Ideal fluid flows using complex analysis
Professor Amit Gupta
Department of Mechanical Engineering
Indian Institute of Technology Delhi
Lecture 12
Introduction to conformal transformation

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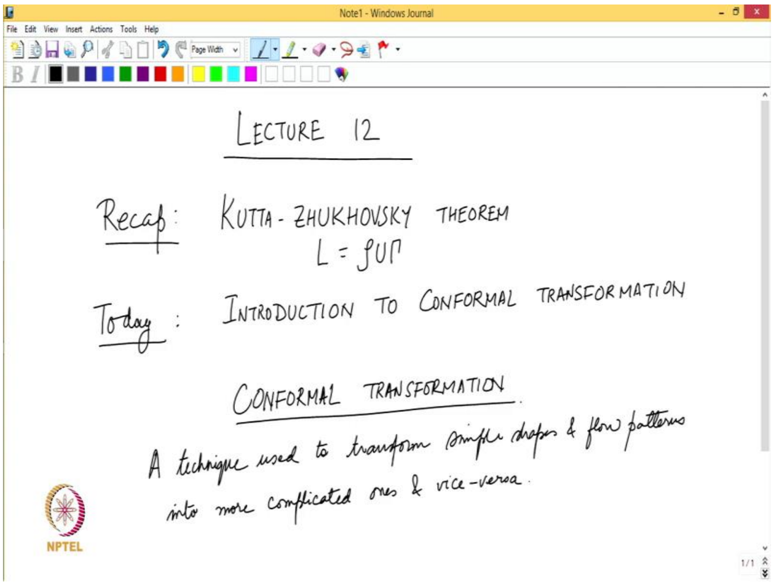

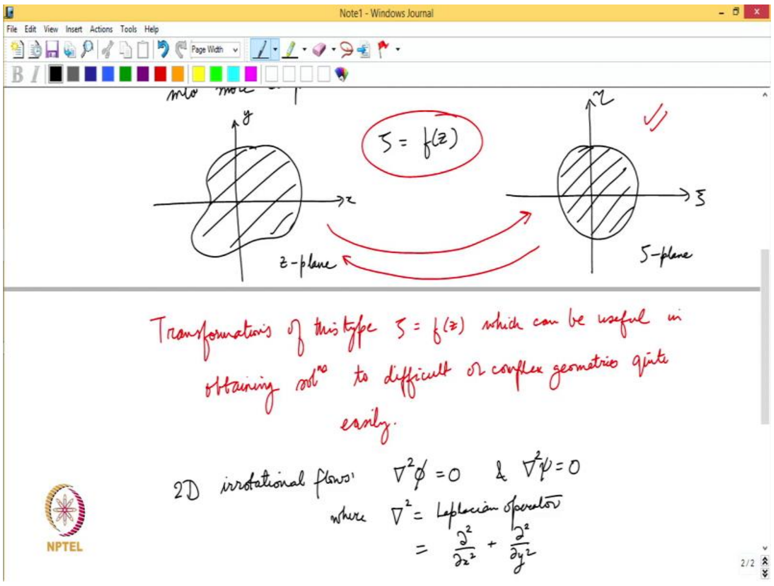

So, in the previous lecture, we came across the Kutta-Zhukhovsky theorem, which define the lift force per unit depth in terms of the circulation. So, we said left is rho U gamma, and we prove this for a simple case of flow pasture circular cylinder, and we did it two ways. One was essentially integrating the pressure distribution and the second was using the Blasius theorem.

So, in today's lecture, I will talk about another important technique, which is called as conformal transformation, and how this can help us obtain a variety of flows in much more complex scenarios. So, let us begin our lecture today. So, we will talk about conformal transformation. And what is conformal transformation first of all, so, this is a technique which is used to transform simple shapes or flow patterns into much more complicated ones and vice versa.

So, this is a technique which is used to transform you can say simple shapes and flow patterns into more complicated ones and we can do this vice versa. So, it can even be that we can go from a more complicated problem to a simpler one. But essentially it is more commonly used where we have a simple solution or a simple problem and we want to use that to derive a solution to a much more complex problem.

So, the way this works is, let me start with a simple case say we have a shape in the z plane or the z plane say this is a z plane and we have some complex shape in this plane. The solution to which we may not know or may know it either way, let us say we do not know the solution to this problem. But we know the solution to a simpler problem. For instance, we know the solution to say a problem which has a simpler geometry, say a circular geometry.

And this is what I am going to call as a zeta plane which has its axis as xi and eta. So, conformal transformation is about finding a suitable function, which says zeta to be f of z, which transforms this function or this type of an object to a shape of this type, or even vice versa where maybe we have the particular problem, which we know the solution to an easier to solve problem and you want to transform it back into a much more complicated shape.

So, what we are looking for are these transformations of this type. Let me write this down. So, we are looking for transformations of this type which I have written as zeta equal to f of z or f of z, which can be useful in obtaining solutions to difficult or complex geometries quite easily. So, you can imagine that if I have the solution to maybe the right problem, I can use a conformal transformation.

And I will talk about a little more about informal transformation very soon, but I could use some kind of transformation to revert or to bring it into a more complex scenario. Now, because this is transforming, not just the geometry, the flow will also have to transform and we are dealing with 2d irrotational flows.

We are dealing with 2d irrotation flows, which satisfy the equations del square phi is 0 and del square psi is 0 where del square is called as a Laplacian operator and it is given as say in this z plane this would be d2 dx square plus d2 dy square. So, we need to identify for instance, if we have a flow and a shape in the z plane and we do a conformal transformation and take it into the zeta plane then how does a geometry get transformed and how does a flow get transformed and the flow parameters are basically phi and psi.

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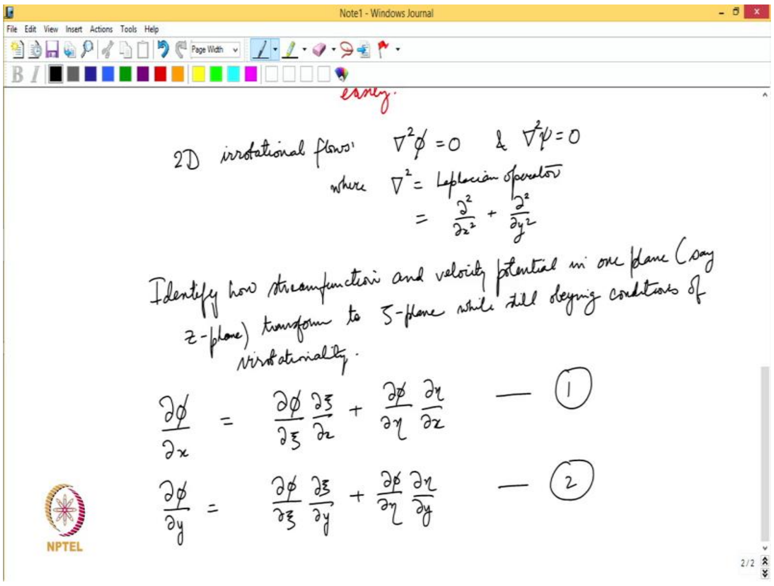

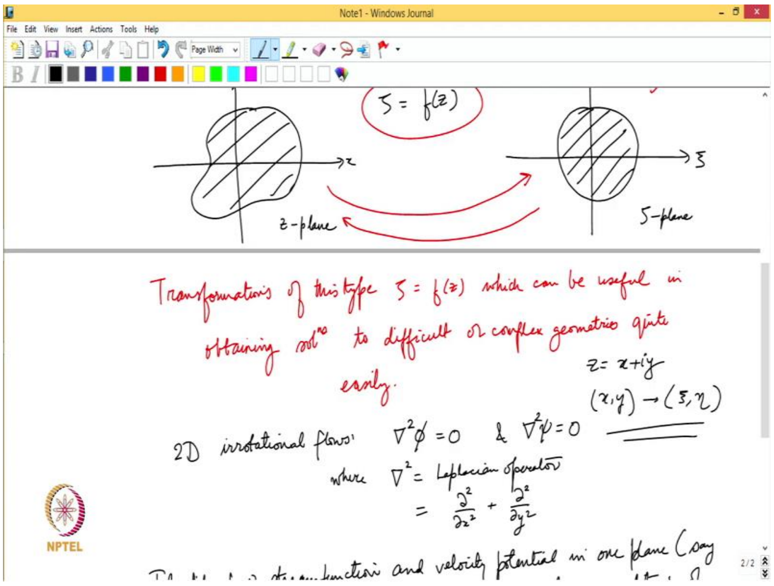

So, we need to identify how these stream function and the velocity potential in say one plane for instance for my case says z plane or z plane how these transform to the zeta plane while still obeying, which is the most important part, conditions of irrationality. So, you want the same rules to be still applicable irrespective of the fact that we are transforming the given problem.

So, what we need to check for whether the function in a specific problem let us say z problem, when it goes to zeta problem, it still satisfies the condition of irrationality is to check for how the Laplacian operator transforms when we go from the z plane to the zeta plane. So, we would need the following. So, we would need to calculate d2 phi dx square and d2 phi dy square, and similarly, d2 psi dx square and d2 psi dy square.

And we need to transform that to the zeta plane. For that you will need to calculate these derivatives for instance d phi dx because I know the second derivative of phi with respect to x will give me d2 phi d square, which is an essential component of the first part of the Laplacian operator. So, d phi dx for instance, would be if I use a chain rule, now, recall that our system is getting transformed we are going from z to zeta.

So, we will have d phi d x to be d phi d xi d xi dx plus d phi d eta d eta dx. So, this is equation 1 for our case. And similarly, I can write d phi dy, just the first derivatives for now, this will become d phi d xi d xi dy plus d phi d eta d eta dy and say this is equation 2. This is just by a chain rule. Essentially what I am doing is if you look at this problem, we have zeta is f of z, z is a number in the complex plane which is x plus iy, we are transforming it to the zeta plane. So, that when we go to the zeta plane x comma y changes to xi comma eta, we get new coordinate xi and eta.

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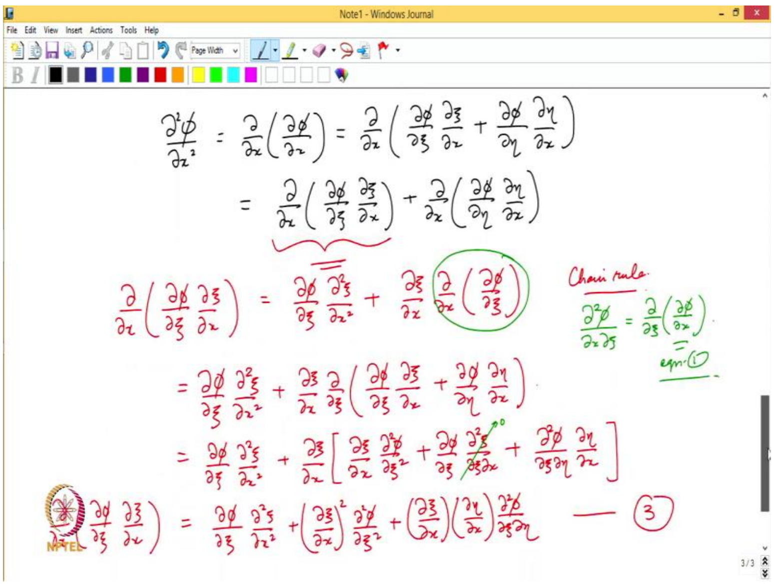

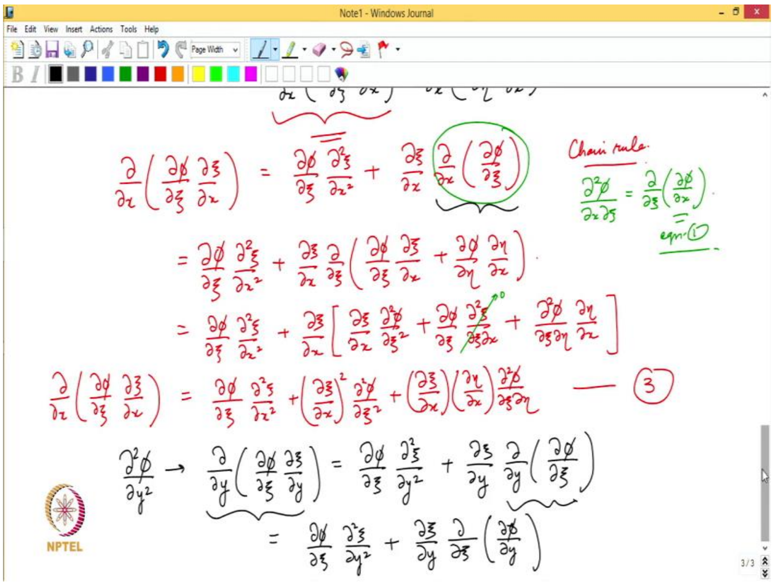

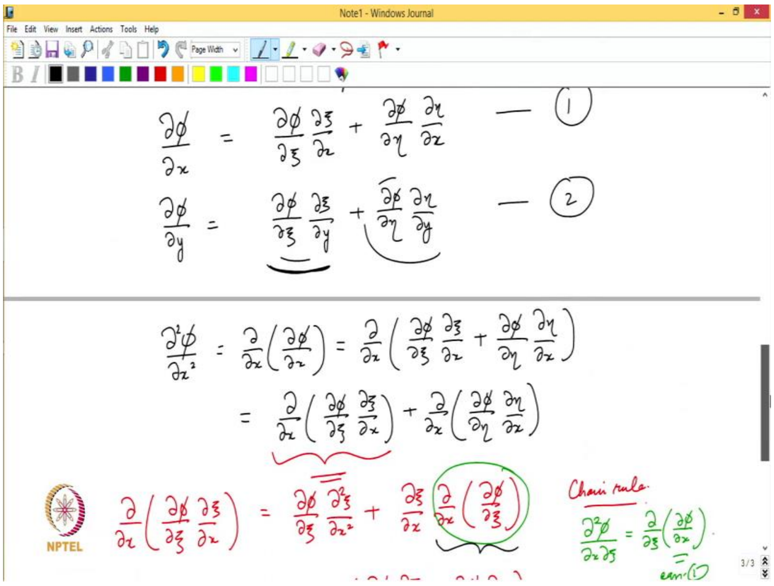

Now, d phi dy is also done. Now, what I am going to do is try and demonstrate how the Laplacian will change. So, say we want to calculate d2 phi dx square, which would be d by dx of d phi dx. Now, this I am going to now use, I am going to use equation 1 to define what d phi dx will be in terms of xi and eta. So, we can write this as d by dx of d phi d xi d xi dx plus d phi d eta d eta dx.

Which we can also write as if we split the derivative into two parts, we could write this as d by dx of d phi d xi d xi dx plus d by dx of d phi d eta d eta dx. I am hoping that you are with me till this point. Now, the process from here is slightly long, it is quite doable. But for the purpose of this demonstration, I am going to try and stick to some aspects only. Try and show you how certain terms evolve. And then I will sort of generalize in the end.

So, for now, let me just consider this part. Let us only focus on this part for now. So, say we would write this as d by dx of d phi d xi d xi dx, I just want to look at how this will simplify. So, I can use a chain rule here. And the way I would use the chain rule is let us say we do the first we keep the first function constant, we do the derivative the second one, so we will get d2 xi dx square plus, we will have d xi dx into d by dx of, we will get d phi d xi, this is just using the chain rule.

Now, what I am going to do is I see that there is the derivative of respect to phi or this term is essentially, if you want to put it, this would be d2 phi dx d xi, which I could also write as d by d xi of d phi dx. And I could use d phi dx, which I have in equation 1. So, d phi dx is given an equation 1. So, let us use this and simplify what we have on the right side. So, we will have d phi d xi d2 psi dx square plus we will have d xi dx.

And now we will have d by d xi of d phi dx which is d phi d xi d xi dx plus d phi d eta d eta dx. This is just from equation 1. You can just verify. Now, we take the derivatives. So, we will have d xi dx, so d by d xi operated on the first function which is d phi d xi d xi dx, so this will be d xi dx, this would be d phi d xi dx times d2 phi d xi square plus if I go by chain rule we will have d phi d xi into d2. Let me write this down.

So, we can d phi d xi into d2 psi d xi dx, which is actually if you are if you notice carefully this should be 0 because d xi will be 1. So, this is going to be d of a constant by dx which is 1 or which is 0, plus we will have d2 phi d xi d eta d eta dx. And finally, we will have another term, which would sort of go by d by d xi of d eta dx, but e times xi being independent variables, that term will also give you a 0.

So, finally, what we have is if we just now open up the brackets, we will have the following derivatives, this is what we will have for d by dx of d phi d xi d xi dx. So, let me say this is equation 3. As you can see, it is growing now, for example, one term only in that d2 phi dx square gave me so many terms, we could do the other part as well. But let us now focus on d2 phi dy square.

If I look at equation 2 now, where we have written d phi dy, so d2 phi dy square will have the first function this one with d by dy, and then this one with d by dy. So, just for convenience, or for some variety, let me just say I will take this one with d by dy. So, d2 phi dy square will lead me to, if I take this derivative, it will lead me to d by dy of, as I said the first term, which is d phi d xi d xi dy. So, we will have d phi d xi d xi dy.

So, let us see what this gives us. So, if I take these derivatives now, we will have d phi d xi d2 xi dy square plus d xi dy and then we will have again d by dy of d phi d xi, where I again note that this is very similar to what I had here. It is the same idea that I can now swap the order of the derivatives, I can take the y derivative first and then the xi derivative. So, I could write this as d phi d xi d2 xi dy square plus d xi dy dy d by d xi of d phi dy and then I use d phi dy from equation 2.

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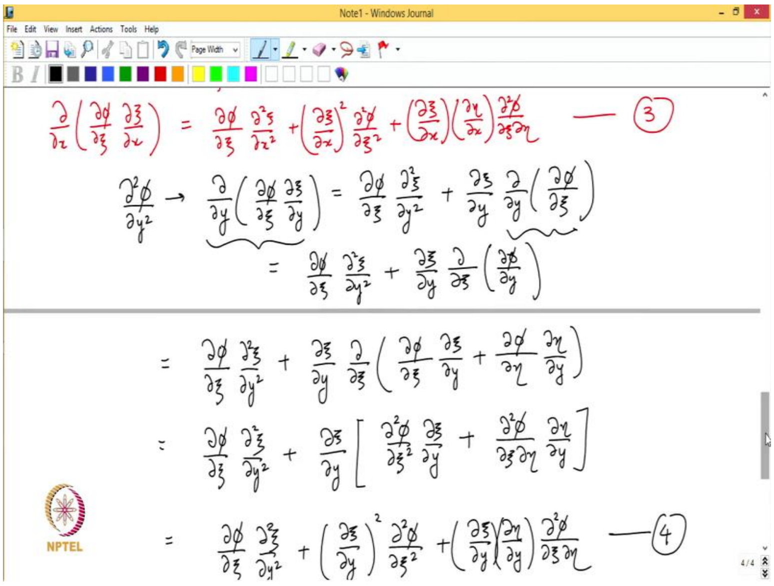

So, we use d phi dy from equation 2. So, we have d phi d xi d2 xi dy square plus d xi dy of d by d xi of d phi d xi d xi dy plus say d phi d eta d eta dy which we can just take one more maybe a couple of steps to simplify so that some pattern emerges. So, we have d xi dy, this will be say d2 phi d xi square times d xi dy plus d2 phi d xi d eta d eta dy. And which we can now collect together to get the following.

So, we will have d xi dy square d2 phi d xi square plus d xi dy d eta dy into d2 phi d xi eta. Say this is our fourth equation. So, you see that when I do this, we now have at least some parts of the picture, we have some derivative, we have calculated some derivative here we have determined this part of the derivative here, the only two terms that remain are essentially the following.

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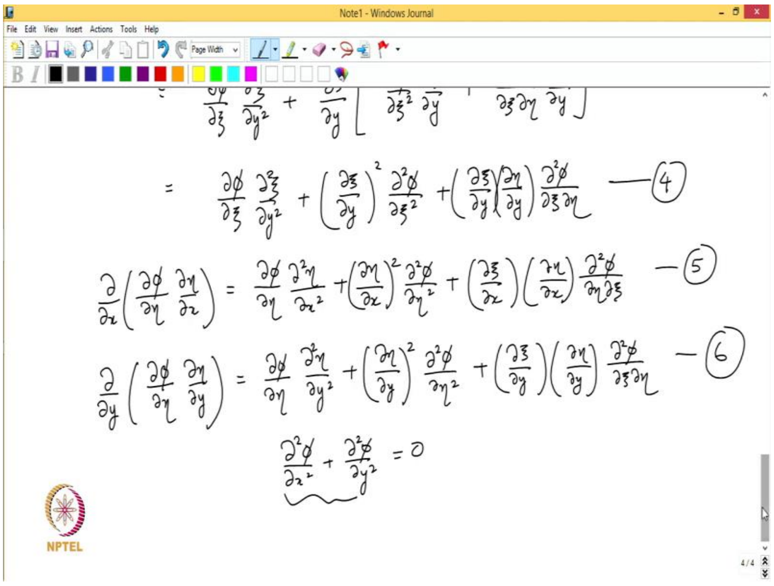

We have not done d by dx of d phi d eta d eta dx. But this is quite easy now, if I can write this for you straight away, this should be d phi d eta d2 eta dx square plus d eta dx square d2 phi d eta square plus we will have d xi dx d eta dx into d2 phi d eta d xi. So, this is the fifth term which will come from the d2 phi dx square, the second part of that expression.

And similarly, if you look at d2 phi dy square, this other term that I have not worked out yet is d by dy of d phi d eta d eta dy which again I can write as just to save some time, which I am sure you can work out now on your own, this will be d phi d eta d2 eta dy square plus d eta dy square d2 phi d eta square plus d xi dy d eta dy times d2 phi d xi d eta.

So, long derivation I understand but we have now assembled all parts to calculate what we said was d2 phi dx square plus d2 phi dy square equal to 0. So, we have now have everything in this equation in terms of xi and eta. So, what we can now do is assemble it.

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$$\frac{\partial^2\phi}{\partial x\,\partial\xi} = \frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial x}\right) \quad \text{eqn (1)}$$

$$= \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}\right)$$

$$= \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\xi}{\partial x}\left[\frac{\partial\xi}{\partial x}\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial\phi}{\partial\xi}\cancel{\frac{\partial^2\xi}{\partial\xi\,\partial x}}^{\,0} + \frac{\partial^2\phi}{\partial\xi\,\partial\eta}\frac{\partial\eta}{\partial x}\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x}\right) = \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} + \left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial\xi^2} + \left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right)\frac{\partial^2\phi}{\partial\xi\,\partial\eta} \quad \text{— (3)}$$

$$\frac{\partial^2\phi}{\partial y^2} \rightarrow \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial y}\right) = \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial\xi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial\xi}\right)$$

$$= \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial y}\right)$$

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$$= \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial y}\right)$$

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$$= \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial\xi}{\partial y}\left[\frac{\partial^2\phi}{\partial\xi^2}\frac{\partial\xi}{\partial y} + \frac{\partial^2\phi}{\partial\xi\,\partial\eta}\frac{\partial\eta}{\partial y}\right]$$

$$= \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial\xi^2} + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right)\frac{\partial^2\phi}{\partial\xi\,\partial\eta} \quad \text{— (4)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}\right) = \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial x^2} + \left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2\phi}{\partial\eta^2} + \left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right)\frac{\partial^2\phi}{\partial\eta\,\partial\xi} \quad \text{— (5)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}\right) = \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial x^2} + \left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2\phi}{\partial\eta^2} + \left(\frac{\partial\xi}{\partial x}\right)\left(\frac{\partial\eta}{\partial x}\right)\frac{\partial^2\phi}{\partial\eta\,\partial\xi} \quad \text{— (5)}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial y}\right) = \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial y^2} + \left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial^2\phi}{\partial\eta^2} + \left(\frac{\partial\xi}{\partial y}\right)\left(\frac{\partial\eta}{\partial y}\right)\frac{\partial^2\phi}{\partial\xi\,\partial\eta} \quad \text{— (6)}$$

$$\underbrace{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}}_{} = 0$$

So, let us assemble, let us put d2 phi dx square and then d2 phi dy square. So, when I put d2 phi dx square now I want you to understand is some kind of a pattern that is developed now, when I write, for instance, the first expression here. This equation 3 will bring the first part of it. So, we will have terms of this type, we will have terms of this type and then this one, so we will have basically some terms which will have first derivatives of phi, but second derivatives of xi or eta.

We will have few terms which will have second derivatives of phi and then squares of the derivatives of xi or eta and then some kind of cross derivative terms. That is true in all expressions that you see here. First derivative multiplied by second derivative and then the derivative square multiplied by second derivative of phi and then cross derivatives. Same thing has happened here. So, you can see that it is the pattern will, there is some pattern to it at least.

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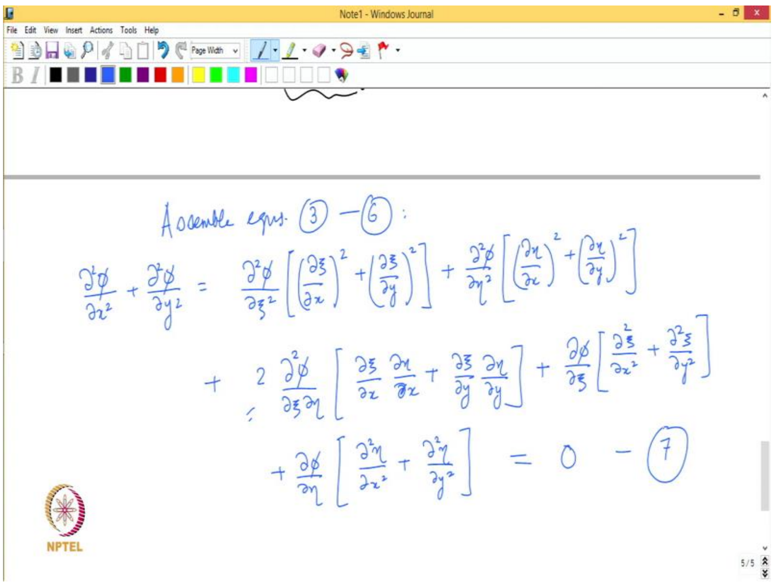

$$= \frac{\partial \phi}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \xi}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial^2 \phi}{\partial \xi^2} + \cancel{\frac{\partial \phi}{\partial \xi}\frac{\partial^2 \xi}{\partial \xi \partial x}} + \frac{\partial^2 \phi}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right]$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial x}\right) = \frac{\partial \phi}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 \phi}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 \phi}{\partial \xi \partial \eta} \quad \text{— (3)}$$

$$\frac{\partial^2 \phi}{\partial y^2} \rightarrow \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial y}\right) = \frac{\partial \phi}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial \xi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial \xi}\right)$$

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$$= \frac{\partial \phi}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial \xi}{\partial y}\left[\frac{\partial^2 \phi}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial^2 \phi}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y}\right]$$

$$= \frac{\partial \phi}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 \phi}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 \phi}{\partial \xi \partial \eta} \quad \text{— (4)}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial x}\right) = \frac{\partial \phi}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 \phi}{\partial \eta^2} + \left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 \phi}{\partial \eta \partial \xi} \quad \text{— (5)}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial \eta}\frac{\partial \eta}{\partial y}\right) = \frac{\partial \phi}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} + \left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 \phi}{\partial \eta^2} + \left(\frac{\partial \xi}{\partial y}\right)\left(\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 \phi}{\partial \xi \partial \eta} \quad \text{— (6)}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

So, if you assemble everything now, all equations. So, if we assemble equations 3 to 6, we will have d2 phi dx square plus d2 phi dy square to be the following. We will have d2 phi d xi square which comes from let me show you here d2 phi d xi square, which comes from here this middle term. So, we will have d2 phi d xi square and similarly we will have d2 phi. Look at its coefficient it is d xi dx square.

Similarly, we have another d2 phi d xi square, which will give me d xi dy square as a coefficient. So, we will say this is d2 phi d xi square times d xi dx square plus d xi dy square that is 1 part. In a similar manner, we will have a d2 phi d eta square. So, let us look at this one, we will have d2 phi d eta square, which has coefficient here d eta dy square and then we will have d eta dx square. So, we can write that here.

So, we will have d eta dx square plus d eta dy square. Then we will have some cross derivative terms, which will add, so we will have 2 d2 phi d xi d eta into d xi dx d eta dx plus d xi dy d eta dy. So, we can just compare where this came from. This is coming from here and here, each of these will have this same contribution. So, we have now seen that it is two of each type, so two of d xi dx d eta dx, and then two of d xi dy d eta dy. So, that is why we have a coefficient 2 here, which is justified.

And plus, now we will have single derivative terms, such as this one. Let us look at, this one, you have d phi d xi d2 psi dy square, and then we will have, similarly, we will have d phi d xi, for instance, d2 psi dx square. So, these terms will also now come together. So, we will have d phi d xi times d2 psi dx square plus d2 xi dy square plus d phi d eta d2 eta dx square plus d2 eta dy square.

Now, this should be 0 because the left-hand side is 0, by transformation this is what we have obtained say this is equation 7.

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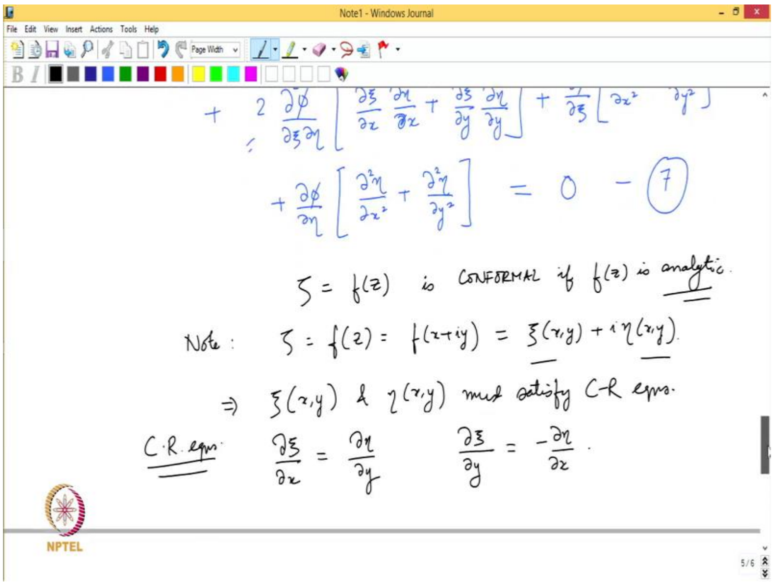

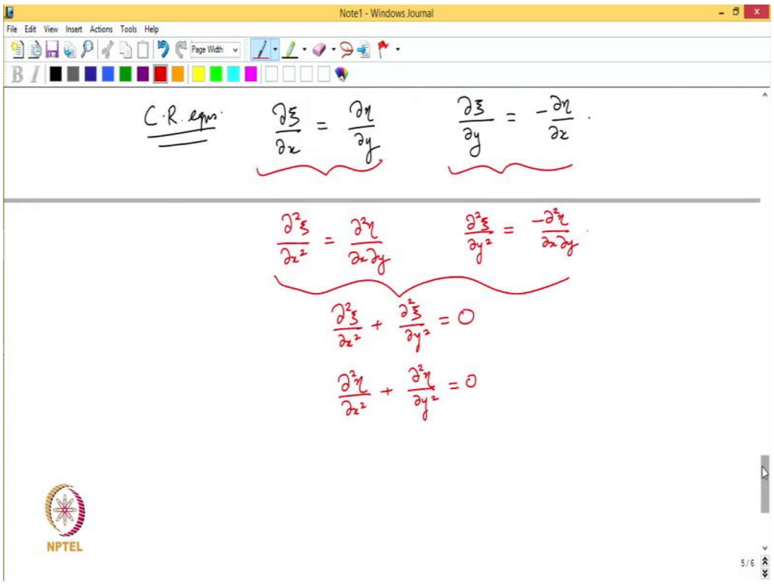

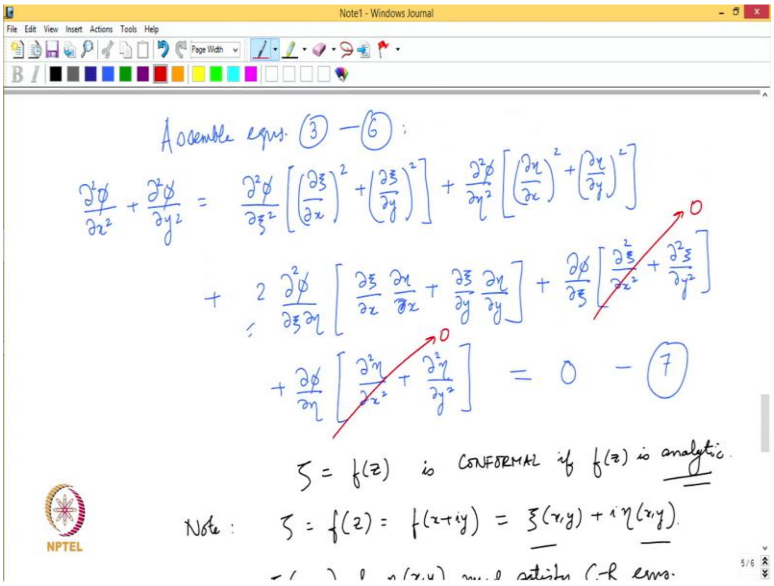

Now, we talk about what was the nature of this transformation. Notice that the transformation that I have invoked is zeta is some f of z that was a transformation that we started with. Now, we say that this transformation is conformal. We said this is conformal if f of z is an analytic function. So, the transformation will be conformal or it would be called a conformal transformation if f of z is analytic. Now, note that f of z which is z being in the, in terms of x and y being x plus iy, this is being transformed to a new set of coordinates in this zeta plane, which I am writing it as xi of x comma y plus eta of x comma y.

So, if the function f of z is analytic, then the component is zeta the real and the imaginary part they must satisfy the Cauchy Riemann equation. So, for analyticity we know that xi x y and eta x y must satisfy the Cauchy Riemann equations, that is how we say our function is analytic. In fact, that is the whole premise of us using complex analysis in this course. Now, what are the Cauchy Riemann equations for zeta or the components of zeta which are psi and eta.

The Cauchy Riemann equations here would be that d xi dx must be the same as d eta dy that is the first Cauchy Riemann equations. The second one is d xi dy is minus d eta dx. If our transformation is such that it is analytic, then we call it as a conformal transformation. And for the transformation to be conformal, we need to have its real and imaginary parts satisfy the Cauchy Riemann equations.

Now, I can use this to say something about what I have derived in equation 7. So, for instance, if I invoke this equation here, the first identity, then I can take a second derivative of this with respect to x. So, I will have d2 xi dx square to be d2 eta dx dy. And if I take maybe a second derivative of this with respect to y, we will have d2 xi dy square to be minus d2 eta dx dy.

And I could merely add these two now, to get the d2 xi dx square plus d2 xi dy square is 0. So, for our function, which is an analytic function, d2 xi dx square plus d2 xi dy square is 0, which means we do not have this contribution showing up in this equation 7, this will go to 0. Similarly, I could have proven that d2 eta dx square plus d2 eta dy square is also 0.

That is again something that you can prove by taking, let us say derivative with respect to y of the first Cauchy Riemann equation and then derivative with respect to x of the second Cauchy Riemann equation, you will get the same identity. So, that means this is also going to 0.

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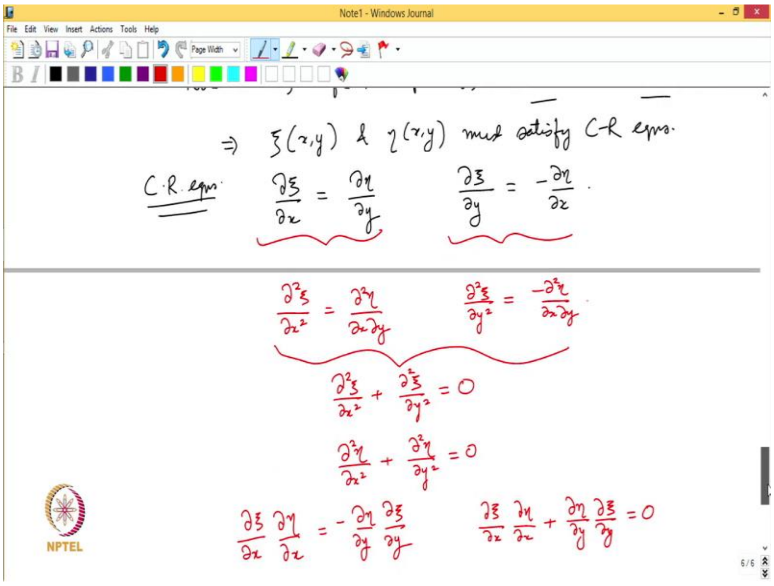

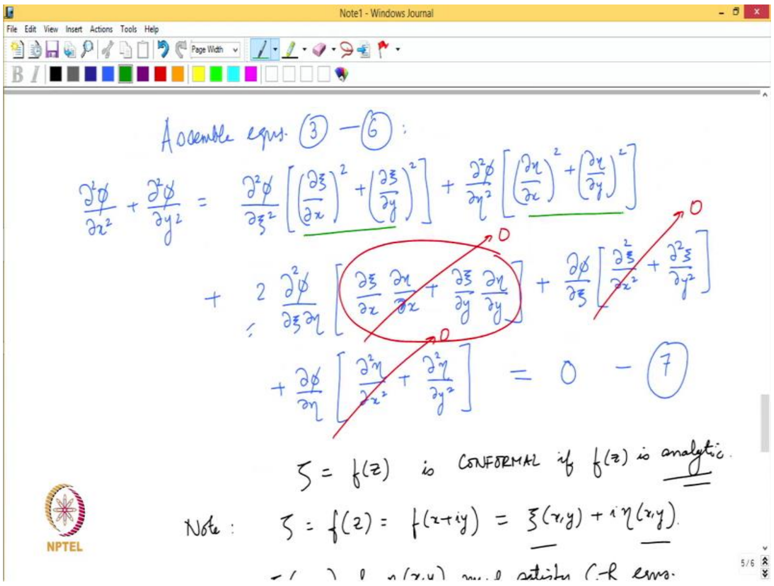

Now, let us look at the other three terms left in this equation 7. To say something about them. Let us consider multiplying the two Cauchy Riemann identities. So, if I say these d xi dx times say d eta dx, where I know that d xi dx is d eta dy and d eta dx is minus d xi dy.

Now, notice that appears here in this coefficient of the mixed derivative. What we are saying is d xi dx d eta dx plus d eta dy d xi dy is 0. That simplifies my equation 7 because then this is also going to 0, the third term here is also gone now. Now, let us look at the last thing that remains, which is what about these coefficients? This and this what happens to them. Now, to understand how they transform how the change, let us take the squares of the Cauchy Riemann equation.

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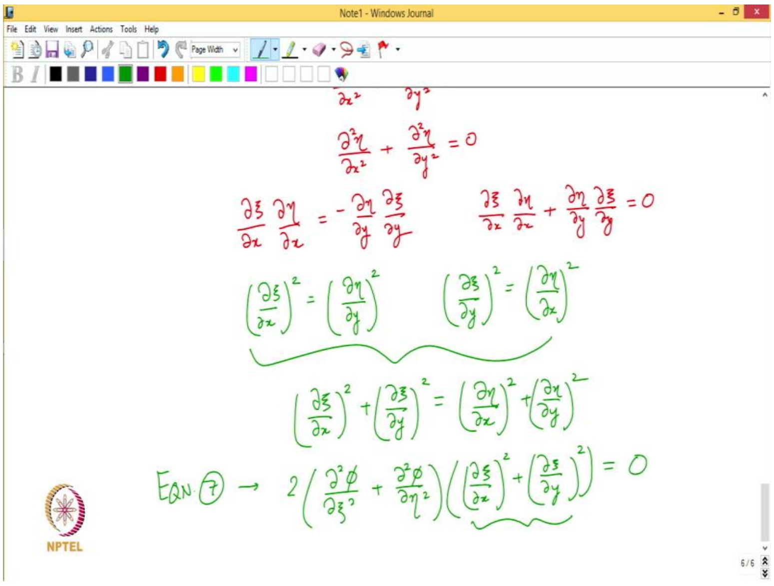

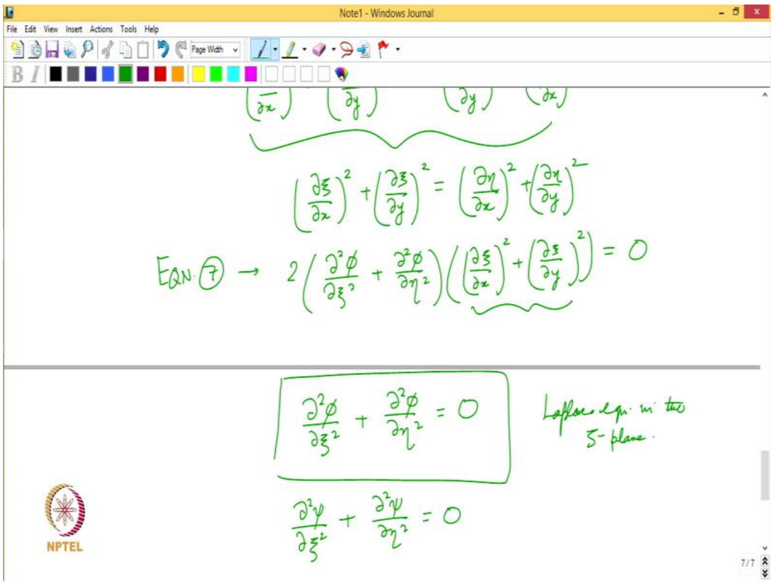

So, I can say that d xi dx square will be d eta dy square and d psi dy square will be d eta dx square. If I add these two equations, I will have d xi dx square plus d xi dy square to be the same as d eta dx square plus d eta dy square. Using this now, if I look at equation 7, which had these squares as the coefficients, we can now write equation seven in the following way. We can now write the equation 7 is 2 times say d2 phi d xi square plus d2 phi d eta square times say d xi dx square plus d xi dy square is 0.

Now, I note that this cannot be 0 because it is definitely positive that is a sum of squares. And so, clearly d2 phi d xi square plus d2 phi d eta square must be 0. And now, this is basically the Laplace equation in the zeta plane. So, what we have now proved is that the Laplace equation that we had in the z plane or z plane transforms to Laplace equation in the zeta plane.

And similarly, I could also prove that for the stream function we will have d2 psi d xi square plus d2 psi d eta square is 0 that will also be the same basic derivation. So, that can also be proved now. So, we have a key message here now, and the message is since phi and psi satisfies the Laplace equation in both zeta and z plane. Hence, a complex potential which lets in z plane is also a valid complex potential in the zeta plane and vice versa.

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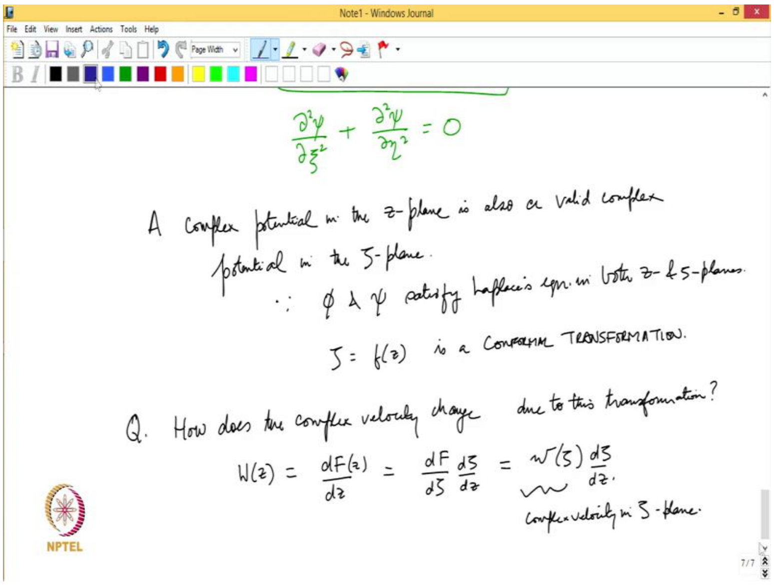

So, a complex potential in the z plane, a valid complex potential is also a valid complex potential in the zeta plane because phi and psi satisfy Laplace equation in both z and zeta planes and that is the whole idea of using zeta equal to f of z as a conformal transformation. So, what we have now come up with is that if we have a solution for some simple body, let us say in the z plane or zeta plane does not matter.

If the solution is available, which is in terms of phi and psi of course, then the solution to a more complex body can be obtained by just simply substituting zeta equal to f of z in the complex potential. So, we will take an example of this in the next lecture, but for now, let me talk about one aspect of this transformation, which is that how does the complex velocity transform, how does the complex velocity change due to this transformation.

So, recall that we defined complex velocity w as dF dz where F is a complex potential. By chain rule I could write this as dF d zeta into d zeta dz. Now, I note that F is a valid complex potential in both z and zeta planes because that is what we have derived now. So, dF d zeta could be written as maybe complex velocity in the zeta plane that is dF d zeta into d zeta dz. This is the complex velocity in the zeta plane.

So, what this equation tells us is that the complex velocities are proportional to each other. When you transform from one plane, let us say z plane to the zeta plane. So, they would be proportional with the constant of proportionality being the derivative d zeta dz. To show you how this works let us take an example.

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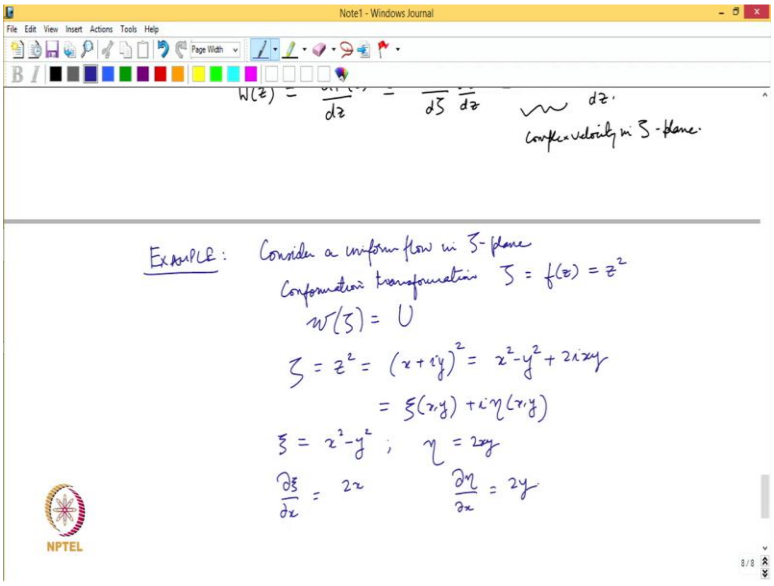

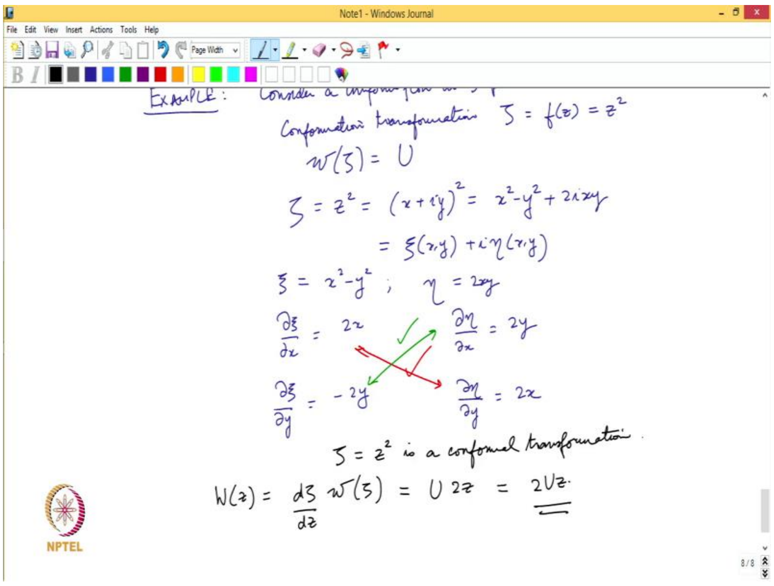

Let me take the example of the simplest possible velocity that we have encountered so far in this course. So, consider we have a uniform flow say in the zeta plane and we have a conformal transformation of the following type. So, the transformation is zeta is which is f of z is z square. So, you want to calculate now that if you have a uniform flow in zeta plane what would it look like in the z plane.

So, clearly, this w in the zeta plane would be just U, say U is the magnitude of the uniform flow. So, w zeta is U. Now, note that the transformation is zeta is z square, which could be written as x plus iy square, which would be x square minus y square plus 2y x y and which should be xi x comma y plus Iota eta x comma y. So, clearly xi is x square minus y square and eta is 2xy for this transformation.

Now, if I take the derivative just to check whether this transformation satisfies the Cauchy Riemann equations, you can say d xi dx is 2x, we can say d eta dx is 2y then d xi dy is minus 2y and d eta dy is 2x. But do you notice that d xi dx is the same as d eta dx that is satisfied that is done and then you can say that d xi dy is minus of d eta dx. So, the Cauchy Riemann equations for this problem are satisfied and so, zeta equal to z square is a conformal transformation.

So, it can be classified as a conformal transformation, because its components satisfy the Cauchy Riemann equations. Hence, we can now write that the complex velocity in the z plane would be d zeta dz into the complex velocity in the zeta plane and the complex velocity in zeta plane has given us U d zeta dz is 2z. So, wz is 2Uz. So, what we see in this example is that the complex velocity which was U in the zeta plane transforms to complex velocity in the z plane of the form 2Uz, not just U.

And so, this is basically one way. This example shows us how these velocities will be transformed, we will come up with a conformal transformation. So, I will stop this lecture here. In the next lecture, we will finish up some remaining attributes of conformal transformation and that would basically also complete our discussion for this course. So, thank you.