Computational Fluid Dynamics for Incompressible Flows
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Lecture 2
Finite difference formulations of Parabolic Equations: Implicit Methods

Hello, everyone so in last class we considered a parabolic equation and we have learned finite difference formulations to discitis this parabolic equations. And mostly we considered in last lecture this explicit method where only one unknown was there. So, in today's lecture, we will consider the same parabolic equation and we will discitis this equation using implicit methods. So, today is lecture two finite difference formulations of parabolic equations and we will learn implicit methods.

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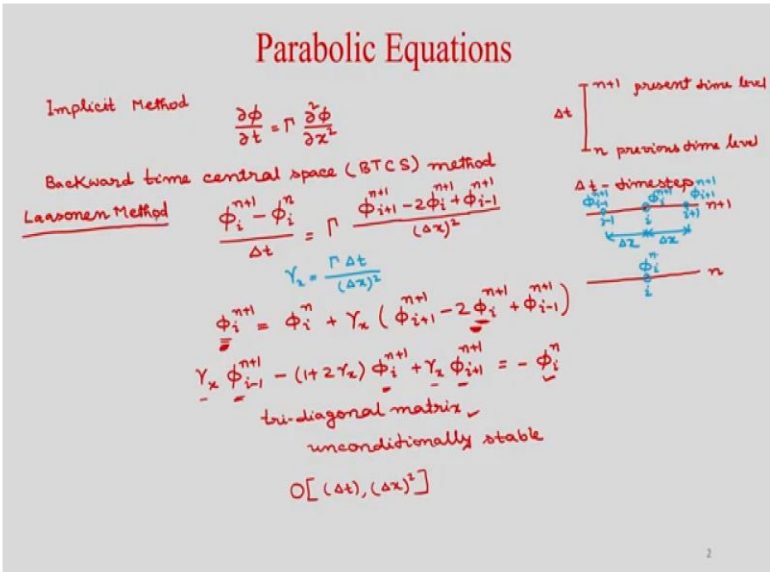

So, this model governing equation whatever we considered that is del phi by del t is equal to gamma del 2 phi by del x square. So, now we will use implicit methods so in implicit method you know that there will be more than one unknowns. Implicit method, the first method what we will learn that is known as Backward Time Central Space method.

So, that is known as BTCS, so Backward Time Central Space, so commonly known as BTCS method. So, that you can see that we will use that finite difference formulation for this first derivative as backward finite difference approximation. And the special second derivative what we have in right hand side that will use central difference method.

So, obviously the order of accuracy will be faster in time and second order in space, so you can see so which we will discitis this equation is so del phi by del t will take phi i n plus 1, so same thing i is the grid index. And the time level we are denoting with the superscript n or n plus 1 where you know that n is your previous time and n plus 1 is your present or current time.

And difference between this time is known as time step and that is your delta t. So, delta t is your time step, so this now we will use first order accurate scheme which is your backward finite difference formulation for this first derivative with respect to time. So, that we have discitis, now gamma and central difference will use in space.

So, it will be phi i plus 1 minus 2 phi i plus, phi i minus 1 divided by del x square. So, as it backward in time all these phi will take from the time level n plus 1 which is your present time. So, you can see that we have more than 1 unknowns because at i i plus 1 and i minus 1 at these discrete points we have unknown phi at time level n plus 1. So, if you see the grid points so you can, so this is your n th level grid and if it is n plus 1 level grid.

So, this is your i this is your i plus 1 and this is your i minus 1. So, this is your obviously you have a uniform grid size so that is your delta x and this is your also delta x. So, it is at level n plus 1 obviously then we will have phi i at n plus 1, phi i plus 1 at n plus 1 and phi i minus 1 at n plus 1. So, this the time level n plus 1 we are considering and n th level only one known values required. So, that is at point i and its it value at phi i in n time level.

So, all the unknown you take in left hand side and obviously there are 3 unknowns so you will get the algebraic equation like so if you define the diffusion factor which is known as gamma x as gamma delta t by delta x square. So, this is your known as diffusion factor coefficient, so gamma delta t by delta x square. And now, we can write it as, so we can see phi i n plus 1 is equal to phi i n plus gamma x into phi i plus 1 n plus 1 minus 2 phi i n plus 1 plus phi i minus 1 n plus 1.

So, all these n plus 1 time level you can take in left hand side so you can see phi i already in the left hand side it is there. So, in the left hand side of these algebraic equation you can see that phi i at time level n plus 1 is there, so this is your term and right hand side also you have phi i n plus 1. So, now you can rearrange it and you can write it as gamma x, so phi i minus 1 n plus 1 and if you now take phi i so this we are taking in the left hand side it will be. So, minus 1 plus 2 gamma x phi i at n plus 1. And you have another term plus gamma x phi i minus 1, sorry i plus 1 n plus 1.

So, you will have in the then right hand side as phi i will go this side so it will be minus phi i n. So, this is the final algebraic equation so at each discrete point you can write these algebraic equations. So, obviously if you form the matrix then you will get a tridiagonal matrix. So, for all the points interior points if you write this in a matrix format then you will get tridiagonal matrix.

Because you can see that you have phi i at n plus 1 phi i plus 1 n plus 1 and phi i minus 1 at n plus 1. So, you can see that in the left hand side there are 3 unknowns at n plus 1 time level. So, that is your present time level where we need to find the values at n plus 1 time level and 3 discrete points phi i phi i minus 1 and phi i plus 1.

So, here these 3 discrete points we need to find the value of phi. So, obviously you can see it will form tridiagonal matrix and right hand side these value is known from the previous time level n. And obviously you can see this is your coefficient of this phi so it is gamma x it is minus 1 plus 2 gamma x and this is plus gamma x. And if you can form a tridiagonal matrix considering this equal at all grid points then you will get a tridiagonal matrix and it is easy to solve using TDMA Tri Diagonal Matrix Algorithm which is famously known as Thomas algorithm.

So, if you see the stability analysis of this finite difference formulation then solution is unconditionally stable. So, this is the advantage of this implicit method, so it is unconditionally stable, unconditionally stable. So, that will show later and what is order of accuracy? Order of accuracy is your delta t faster at n time and delta x square second order in space.

So, this is simplest method where 3 unknowns are there and easy to solve this equation and it is sometime it is known as Laasonen method. So, sometime it is known as Laasonen method. And famously known as BTCS Backward Time Central Space method.

So, obviously you can see as it is implicit method there is no time restriction implicit method and it is unconditionally stable. So, obviously you do not have any time step restriction, so delta t you can choose a larger value but obviously in terms of the accuracy there is some limitation because if you choose higher delta t obviously truncation error also will increase.

Because delta t tends to 0 then only the truncation error will tend to 0. So, obviously you have some limitation in choosing the delta t but you can go larger time step if you use implicit method. But in explicit method we have seen that there is some restriction where gamma x should be less than equal to half for this finite difference formulation of this equation.

So, now we will modify this equation and we will get one finite difference formulation which is second order accurate in time and space and popularly this scheme is known as Crank-Nicolson scheme. It is also implicit scheme and it is known as Crank-Nicolson scheme.

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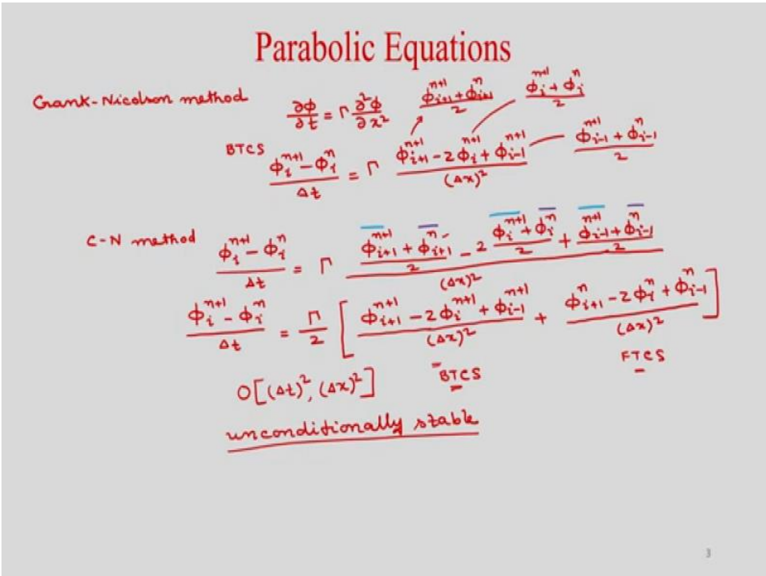

So, Crank-Nicolson method or scheme, so here whatever we have used in BTCS scheme let us write first BTCS we have used so what is the governing equation? It is del phi by del t gamma del 2 phi by del x square. So, in BTCS you have used phi i n plus 1 minus phi i n divided by delta t is equal to gamma into phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square. So, this is your just BTCS scheme whatever we have discussed just now and all these you are taking from n plus 1, n plus 1, n plus 1.

So, here what you will do, now instead of taking this diffusion term this dependent variables at n plus 1 time level will take the average of phi i n and phi i n plus 1 that means the values of phi at n th time level and the n plus 1 time level, so the average value you have to take. So, what we will take so these value we will take as phi i plus 1 plus phi i plus 1 at time level n plus 1 and n divided by 2.

So, average value you are going to take. Similarly for this point also you will take phi i plus phi i n plus 1 n divided by 2 and this also we will take phi i minus 1 plus phi i minus 1 at n plus 1 and n by 2. So, now we are modifying this BTCS scheme, in the diffusion term the value of phi which we took at n plus 1 time level now we are taking an average value of phi n plus 1 and phi n at each discrete points. So, that we are taking.

So, if you write this Crank-Nicolson method which is known also CN method, so if you write it so what you are going to get? Phi i n plus 1 minus phi i n divided by delta t is equal to so now you are replacing these as phi i plus 1 n plus 1 plus phi i plus 1 n divided by 2, divided by delta x square minus 2 into phi i n plus 1 plus phi i n divided by 2 and plus phi i minus 1 plus phi i minus 1 n plus 1 n divided by 2.

So, we have taken the average value, so now you rearrange it so if you rearrange it in this manner then you can see that you will get phi i n plus 1 minus phi i n divided by delta t is equal to. So, you can see that everywhere 2 is there so that you can take it outside gamma by 2 and you take all n plus 1 together so this term, this term, this term and n term you write separately, phi i n this and this. So, if you write it then you will get phi i plus 1 n plus 1 minus 2 phi i n plus 1 and now this term, so this is your phi i minus 1 n plus 1. So, now divided by delta x square.

So, all the n plus 1 time level value we have taken together and similarly you will take all n th time level these terms together. So, you will get phi i plus 1 n minus 2 phi i n plus phi i minus 1 n divided by delta x square. So, can you see or observe, can you observe something from this equation. You can see that this term is discretization like BTCS implicit method whatever we have used backward time central space method so similar to that so it is similar to BTCS.

And you can see these all we have taken from the previous time level n which is actually explicit method and we have used FTCS, Forward Time Central Space, so it is kind of Forward Time Central Space method. So, it is you can see Crank-Nicolson method when we are using so you can see when we are using the Crank-Nicolson method it is some average of BTCS and FTCS because right hand side if you see and observe you say see it is average 1 by 2 and it is BTCS type discretization and it is FTCS type discretization.

Why it is BTCS? Because all are at n plus 1 time level, you see this 3 discrete points i plus 1 i, i minus 1 or at the present time level n plus 1. So, it is kind of BTCS method and this one you can see all this discrete points i plus 1, i and i minus 1 the value of phi at n th time level and it is kind of explicit method whatever we learnt FTCS so similar to that. So, and 1 by 2 it is here so you can see it is average of BTCS and FTCS scheme.

The advantage of this discretization is that it is a second ordered time accurate. Anyway we are using central difference method in space so obviously it is delta x square but time level also it is delta t square. So, the accuracy is order of delta t square and delta x square, so second ordered accurate both in time and space and it is unconditionally stable. So, if you do the stability analysis of this finite difference formulation you will find that it is unconditionally stable.

So, you do not have any restriction to choose the time step delta t. So, it is unconditionally stable, so it is unconditionally stable. So, you can see that for this discretization method Crank-Nicolson it is second order time accurate and unconditionally stable. So, how you can show that is a second order time accurate.

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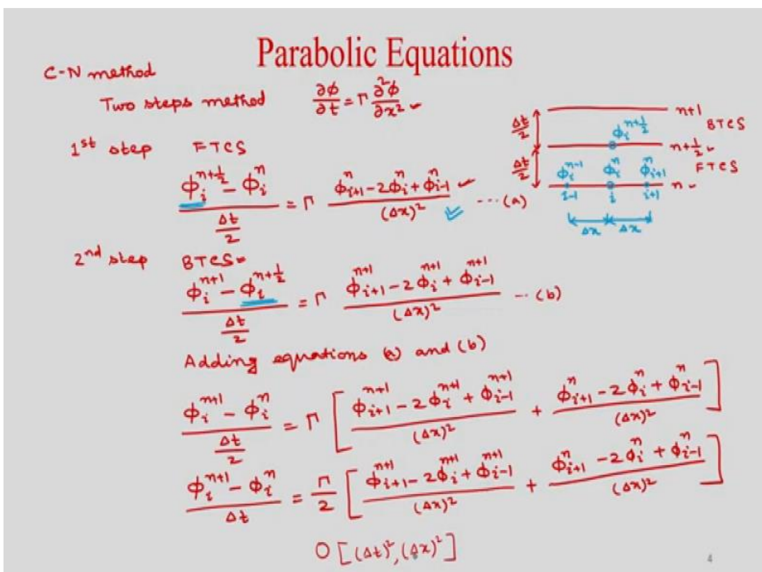

So, what method you can see now you can behave it as a 2 step methods, 2 steps method. We are taking about only Crank-Nicolson method, so now you can see as 2 step method let me write the equation first del phi by del t is equal to gamma del 2 phi by del x square. So, this is your n into n plus half and n plus half to n plus 1.

So, you are going in 2 steps from 0 to delta t, so this is your delta t by 2 this is the time step delta t by 2 and this is also when you are going from n plus half to n plus 1 this is also del t by 2. So, now you see that this Crank-Nicolson method you can think as a 2 step methods so where you will use first from time level n to n plus half as FTCS which is your explicit method and from n plus half to n plus 1 you will consider as BTCS which is your implicit method.

So, you can see here so you are moving from n to n plus half and the time step is delta t by 2 and you use FTCS method here. And when you are moving from n plus half to n plus 1 so again the time step is delta t by 2 and you can use BTCS, which is your implicit method.

So, if you use that way then you can discretize this model parabolic equation as, so first step so you are using FTCS scheme. So, how we can discretize so you are going from n to n plus half so it is n to n plus half divided by del t by 2, so which is your time step is equal to now right hand side you have gamma so this discretization is phi i plus 1 minus 2 phi i plus phi i minus 1 divided by del x square.

So, this we are using FTCS so all will be at n th time level. So, if you see the grip points so here you can see so at n to n plus half you are moving so now you use this as point i this is your i plus 1, this is your i minus 1 and constant step size delta x you are using. So, the values of phi i at n you are using phi i plus 1 at n you are using and phi i at n minus 1 you are using and from there you are calculating the value at phi i at n plus half.

So, that from this discretization you are finding. Then in the second step, so in the second step now use BTCS. So, now you are moving from n plus half to n plus 1 so obviously when you are using BTCS the special derivative all you will take from n plus 1 time level. So, it will be phi i n plus 1 minus phi i n plus half divided by del t by 2 is equal to gamma phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square and all you will take at n plus 1.

So, all you will take from n plus 1 time level so this is your BTCS method you are using. So, now if you, you can see that if you simply add these 2 equations so if you add this 2 equation what you will get so you see phi i n plus half, phi n plus half you will cancel out. So, if you say that this is your equation a and this is equation b then adding equation a and b what you will get you see? You will get phi so phi i n plus half so this 2 will get cancelled.

So, this phi i n plus half and this phi i n plus half we will get cancelled so you will get phi i n plus 1 minus phi i n divided by del t by 2 is equal to gamma, so you are adding it simply we add it so phi i plus 1 n plus 1 minus 2 phi i n plus 1 plus phi i minus 1 n plus 1 divided by delta x square. And phi i plus 1 minus 2 phi i plus phi i minus 1 so this you at so these you are taking from n plus 1 because this is your writing so delta x square.

So, now this half you can take it this side so it will be phi i n plus 1 minus phi i n divided by delta t so delta t you are taking here and half we are taking in the right hand side. So, you can write gamma by 2 and phi i plus 1 minus 2 phi i plus phi i minus 1 divided by del x square so all at n plus 1 time level plus phi i plus 1 minus 2 phi i plus phi i minus 1 n, n, n delta x square. So, you can see that you can think this Crank-Nicolson method as a 2 step computations, so one computation you are doing from n to n plus half as FTCS then n plus half to n plus 1 as BTCS scheme.

So, you can see that obviously it is unconditionally stable we have already discussed and the order of accuracy is delta t square and delta x square, so order of accuracy is delta t square and delta x square. So, we will so in today's lecture that this Crank-Nicolson method is second order accurate in time and space but before that let us discuss another method which is known as beta scheme or beta method.

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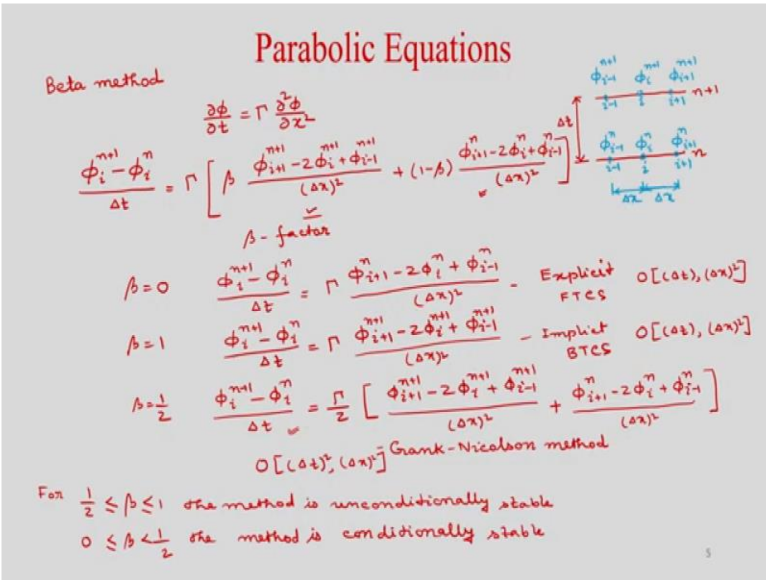

So, this is a general form of finite difference equation of model equation del phi by del t is equal to gamma del 2 phi by del x square so our governing equation is del phi by del t is equal to gamma del 2 phi by del x square. So obviously we have different time level one is n which is known as previous time level this is your n plus 1 it is your current time level and the time step is delta t, time step is delta t.

So, we will now use a general form to discretize this equation like this. So, we will discretize as phi i n plus 1 minus phi i n divided by delta t, so the i discrete point obviously if you can see that it will be your i, this is your i plus 1 and this is your i minus 1. So, the step size is constant this is your delta x.

Similarly you have here phi i plus 1 which time level n phi i n and phi i minus 1 n and similarly at n plus 1 time level also you will have the discrete points i plus 1 and i minus 1 so you have phi i n plus 1 phi i plus 1 n plus 1 and phi i minus 1 n plus 1. So, with this now if you discretize this equation using beta method we will write in this so diffusion coefficient gamma then we will take a factor beta then we will take the implicit kind of discretization of this special derivative phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square.

And we will take this discretization at n plus 1 time level n plus 1 and the remaining 1 minus beta we will take in n th time level so it will be phi i plus 1 minus 2 phi i plus phi i minus 1 delta x square and all these at n th time level. So, what we have done in beta method that we have taken a factor beta where beta times the del 2 phi by del x square discretization we have taken at the implicit manner.

So, all these phi we have taken as n th plus n plus 1 time level and the remaining 1 minus beta times the discretization of this special derivative del 2 phi by del x square and all these dependent variables phi at discrete points i i plus 1 and i minus 1 we have taken at the time level n. So, that is your previous time level and in the beta times we have taken the value of dependent variable at n plus 1 time level which is your current time level.

So, now you can see that is a general method we have written in a factor which is beta is a factor. So, now you can see if beta is equal to if you put 1 then what you will get? And if put beta is equal to 0 what you will get? And if beta is equal to half then what you will get? So, all these things already we have discussed.

So, let us say that if you have beta is equal to 0. So, if beta is equal to 0 then you can see the first term here so this first term will get 0 and you will get only the second term and you will get in this form phi i n plus 1 minus phi i n divided by delta t is equal to gamma phi i plus 1 n minus 2 phi i n plus phi i minus 1 n divided by delta x square. So, what type of discretization it is? So, obviously only 1 unknown is there so it is FTCS, Forward Time and Central Space.

So, explicit method and it is known as FTCS, so now if you put beta is equal to 1 special cases we are just discussing. So, beta is equal to 1 then obviously you will get phi i n plus 1 minus phi i n divided by delta t is equal to gamma phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square and if beta is equal to 1 then this term will get 0 so only this term will remain so you will get n plus 1.

So, what is this discretization? So, this discretization is obviously implicit method because more than 1 unknown is there and commonly this method is known as BTCS, Backward Time Central Space. So, it is implicit and it is known as BTCS. And if you put beta is equal to half then you can see.

So, if you beta is equal to half then you are going to get phi i n plus 1 minus phi i n divided by delta t is equal to, so beta is equal to half 1 minus half will be half so you take common gamma by 2, half we have taken common so you will get phi i plus 1 n plus 1 minus 2 phi i n plus 1 plus phi i minus 1 n plus 1 divided by delta x square.

So, now the second term you will get, so 1 minus beta half we have taken outside these brackets so i plus 1 n minus 2 phi i n plus phi i minus 1 n divided by delta x square. So, you know that it is Crank- Nicolson method so this is your Crank- Nicolson method. So, obviously you know that FTCS is conditionally stable and the condition is gamma x would be less than equal to half where BTCS is unconditionally stable and also the Crank-Nicolson method is unconditionally stable.

But FTCS and BTCS both are first order accurate in time and second order in space but Crank-Nicolson is second order accurate in both time and space. So, it is order of delta t and delta x square this is our order of del t and del x square but Crank- Nicolson method is order of del t square and delta x square.

So, you can see that it is a general method where we have used 1 beta factor and with change of beta you can get different scheme which may be explicit or implicit or first order accurate of second order accurate depending on the value of beta. So, in general we can write for half less than equal to beta less than equal to 1 the method is unconditionally stable, unconditionally stable.

But if beta varies between less than half and greater than equal to 0 then the method is conditionally stable. So, now will show that Crank- Nicolson method is second order accurate in time and space so how will sum? Because from whatever discretization we have written here form there you cannot easily tell that it is second order or accurate in time. Obviously space you can see that space you can say that it is second order accurate in space but from this discretization of this time derivative easily you cannot say that it is a second order accurate in time.

So, let us derive the truncation error and from truncation error we will be able to find that what the order of accuracy, so truncation error obviously you know the difference between the partial differential equation and the finite difference equation. So, now let us see it.

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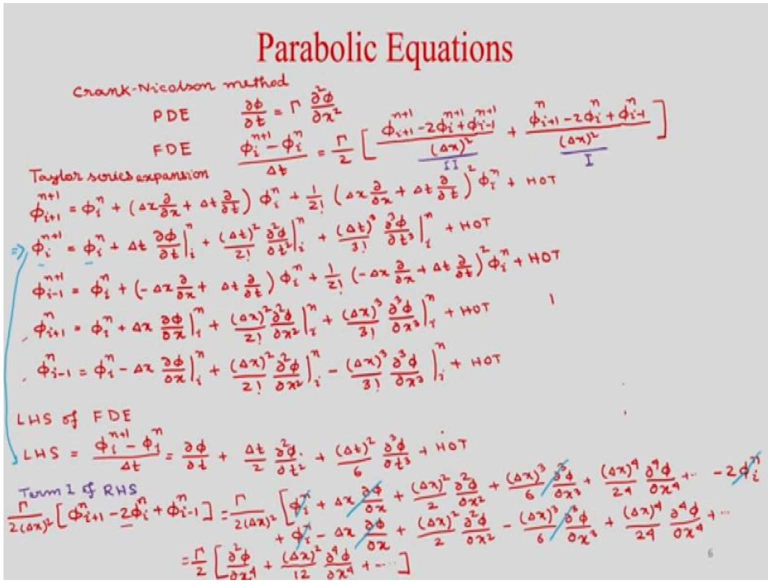

So, what is the discretization of Crank- Nicolson method we have done? So, first let us write that, so the equation is del phi by del t so this is a gamma del 2 phi by del x square. So, this is your partial differential equation and what is your finite difference equation so after discretization using Crank- Nicolson method whatever you get that will be finite difference equation so that will be phi i n plus 1 minus phi i n divided by delta t is equal to gamma by 2.

So, phi i plus 1 minus 2 phi i plus phi i minus 1 divided by delta x square and at time levels let us say n plus 1 and plus you have phi i plus 1 minus 2 phi i plus phi i minus 1 this is your n, n, n. So, this is your Crank- Nicolson method. Already we have discussed in detail so now we want to find what is the truncation error?

For that what we will do each dependent variable now we will expand using Taylor series. Then we will see the difference between these partial differential equation and the finite difference equation and then we will find the truncation error and that will tell us that what is the accuracy of this scheme.

So, now you see if you say phi i plus 1 n plus 1. So, if you use Taylor series expansions, of each dependent variable at particular discrete point and particular at time level. So, you can see this is having i plus 1 and n plus 1 and that means you have x plus delta x and t plus delta t, so it is function of x and t both.

So, x plus delta x and t plus delta t so now if you expand so what you will get? So you will get phi i n plus you can write del x del of del x plus del t del of del t phi obviously at level i and n then you can write 1 by factorial 2. So, then it will be del x del of del x plus del t del of del t square then phi i n and other higher order terms.

So, we are not writing other terms so similarly you do for other so now you write phi i n plus 1 so only it is t plus delta t and Xo it will be simple phi i n plus delta t dell phi by del t at time level discrete point i and n then plus del t square by factorial 2 del 2 phi by del t square. Then you will get del t cube by factorial 3 del cube phi by del t cube i n and other high order terms.

So, that we are not going to write and similarly you write phi i minus 1 and n plus 1. Similarly you write phi i minus 1 n plus 1, so you will get phi i n plus. So, it is for phi i minus 1 so it is x minus delta x and n plus 1 is t plus delta t, so about x and t you are expanding. So, obviously you will get minus delta x because you have delta minus delta x so del, del x plus del t del of del t it is t plus delta t only so phi i n plus 1 by factorial 2 so it will be minus delta x del of del x plus del t del of del t square phi i n plus high order term.

So, now the next will expand phi i plus 1 n. So, whatever are there this dependent variables so phi i plus 1 n. So, here x plus delta x and t so x plus delta x only so it will be phi i n so delta x del phi by del x i n plus del x square by factorial 2 del 2 phi by del x square. Then plus del x cube by factorial 3 del cube phi by del x cube plus high order term.

Similarly, now we will write phi i minus 1 and n. So, now it is minus delta x, x minus delta x so it will be phi i n minus delta x del phi by del x plus delta x square by factorial 2 del 2 phi by del x square i n minus del x cube by factorial 3 del cube phi by del x cube i n plus high order term.

So, now we have expanded all these phi i plus 1, n plus 1 and whatever dependent variables are coming so we have expanded using Taylor series expansion, now let us put it in the finite difference equation. So first, let us consider only the left hand side, if you consider the left hand side then what is there in the left hand side of the finite difference equation.

So, what you will get? Left hand side is phi i n plus 1 minus phi i n divided by delta t. So, what you can write from the second so from the second directly you can write you can see from here. So, from here you can directly write phi i n plus 1 minus phi i n divided by delta t so it will be del phi by del t, we are not going to write i and n because all that derivatives will be at the point discrete point i and the time level n.

So, plus so delta t we have divided so it will be delta t by 2 delta t by 2 del 2 phi by del t square. Then plus you can see from there so delta t you have divided so it will be del t square divided by 6 factorial 3 means 6 so del cube phi by del t cube plus high order term. So in left hand side we have already seen that only these terms will be there and the right hand side you see so let us say this is your let us say this term this term is 1 and this term is 2 .

So, now you see in the right hand side let us say the term 1, term 1 of right hand side. So, what you are going to get? So, it is gamma by 2 phi i plus 1 and you can write delta x square also, so you can write gamma by 2 delta x square and you have phi i plus 1 minus 2 phi i plus phi i minus 1 in whose time level in n.

So, term 1 let us see what will happen so you take phi i plus 1 n so this one then 2 phi i anywhere it is there and phi i minus 1 n, so these two last two this expansion you take and put it here so what you are going to get here? So, you are going to get here gamma by 2 delta x square so phi i plus 1 n so already you have expanded using Taylor series so you put it there.

So, it will be phi i n plus delta x del phi by del x plus delta x square by 2 del 2 phi by del x square plus del x cube by 6. Factorial 3 is 6, del cube phi by del x cube, and other terms high order terms we are not writing then minus, so minus now 2 phi i so minus 2 phi i so minus 2 phi i n then you write phi i minus 1 n the expansion whatever you have written.

So, it will be plus phi i n minus delta x del phi by del x, all these derivative about the point i and n so about discrete point i and the time level n. So, del x square by factorial 2 is 2 del 2 phi by del x square then it will be minus del x cube by factorial 3. So, it will be 6 so it will be del cube phi by del x cube, del x cube and another term you take here so it will be easy so it will be minus del x 4 so what you will get next term? So, factorial 4, so factorial 4, so factorial 4 means 24, so it is 24 del 4 phi by del x 4 plus high order term.

And here also you write another term, so write del x 4 by 24 del 4 phi by del x 4 the other term and minus 2 phi i n. So, now let us see so you can see that this is your phi i n this is your phi i n 2 phi n and it is minus 2 phi n so it is cancelled out. These will cancelled out these also will cancelled out plus and minus.

So, whatever terms you have let us write down here so you will get so delta x square so delta x square you just divide with each term so you are going to get gamma by 2. So, gamma by 2 so you can see here 2 del 2 phi by del x square is there so 2 and divided by 2 is there so it will become and delta x square will be cancelled out so you will get del 2 phi by del x square.

So, next you can see this term so you have 2 into del x to the power 4 by 24 del 4 phi by del x to the power 4. So, if you 2 into this you write then you will get and delta x square will be if you divide then it will be delta x square so it will be delta x square divided by 2, 2 was there so it will be 12, del 4 phi by del x 4. And plus other high order terms so that we are not writing.

So, now in the first term in the right hand side we have done similarly let us do the second term in the right hand side. So, let us do the second term, so in the second term you can see this is phi n plus 1 all n plus 1.

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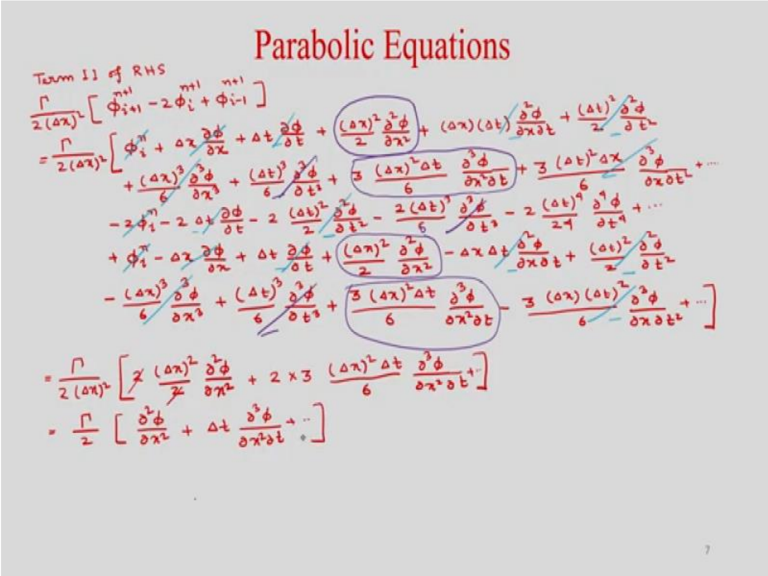

So, you can write term 2 of right hand side so it will be gamma by 2 and delta x square if you take outside. Then it will be delta x square and all at n plus 1 level, phi i plus 1 minus 2 phi i plus phi i minus 1, n plus 1, n plus 1, n plus 1. So, now whatever in earlier slide we have shown that this and this and this so these expansion you put it there.

So, if you put it there what you going to get? So, you are going to get equal to so it will be gamma by 2 delta x square I am just directly putting all these expansion in last slide whatever we have shown. So, you will get phi i n plus del x del phi by del x about point i and time level n I am not going to write this plus del t del phi by del t plus del x square by factorial 2 is 2.

Then del 2 phi by del x square then you write del x into del t. So, it will be del 2 phi by del x del t plus del t square by factorial 2 means 2 it will be del 2 phi by del t square then another cube is there. So, that if you write it here you will get del x cube by factorial 3, so factorial 3 is 6 you can write directly.

Del cube phi by del x cube then plus del t cube by factorial 3 means 6 del cube phi by del t cube then you will get 3 into del x square delta t divided by factorial 3 so it will be 6 it will be 6. Then del cube phi by del x square del t plus 3 into del t square del x by factorial 3 means 6 you write and del cube phi by del x del t square and other high order terms. So, now these we have written the expansion for phi i plus 1 n plus 1.

Now, the term is minus 2 phi i n plus 1 so that we are going to write the expansion so it will be multiplied by 2, because minus 2 phi i n is there. So, whatever you got so minus 2 phi i n minus 2 delta t del phi by del t minus 2 delta t square by factorial 2 is 2 del 2 phi by del t square minus 2 into del t cube by factorial 3 means 6. So, it will be del cube phi by del t cube and minus 2 into del t 4 by factorial 4 so 24 del 4 phi by del t to the power 4. So, and other high order term.

So, this is the expansion for phi i n plus 1. Now, let us write the expansion of phi i minus 1 at time level n plus 1. So, now let us write the expansion of phi i minus 1 n plus 1. So, you can write it as plus phi i n minus delta x del phi by del x plus del t del phi by del t plus del x square by factorial 2 means 2 del 2 phi by del x square.

Then you will get minus del x del t del 2 phi by del x del t plus del t square by factorial 2 is 2 del 2 phi by del t square minus del x cube by factorial 3 is 6 del cube phi by del x cube plus del t cube by 6 factorial 3 6. So, del cube phi by del t cube plus 3 into del x square del t divided by factorial 3 is 6 del cube phi by del x square del t minus 3 into del x del t square divided by 6 del cube phi by del x del t square plus high order term.

So, now let us see that which are terms are cancelled out so one is you can see this phi i n this phi i n and this minus 2 phi i n, so these are cancelled out so del x by del phi by del x this is your plus and this is your minus. Then you can see this is del x del t and this is so this term and this term so this you can cancel then del t by del phi by del t and so this is one term, this is one term, this is one term, this is plus 2 and this is your minus. So, this you can cancel.

Then you can see what are the other terms will be cancelled you just check. So, you can see del t square by 2 del 2 phi by del t square so these, this term, this term and you can see this term. So, now these you can cancel out, because these is minus 2 then you have del x cube by 6 so you can see these and these will cancel out. Then you will get 3 delta x delta x delta t square so this term and this term will get cancel.

Then what are the remaining terms let us see, so you will get the remaining terms is this one, one is this, then you have this term then you have this term you have delta x square del 2 phi by del x square so this term. And you can see this term also you will get cancelled out so you can see this is your, this is your 6 actually. So, you can see this term, this term so it is 2 into this term and it is minus 2 into this term so this will get canceled out.

So, now you can see these are the 4 terms remaining from this expansion and that you can write now is equal to gamma by 2 delta x square, first let us write. Then it will be easy, so you can see this delta x square by 2 delta x square by 2 so there are 2 so 2 times delta x square by 2 del 2 phi by del x square. So, this is also 2 times so you can write 2 into 3 so 2 into 3 delta x square delta t by 6 del cube phi by del x square del t.

So, n plus higher order term we are neglecting. So, you can see you will get gamma by 2 so it will be just, so 2 2 will get cancelled out and delta x square so it is del 2 phi by del x square. So, this three 6 6 will get cancelled out so you will get delta t only and del cube phi by del x square and delta t. So, now we have found the left hand side term and term 1 and term 2 in the right hand side, so if you put all together then what you are going to get.

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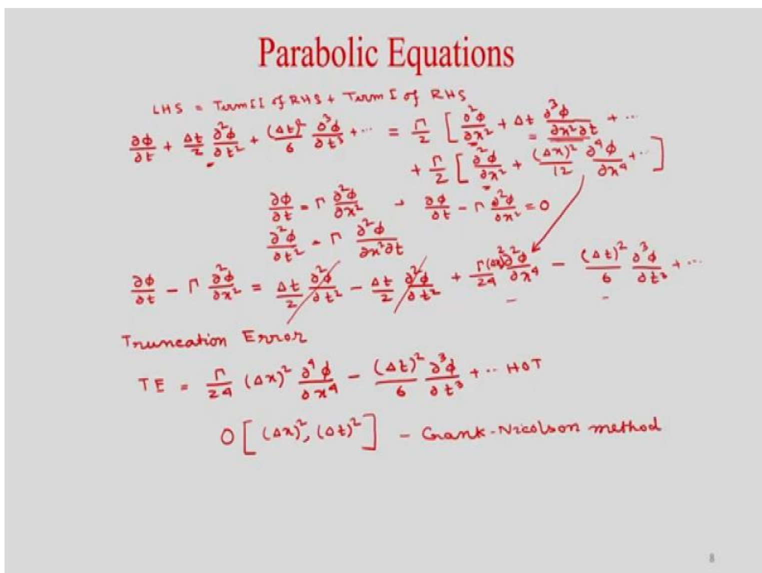

So, in the left hand side whatever you got so you will get so we are now writing left hand side is equal to term 2 of right hand side plus term 1 of right hand side so if you write it then what you are going to get? So, left hand side you can see it is there is del phi by del t plus del t by 2 del 2 phi by del t square plus del t square by 6 del cube phi by del t cube.

So, you can see this is the term, we have written, now other 2 terms, term 1 and term 2 whatever we have derived so let us write it equal to so it will be gamma by 2, gamma by 2. So, what is there in the second term gamma by 2 so it is del 2 phi by del x square del 2 phi by del x square plus del t del cube phi by del x square del t.

This is the term 2 and what is term 1? It is also gamma by 2 so gamma by 2 so term 2 is you see term 2 whatever we have written, so this is the term 2. So, this if you write it here so you are going to get del 2 phi by del x square plus delta x square by 12 del 4 phi by del x to the power 4 plus high order term.

So, now we have written this whatever from the expansion we have got so in the finite difference equation we put and we have written now we will find the truncation error so you can see that our governing equation is del phi by del t is equal to gamma del 2 phi by del x square. Let us take the time derivative of this equation, so del 2 phi by del t square you can write as gamma del 2 phi del x square del t. So, you can see that whatever term is there here that we can replace with del 2 phi by del t square.

So, you replace it there, so what you are going to get? So, del phi by del t del t by 2 all you take in the right hand side, so now you can see this is your gamma by 2 del 2 phi by del x square and it is gamma by 2 del 2 phi by del x square. So, it will be just gamma del 2 phi by del x square, so that you can take in the left hand side. So, it will be minus so this is the so you can see that gamma del 2 phi by del x square t is del 2 phi by del t square so you can write it as del t del 2 phi by del t square.

So, the first term this term we have written here and so is your del t by 2 because gamma by 2 is there, so now you take it in the right hand side so it is v minus del t by 2 del 2 phi by del t square. So, you can see this will get cancel out and this if you take and what is left here you have gamma by 2 so it will be plus gamma by 24 del 2 phi by del x to the power 4. This is there and if you take this term in the minus del t square by 6 del cube phi by del t cube, some other terms.

So, this get cancelled out so you can see that your governing equation is this one, so that means del phi by del t is equal to gamma, minus del 2 phi by del x square, so this is equal to 0 but from the finite difference equation we got this additional term so this is your truncation error. So, now what is your truncation error? So, truncation error now so truncation error TE is gamma by 24 so this delta x square will be there.

Here we miss delta x square, so this term is coming here so del x to the power del x square so it will be del x square del 4 phi by del x to the power 4 minus del t square by 6 del cube phi by del t cube plus other high order terms. So, now you can see the truncation error so the leading term what is there delta x square and delta t square. So, the leading terms, in the truncation error we have delta x square and delta t square that means the order of accuracy of this Crank-Nicolson scheme is delta x square and delta t square so we have proved it now.

So, the accuracy is order of delta x square and delta t square. So, it is just truncation error we have found which is the difference between the partial differential equation and the finite difference equation. So, first we discitis that equation after that we expand it using Taylor series each dependent variables then we put it in the finite difference equation then we the difference we have found as truncation error and the leading terms you can see that it is delta x square and delta t square.

So, obviously the you can see that Crank-Nicolson method is your, Crank-Nicolson method is Crank-Nicolson method is second order or accurate in both time and space. Thank you.