Computational Fluid Dynamics for Incompressible Flows
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Lecture 2
Solution of Navier-Stokes Equations using FDM (Continued)

Hello, everyone. In last lecture in detail, we discussed why we should use staggered grid over collocated grid and we have already discussed that in MAC algorithm we use primitive variable approach and staggered grid. So, today, using this MAC algorithm we will discretize the unsteady two-dimensional Navier-Stokes equations and we will write the discretized equation. Finally, we will discuss about the algorithm to solve this governing equations.

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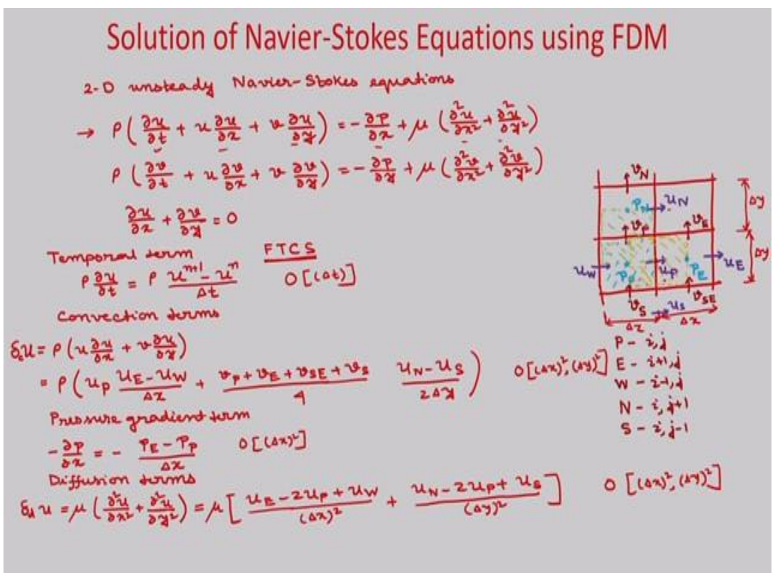

So, let us write the governing equations. So 2-D unsteady, Navier-Stokes equations. So, we are going to write for incompressible flow, so density is constant. So, we can write rho del u by del t plus u del u by del x.

So, we are writing in non-conservative form, v del u by del y is equal to minus del p by del x plus mu del 2 u by del x square plus del 2 u by del y square. So, this is your x momentum equation and this is your temporal term. These two terms are convective terms, this is your pressure gradient term and this is your diffusion terms and mu is your dynamic viscosity, rho is your density of the fluid.

Similarly, you can write the y-momentum equation, rho del v by del t plus u del v by del x plus v del v by del y is equal to minus del p by del y plus mu del 2 v by del x square plus del 2 v by del y square and we have also the continuity equation, del u by del x plus del v by del y is equal to 0. So, you can see that with the known fluid, where density and viscosity are known then you have three unknowns u, v and p and we have three equations.

Now we will use staggered grid. So, in staggered grid we will solve p at the cell center. So, this is your cell for p, this is your main cell and we will solve the velocities in staggered manner. So, you will solve in this phase. So, it is face center, so it is uP and we will solve the v at this face center. So, this is your vP.

So, obviously you can see that you will have here vS and here you will have uE, uW. So, here you will have uW and similarly, here you will have uE and here you will have vN. So, as I discussed in last class that P denotes i, j, indices i, j. E denotes i plus 1, j.

Whereas, w denotes i minus 1, j. North N denotes i, j plus 1 and south S i, j minus 1. So, for each variable you can see that for pressure P velocity u and velocity v. So, we use this indices i, j, for P, i plus 1, j for E, i minus 1, j for W, i, j plus 1 for n, and i, j minus 1 for S.

So, now discretize these equations. First let us discretize the temporal term. So, your temporal term, first we will derive for the x momentum equation then rho del u by del t. So, if you use forward time and central space, because we discussed that in MAC algorithm we will use, forward time and central space. So, rho del u by del t, you can write rho u n plus 1 minus u n divided by delta t.

So, it is forward time and convection terms. So, that is your rho u del u by del x plus v del u by del y. So, we will denote it as delta c u. So convection terms, delta c u. So that you can write, so you can see, so first term. So, u del u by del x, u del u by del x. So, for the, which counter volume you are, which. So, u cell is this one, so this is your u cell. So now we are discretizing this u-momentum equations.

So, obviously you can see that you will have here pE and here you will have pN and v cell is this one. This one is your v cell. So, now we can write u del u by del x for this u control volume. So, for this u control volume what you can write, you can write uP and del u by del x, so del u by del x we need to use central difference. So, you can write uE minus uW divided by delta x. So, all are having this delta x uniform grid if you have.

Then you can write delta x and this is your delta y and for the second one, now v, v at this point in the center for this uP cell, v is not available. So, v you have to interpolate. So, from where you will interpolate? So you have, you can see here you have v. So, this is your vP, so this is your vE, this is your SE, southeast, v southeast. So, v you interpolate at this center for the up cell as vP plus vE plus vS plus vSE divided by 4.

So, now we are interpolating v at the P as vP plus vE plus vSE plus vS divided by 4. So, this is just interpolation, because we are finding the value of v at the center of this up cell, at the up cell here. So, obviously from this 4, v cell we are just interpolating this value at center for uP cell and now you have del u by del y. So, now del u by del y you can see, easily you can find. So, this is your u north and so you can write. So, del u by del y, so del u by del y. So, north so you can write uN minus uS divided by 2 delta y.

So, uS will be somewhere here, another cell if you take. So, it will be uS somewhere here it will be there. So, you can write uN minus uS divided by 2 delta y. So, obviously this is second order accurate. So you can see the convection term we have discretized for the X momentum equation using central difference method. So, the order of accuracy of this discretization is delta x square and delta y square. So, order of accuracy is delta x square and delta y square and this temporal term order of accuracy, order of delta t.

Now let us discretize the pressure gradient term, so pressure gradient term. So, now let us write the pressure gradient term minus del P by del x. So, this minus del P by del x we need to find for the uP cell. Because we are solving the x momentum equation and for that cell we need to know the pressure gradient minus del P by del x.

So, if you see this cell uP cell. So, this is your uP cell, so this uP cell if you see. So, pressure gradient in the x directions, you can easily write P minus pP divided by delta x.

So, this you can write as minus pE minus pP divided by delta x. So, this also central difference you can see. So, order of delta x square and now we have diffusion term.

So, diffusion terms we will denote at delta d u. So, we have mu del 2 u by del x square plus del 2 u by del y square. So, that you can write as, now you see the uP cell. So, in uP cell now you see del 2 u by del x square. So, if you use the second order central difference discretization then easily you can write at uE minus twice uP plus uW divided by del x square.

So, we will write as uE minus twice uP plus uW divided by delta x square. So, central difference we are using for the second derivative and del 2 u by del y square. So, del 2 u by del y square you can see uN minus twice uP plus uS divided by delta y square. So, you can write as uN minus twice uP plus uS divided by delta y square. So, you can see the order of accuracy is delta x square delta y square. So, let us now put altogether in a discretized form for the x momentum equation.

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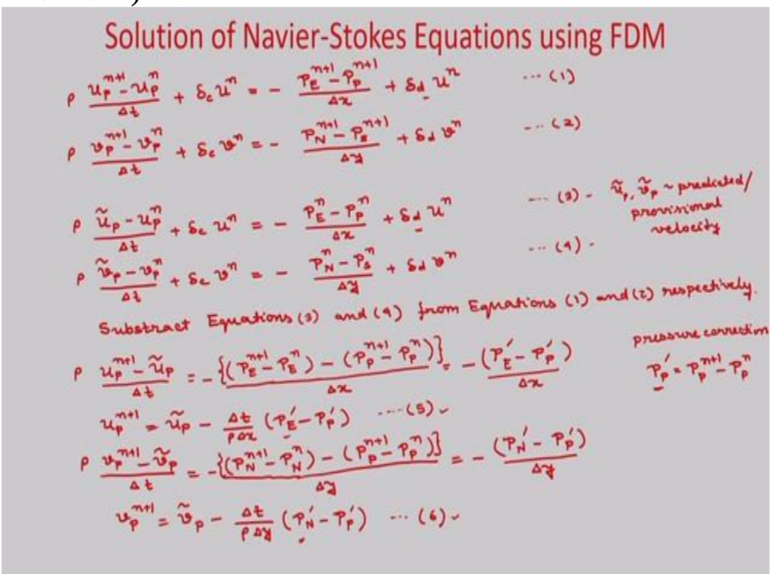

So, we can write as rho uP n plus 1 minus uP n divided by delta t, convection term whatever way we have discretized that in short form, we are writing delta c u n is equal to minus pE n plus 1 minus pP n plus 1 divided by delta x plus delta d u n and similarly you can write for v velocity, vP n plus 1. So, similarly you can derive all these terms and write it for the v momentum equation plus delta c vn is equal to minus pN n plus 1 minus pS n plus 1 divided by delta y plus delta d vn.

So, let us write the equation number. So, this equation number 1 and this equation number 2. So, these are the discretized form of the x and y momentum equations and we have used forward time and central space discretization. So, overall order of accuracy is delta t delta x square delta y square.

Now if you see this governing equation. In the left-hand side, in the temporal term, you have 1 unknown uP n plus 1 and all other velocities are known. But if you the pressure term. So, pressure term you have to take at n plus 1, but at n plus 1 pressure is unknown. So, we have to now simplify it in some way.

So, what we will do now, we will calculate some provisional velocity u tilde assuming the pressure from the previous time level n. n plus 1 is the current time level and n is the previous time level. So, we will find some provisional velocity or predicted velocity using the pressure value at previous time level n.

So, if you write the equation, so we will write rho. So, now as we do not know the pressure at n plus 1, we will use pressure at n and we will get some provisional velocity u tilde minus uP n divided by delta t plus delta c u n is equal to minus. So, pressure now we are taking from the nth level that is why we are writing this tilde velocity, which is your provisional, a predicted velocity from the previous time level pressure.

So, plus delta the u n and for v velocity you can write vP tilde minus vP n divided by delta t plus delta c vn is equal to minus pN at time level n minus pS at time level n minus pS at time level n divided by delta y plus delta d vn. So, these equations number let us give as 3 and 4.

So, you can see that equations 1 and 2 we need to solve. But there are too many unknowns, because you have un plus 1, vn plus 1 we need to solve for and pressure is also unknown at n plus 1. So, what we did, we assumed the pressure from the previous time level and we are solving these equations. So, obviously we will get some provisional velocity and that we are denoting as u tilde and v tilde. So, now you subtract equation 3 and 4 from equation 1 and 2.

So, now subtract equations 3 and 4 from equations 1 and 2, respectively. So what you will get, you see? So obviously, from the first term you will get uP n plus 1 minus uP tilde divided by delta t. The conviction terms are same. So, this will cancel out then we will have only pressure term in the right-hand side.

So, you can see that we will have pE n plus 1 minus pE n and minus pP n plus 1 minus pP n divided by delta x and you see the diffusion term. Diffusion terms are same, so this will cancel out. So, we will get this equation and that we can write the difference between p and plus 1 minus pn as pP prime.

So, we can write minus pE prime minus pP prime divided by delta x. So, you can write now uP n plus 1 you can write as uP n plus 1 is equal to. So, uP tilde minus pE delta t by rho delta x pE prime minus pP prime and similarly if you subtract equation 4 from equation 2, you are going to get rho vP n plus 1 minus v tilde P divided by delta t is equal to minus pN n plus 1 minus pP n minus pP n plus 1 minus pP n divided by delta y and this you can write in terms of pn prime, so this will be n.

So, you will get pN prime minus pP prime divided by delta y and you can write vP n plus 1 is equal to vP tilde minus delta t by rho delta y pN prime minus pP prime. Now let us write the equation number. So, this equation uP n plus 1, let us write equation 5 and this vP n plus 1 as equation 6.

So, after subtracting, we got some relation between the provisional velocity and the corrected velocity, uP n plus 1 in terms of pressure correction. So, now pP prime we have written as pP prime that we have written as pP n plus 1 minus pP n. So, the difference of pressure at n plus 1 and n we are telling that this is known as pressure correction, pressure correction. So, pP prime. So, in terms of pP prime we have written.

So, now you can see that equation 5 and 6, uP tilde and vP tilde you can solve from equation 3 and 4. So, you can get, once you know the uP tilde and vP tilde then if you know the pressure correction, then you will be able to get the uP and vP at n plus 1 time level.

So, now we have to find what is the pP prime. Because that is the pressure correction is unknown. So, now as pressure correction an unknown and there is no separate pressure equation, we need to derive it from the continuity equation. Now we will satisfy the continuity equation in the main cell of pressure P and we will derive the equation for pressure correction.

So, here let us write that u tilde P and v, tilde P are predicted or provisional velocity. So, now we will start from the continuity equation. So, for two-dimensional incompressible flow, continuity equation is del u by del x plus del v by del y is equal to 0 and this will satisfy at the main cell of pressure P.

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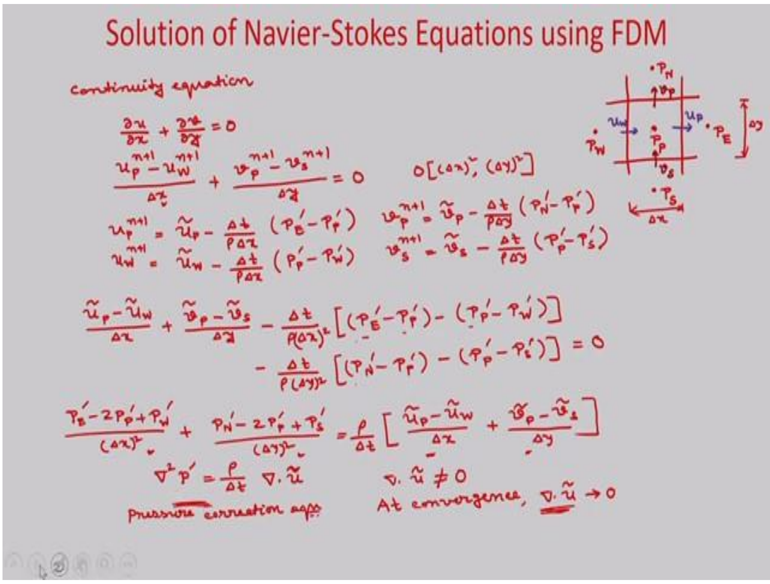

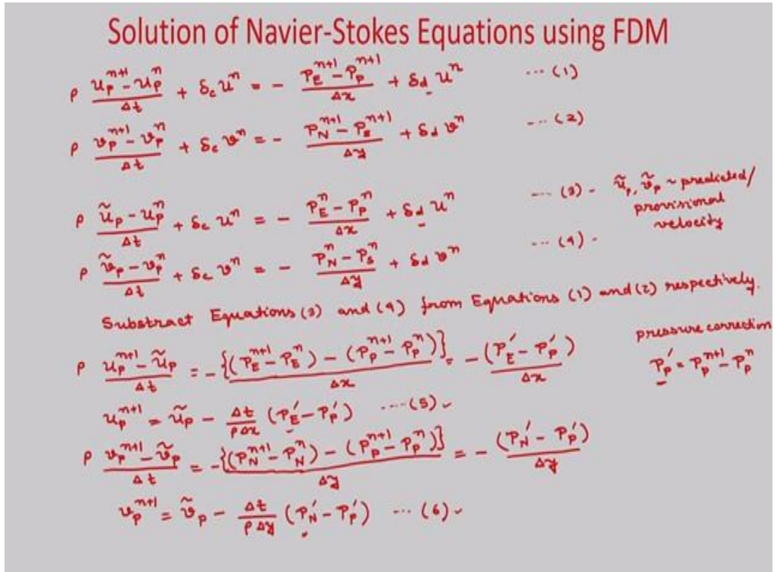

So, if you see the main cell, so this is your main cell where this is the cell centre where pP is there and this your p east, this is your p west, this is your p north and this is your p south and velocities. So, u velocity you have here, so this is your uP, this is your uW and you have v velocity, vP and this is your v south. So, now let us write the continuity equation del u by del x plus del v by del y is equal to 0. So, now you discretize uP n plus 1 minus uW n plus 1 divided by delta x. So, del u by del x, del u by del x we are finding at the cell center of pressure P.

So, obviously about this point if you write, then obviously you can write uP n plus 1 minus uW n plus 1 divided by the distance delta x. So, this is your delta x and this is your delta y. Now del v by del y is similarly about this cell center if you find the derivative del v by del y, it will be vP n plus 1 minus vS n plus 1 divided by delta y is equal to 0. So, it will be vP n plus 1 minus vS n plus 1 divided by delta y is equal to 0 and what is the order of accuracy of this discretization? Obviously delta x square delta y square, because we use central difference.

Now the relation, what we derived for u and v equation 5 and 6 we will substitute there, because uP n plus 1, this is the relation; vP n plus, 1 this is the relation. So uP n plus 1, what is the relation? uP tilde minus delta t by rho delta x pE prime minus pP prime. Similarly, you can write uW n plus 1. So, for the velocity uW, similarly you can write uW tilde minus delta t by rho delta x and what is the pressure gradient for this term? It will be pP minus pW. So pP prime minus pW prime.

Similarly, for v you can write vP n plus 1, already we have derived it, vP tilde minus delta t by rho delta y and pressure gradient is pN prime minus pP prime and similarly, you can write for vS n plus 1 as vS tilde minus delta t by rho delta y plus. So, the pressure difference for this cell will be pP prime minus pS prime.

So, now these you substituted in the discretized continuity equation. So, if you substitute it and rearrange it, you are going to get uP tilde minus uW tilde divided by delta x after rearranging, you will get vP tilde minus vS tilde divided by delta y minus delta t by rho delta x is there and another delta x, we are dividing.

So, it will be delta x square and it will be pE prime minus pP prime minus pP prime minus pW prime and you will get again minus delta t by rho delta y square and you will get pN prime minus pP prime minus pP prime minus pS prime equal to 0. So, after substituting and rearranging you will get this equation.

Now further let us rearrange it. So you can write it as, so the pressure term, pressure term let us write in the left-hand side. So, if you write this, you see pE minus pP prime and minus pP prime. So, it will be minus 2 pP prime and this would become plus pW prime. So, if you write this, so it will be pE prime minus twice pP prime plus pW prime divided by delta x square plus from here you will get pN prime minus twice pP prime minus pS prime divided by delta y square and other terms you take in the right-hand side.

So, you will get rho by delta x, so rho by delta t. So, you will get rho by delta t uP tilde minus uW tilde by delta x plus vP tilde minus vS tilde divided by delta y. So, can you recognize left-hand side and right-hand side? So left-hand side if you see, it is the discretization of the grad square p prime.

Because grad square p prime is del 2 p prime by del x square plus del 2 p prime by del y square. Now del 2 p prime by del x square if you use central difference, you are going to get this, you will get this discretization p prime minus 2 pP prime plus pW prime by delta x square. Similarly, del 2 t by del y square, if you discretize it, you will get pN prime minus twice pP prime plus pS prime divided by delta y square.

So, this is nothing but grad square p prime and right-hand side you see, this is nothing but the divergence of u tilde. So, that means whatever provisional velocity you are getting, for that if you satisfy the continuity equation then you will get rho by delta t divergence of u tilde.

So, initially provisional velocity, it will not satisfy divergence of u will not be 0. But at convergence, divergence of u prime will tend to 0. So, this will satisfy the continuity equation u tilde. So, this is the equation for the pressure. So, it is known as pressure Poisson equation. So, this is known as pressure correction equation.

So, this is known as pressure correction equation and this is your pressure Poisson equation and you can see, so you can see that this equation is difficult to solve because you have a pentagonal metrics. So MAC algorithm suggest that, you assume the corrections of neighboring terms as 0. So, in the MAC algorithm suggest that the pressure correction in the neighboring cells at 0. So, this is the assumption.

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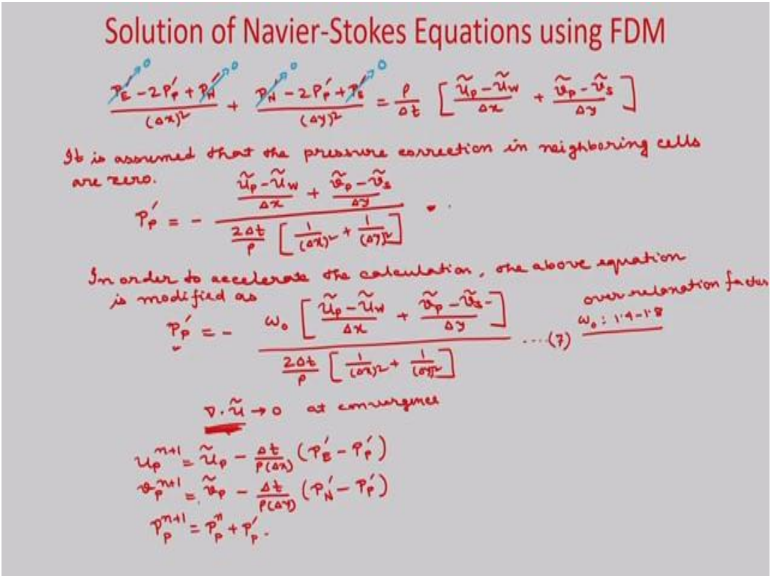

So if you do that, then you can rewrite this equation. So, now in MAC algorithm simplified it as, so if you are dropping the, or you can write this. So you have pE prime minus twice pP prime plus pW prime by delta x square plus pN prime minus twice pP prime minus pS prime divided by delta y square is equal to rho by delta t uP tilde minus uW tilde divided by delta x plus vP tilde minus vS tilde divided by delta y. It is assumed that the pressure correction in neighboring cell are 0.

So, you can see, so if you assume this. So, this will be 0, this will be 0, this will be 0 and this will be 0. So, essentially you will get pP prime is equal to minus uP tilde minus uW tilde divided by delta x plus vP tilde minus vS tilde divided by delta y and you will have two delta t by rho, 1 by delta x square plus 1 by delta y square.

So, simplifying this pressure correction equation, what was pressure Poisson equation. Now you have written a formula to find the pressure correction p prime. Because only 1 is unknown p prime. So, pP prime, so this is a formula. So, pP prime you are finding from the known delta t density, then delta x delta y and the divergence of u tilde and you can see that at convergence, divergence of u tilde will be 10 to 0 that means that time pressure corrections will be 0 and if pressure correction is 0. So, p n plus 1 will be pN.

So, now this equation, you can actually use some over relaxation factor for further accelerating the solution. So, in order to accelerate the calculation, the above equation is modified as, so we will use some over relaxation factor minus w naught uP tilde minus uW tilde divided by delta x plus vP tilde minus vS tilde divided by delta y divided by 2 delta t by rho 1 by delta x square plus 1 by delta y square.

So, omega naught you need to find by (())(38:22) method, but mostly you can use it between 1.4 to 1.8. You can use around 1.4 or 1.5 to 1.8. So, this is your over relaxation factor. So, this equation now let us give as number, so earlier we used 1, 2, 3, 4, 5, 6. So, this you can write as 7. So, you see in this equation, in this equation, we are telling that if divergence of u tilde will be 10 to 0 at convergence. So, that means this criteria you can use as the convergence criteria for the inner loop, for the inner loop you can use this as convergence criteria.

So, now you can see that once you solve the u tilde v tilde then that you can use to find the pP prime, because u tilde, if you v tilde you know then you can find the pP prime and once the p prime is known, now you can update the velocities as uP n plus 1 is equal to uP tilde minus delta t by rho delta x pE prime minus pP prime and vP n plus 1 as vP n, vP tilde minus delta t by rho delta y, pN prime minus pP prime.

So, this way you can solve to find the updated velocities u n plus 1 and v n plus 1 and if you want to find the pressure. So, pressure also you can find, you see, p n plus 1 is equal to p n plus p prime. So, this is the equation, p n plus 1, so p if you write cell p so it will be pP n plus 1 is equal to pP n plus pP prime. So, at convergence, pP prime will be 0. So, pP n plus 1 is equal to pP n. So, obviously whatever u tilde you will get that will be u n plus 1. So, now let us write the algorithm. So, whatever we discussed and discretized these equations, how we will solve using MAC algorithm.

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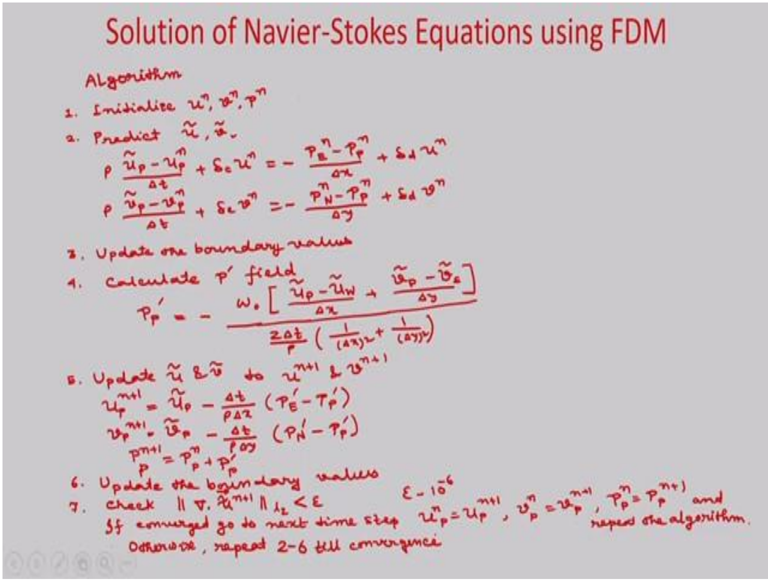

So, the algorithm will be like this. First, you initialize, un, vn, pn and also obviously you need to generate the grid and you have to also initialize this value at t is equal to 0. So, time t is equal to 0, you need to initialize the values. Then you predict u tilde and v tilde. So, what are the governing equations you have already derived?

So rho uP tilde minus uP n divided by delta t plus delta c un is equal to minus pE n minus pP n divided by delta x plus delta d un and for v, vP tilde minus vP n divided by delta t plus delta c vn is equal to minus pN n minus pP n divided by delta y plus delta d vn. So, if you solve these equations, you are going to get u tilde and v tilde which are the provisional velocities.

Once you know the provisional velocities, now update the boundary conditions. If it is Dirichlet type boundary conditions it will not matter. But if we have some Neumann type boundary condition it is better to update the boundary values. Then you solve for the pressure correction.

So, that is the formula, so calculate so you need to solve, you need to calculate only. Because calculate p prime field. From where? So, you have the equation, pP prime is equal to minus omega naught is your over relaxation factor, uP tilde minus uW tilde divided by delta x plus vP tilde minus vS tilde divided by delta y divided by 2 delta t by rho 1 by delta x square plus 1 by delta y square.

So, now once you know the pP prime. Now you can update the velocities. So, in next step, you update u tilde and v tilde to u n plus 1 and v n plus 1. So, you know uP n plus 1 is uP tilde minus delta t by rho delta x pE prime minus pP prime and uP n plus 1 is equal to, vP n plus 1 is equal to vP tilde minus delta t by rho delta y pN prime minus pP prime.

So, you know p prime, so and also u tilde already have solved. So, you update the velocities at n plus 1 and also, if you want to find p n plus 1 that also you can do Pn plus p prime. But although this your, so pP n plus so this is p prime. So, P n plus 1 p is equal to pP n plus p prime. So, this once you know, you again update the boundary values, update the boundary values.

Now, check for the convergence, for the inner loop. So, check divergence of u tilde and this is a vector n plus 1 l2 less than epsilon, where epsilon is a pS and some convergence criteria maybe 10 to minus 6 or minus 4. So, that you check. So if converged, if converged, go to next time step and assign unP is equal to uPn plus 1.

So, this is now as initial guess whatever you have used at t is equal to 0. Now for the next time step this is the value you will use, vP n plus 1 and pP n as pP n plus 1. So, if converged, go to the next time step and repeat, next time step and repeat the, so if converged go to next time step and you put it and repeat the algorithm. If not, so otherwise repeat 2 to 6. So, you do not update the values, you repeat 2 to 6 for convergence or till convergence.

So, you can see, so this is the algorithm you have to follow to solve these equations. So, first you solve the provisional velocities, once provisional velocities you know, you can solve for the pressure correction equation.

Once you know the pressure correction equation, you can solve for the updated, you can update the velocities u n plus 1 and v n plus 1 and if check for the convergence, if it is converged then you go to the next time step and update uN as u n plus 1, vN as v n plus 1 and pN as p n plus 1 and repeat the algorithm and if not, then you have to satisfy divergence of u tilde. So, that you have to repeat 2 to 6.

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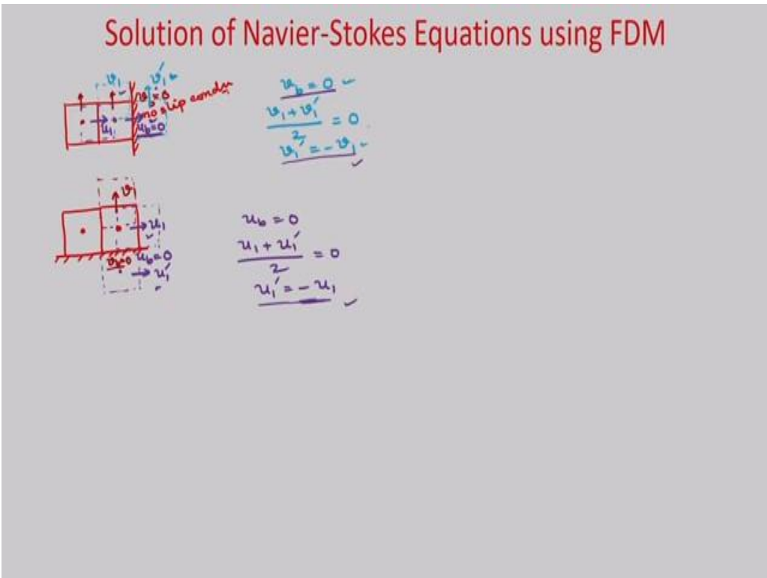

Now quickly we will discuss about the boundary conditions, because you have seen that for this MAC algorithm, you have no-slip boundary conditions let us say on this right wall. So, no-slip condition, no-slip condition you have, now for u velocity if you see, u velocity if you have two cells here, where do you solve for u velocity?

So these are pressure for pressure you have these values and you have v velocity is here. So, if v velocity is here, now how we will find or apply the boundary conditions? So obviously vP at boundary, vb as 0. So now you take a fictitious cell here, so you use some fictitious cell and this is your fictitious cell value.

Let us say this is v prime 1 and let us say this is your v1. Then you see so if vb is 0, so if vb is 0 then you can write v1 plus v1 prime divided by 2 is equal to 0. So, that means v1 prime will be minus v1. So, when you are solving for this v velocity cell, you need the value from the neighbor and this neighbor value is v1 prime and v1 prime is your some ghost cell value.

So, v1 prime you can assign from the boundary condition, if it is no-slip condition as v1 prime as minus v1 and because at the boundary vb is 0. So, when you are solving for v1 you will get the v1 prime.

Similarly, if you have a bottom boundary, so now you have the velocities, u velocities.. So, this is your p and you have u velocity here. So, you have u velocity here, now you have ub is 0 that is your boundary condition and you need to now, let us say this is your u1, you use some fictitious cell or ghost cell where you have u1 prime.

So this is fictitious. You do not need to generate the mix for it, so just you need the array for this. So for this now to satisfy the no-slip condition at the wall, ub is equal to 0. So, obviously u1 plus u1 prime divided by 2 will be 0 and u1 prime will be minus u1.

So, when you are solving for this cell, for u velocity cell, you need the neighbor values of u, u1 prime and that u1 prime you can assign as minus u1. For v velocity you do not need. Because v velocity exactly satisfies this. Because v velocity you are solving here. So, this is your v1 and this is your vb is equal to 0. So, obviously, v1 when you will solve for this v1 cell then obviously you can use vb is equal to 0.

So, similarly, when you are solving for u. So, when you are solving for this v. So, you see u where you are solving, here you are solving u here and u here. So, when you are solving for u here. Let us say this is your u1. So, if you are solving u1 and this is your ub is equal to 0 because no-slip condition.

Then when you are solving for this cell of u1, you need the value of ub only. So, ub directly you will get, but when you are solving for b, you need the value from v1 prime and that is unknown. So, that we are using, satisfying the no-slip condition at the boundary so vb is equal to 0 and v1 prime is equal to minus v1.

Similarly, when you are solving for top and bottom boundary. So, this will be when you are solving for this velocity, you need to know the value at u1 prime which is your fictitious point and that you can assign u1 prime is equal to minus e1. So accordingly, you can derive for other boundary conditions to solve this discretized Navier-Stokes equations.

So in today’s lecture, so we started with the Navier-Stokes equations. Then we use forward time and Sintel space method to discretize the u and v momentum equations. So, as pressure at n plus 1 is unknown. So, we solve for the provisional or predicted velocity u tilde and v tilde taking the pressure at previous time level n.

So, when we subtracted this equation u n plus 1 and u tilde, we got the values u n plus 1 in terms of u tilde and the pressure correction where p n plus 1 is equal to pn plus p prime and p prime known as pressure correction. So, to update this values, u tilde is known but pressure correction is unknown. So, we used continuity equation to get the pressure correction equation. So, we got a pressure Poisson equation.

So in MAC algorithm, it simplifies as assuming the neighbor pressure correction velocity as 0. So, we got a formula to find the p prime in terms of the divergence of u tilde and we have the density rho delta t and delta x and delta y. Now at convergence/divergence of u tilde will be 0.

Because it will satisfy the continuity equation. So, now once you find the pressure correction that you can use to find the corrected velocities u n plus 1 and v n plus 1. So, and also we discuss the solution algorithm and at last we discussed about the boundary conditions for this staggered grid, thank you.