we are taking from r plus 1 to n; so all these guys V j, where j is between 1 and r that is going to vanish here. So the only term which was (Refer Time: 17:47) where a i is coming from. So, this is a i v i star v i which is a i, that is only a term will survive. So, you see we have proved a i 0 and obviously we have used fact that the i is between r plus n and n. So, we have checked that these coefficients are not there. Therefore, e f is a combination of first star v 1 star to v r star.

So, that proves that v 1 star etcetera to v r star is basis of image of f star. Altogether we have proved very important result which we will be very very useful later rank of f and rank of dual are same. When I do a matrix and they ranks etcetera this is analog of fact that matrix and is transfers have the same rank. Or in other words row rank equal to column rank etcetera, etcetera so on.

(Refer Slide Time: 19:20)



Now I will start a new topic which is called quotient spaces. Before I start formally, I want give little bit motivation and some comments. So, quotient spaces in general very important concept this comes in very subject. For example in the sets, we have a set x and usually on the set there are equivalence relations. Let us call it tilde, then we have quotient set x by tilde. This is the set of all equivalence classes, equivalence classes are also denoted like this. So, these are equivalence classes of this. Because x is decompose into the equivalence classes this is the disjoint union.

So, we are looking at the equivalence classes and collecting the different equivalence classes. You need to equivalence classes either they are equal or they disjoint. So, this quotient set is called a quotient set of x with respect to the relation with respect to the equivalence relations tilde. And we have also natural map here q, q is a quotient map. Any element x will go to equivalence class. Of course, many of them go to same equivalence class. So, the fibers of this map are precisely equivalence classes and this map is surjective.

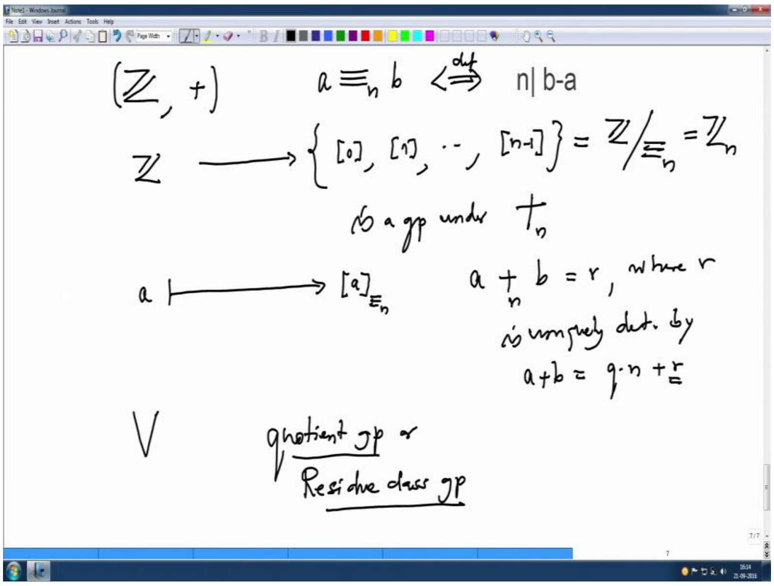
So, one usually studies set with equivalence relation with these quotient map and these quotient set because when we want to induct or for some reason you know all equivalence classes very well then you can put information together to get information about x. But here there is no structure on x other then it is a set. When we have now more structure on x; for example, x is a group or x is a monoid or x is group or x is a ring or x is vector space, then you have to worry about whether I can pass on their structure to the quotient set or not. And this is not usually the case. So, you have to give some condition on the equivalence relation so that our structure is passed on the quotients.

Just to give an example; suppose when we have a group G, and a sub group H sub group. Now of course, H will define the equivalence relation. Actually H defines two equivalence: relation one is called left one, another is called a right one. So, they return like this which side you right H of the tilde. And this first one defined by A b; that is if a b inverse belongs to H. And this one is x y if x inverse y belong to H. I may be I might made some error in this, but we will cross check it. So, this equivalence relation the equivalence classes will be the left cosets H b.

So, this is the equivalence classes the first one. And these equivalence will be the right cosets and left cosets; this is the right cosets and that is the left cosets x H. In general these two cosets are not different. So, equivalence classes are different. Now the problem is when I take the quotient set with respect to this one I get left cosets, so left cosets. And with respect to this one, this one I get left cosets one of my left coset other I get right cosets. So, even sets these are different, but you would like to put group structure on these quotient set. That means, you would like define a group of operation on this set so that the quotient map becomes group on Homomorphism. So, there is a quotient map here, there is also quotient map here, and we want to put group structure here such that this quotient map is group of Homomorphism. Then only it is useful, because you would like to pass on from G to this.

And this does not always happen. If one analyze this you come to condition that H is so called normal subgroup. This is the condition you get it, which is always true for Abelian groups. You remember normal definition is for any g in G g H g inverse should be a sub group of H again this is condition for all g then you call it normal. If you analyze this, this is the precise meaning of fact that this equivalence relation comparable with the group operation or it is sometimes it is also called congruence relation. So anyway, for Abelian group is always true. For example, we have done this construction very often for the additive group of integers.

(Refer Slide Time: 25:51)



This is Abelian. And we have typical equivalence relation congruence module n. Congruence module n in the equivalence relation which is defined by a congruent b module congruent module n if and only if the definition was b minus a divides n.

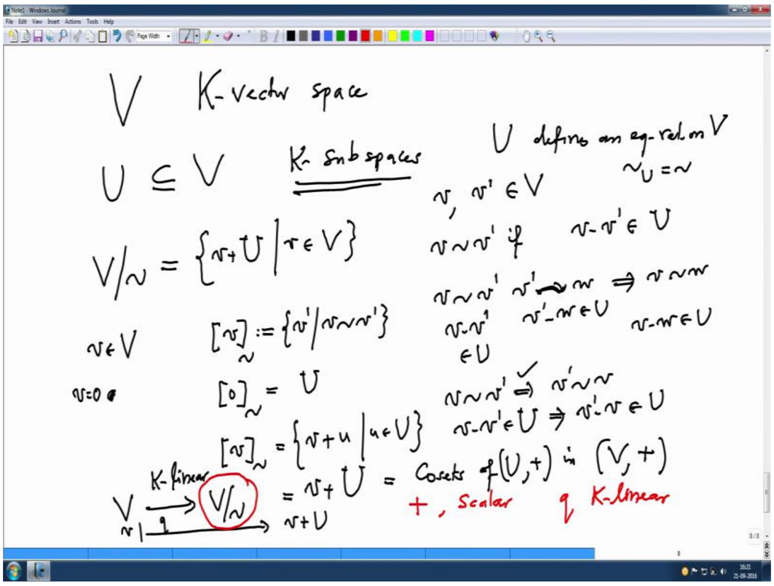
And now we know what the equivalence classes of this. Equivalence classes precisely the residue classes; that is residue class of 0, residue class of 1 and so on. Residue class of n minus 1 and this is the quotient set, this is precisely z, this quotient set and this is the group; again under addition module n. So, this is a group under addition module when I write this that simply means you take integers a and b add them usual and take the reminder after dividing by n. So, this is r where r is uniquely determine by the division logarithm a plus b equal to q times n plus r; this r is the remainder after division by m.

So, this is a group again and this group we have being calling a z module. And now the natural map from here to here z to z module n is any integer a going to its remainder after dividing by n. So, that is residue class of a residue class of a module n and we have seen this is a group of morphism. So, this is very important when you want do calculations module n, because from integers pass on these groups and you do your calculations in this group.

In short a vectors spaces I am also looking for such thing if I have a vector spaces V I want to construct a new vector spaces which is like these model. So, those we V call quotient spaces, this is also called quotient group of set; which called quotient group or it is called residue class group. And such a construction I want to do it for a vector space, one can do it for rings also, you can do it for almost very object you can ask question whether given in object with some structure if they equivalent relation can we pass on to the quotient set the similar structure. Or what are conditions on the equivalence relation we may require to pass on this structure from the original set to the quotient set.

So, with this motivation I just want to start like I studied for a group.

(Refer Slide Time: 29:20)



Now, let us take V to be K vector space and let us take U to subspace K subspace. Then definitely these U will define the equivalence solution, U defines an equivalence relations on V what is it. That means, two vectors V and V prime; let me call it the notation for equivalence relation define by U is tilde subspace U. I will fix U, so I will just write this by tilde the definition of tilde is U V V prime in V, then you say that V is related to V prime if is a definition that V minus V prime belongs to U.

Now let us check orally that this is indeed equivalence relation. Remember U is a sub space. So, first is flexibility that is V related to V, but then V minus V is 0 all that you need to check 0 belongs to u, but that is part of definition of a subspace. Similarly, transitivity: V related to V prime and V prime related to say W. This means V minus V prime belongs to U, this means V prime minus W belongs to U but if these two are in U then they sum is also in U because it is a subspace. That means, V minus W belongs to U, and that means V is related to W.

So, we have checked flexibility transitivity and now you want to check that is symmetric. So, symmetric means V related to V prime, V prime related and V prime related to V that should imply that. But this means V minus V prime is a u, but somebody we knew then minus of that is also U because it is a subspace. So, V prime minus V is also U because it is subspace, so therefore this is also correct. So, we have checked that this an equivalence relation. And that is the quotient space now, V what, tilde is what; these are the equivalence classes. So, let us see; what are the equivalence classes. So, far if you fix V in V the equivalence class of V and tilde is by definition precisely all the vectors in V which are related to this given V; that means all this V prime such that V is related to V prime. These are the equivalence classes. For example, if V is 0 then what is the equivalence class of 0? that is precisely U.

For arbitrary V what is the equivalence class of V; that is all those vectors which I can write V plus somebody is in U, this guys is a precisely equivalence class of V because if U is in mu then this V plus U related to U because if I subtract V from U here you get U is in and everybody like that. So, this is the notation for these I want to write it has V plus U. This by definition takes all the vectors U and I added to given V. So, these are visibly they are cosets; cosets of U plus in V plus. So, that is the quotient set. This is nothing but cosets; that is V plus V where V belongs to V. And you can also analyze when are the two cosets equal that is precisely when the difference is or the representatives belongs to U.

And our problem is now to define vectors spaces structure on this quotient set so that the natural quotient map is a K linear map, this map should be K linear. And this is the natural map q, not arbitrary map this is a map V goes to V plus U. And we want to put a structure here now. We are looking for; on this set we are looking for addition so that it becomes Abelian group. We are also looking for a scalar multiplication so that it becomes vector space and with these structures this q map should be K linear all this thing we need to check. But thing are so natural that if one defines it correctly is clear that is K linear.

So, I will stop and we will continue after the break.