Fundamentals of Convective Heat Transfer
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Module - 04
Convective Heat Transfer in External Flows - II
Lecture – 10
Momentum integral equation for flat plate boundary layer

Hello everyone, till now we have solved the boundary layer equation using similarity method, where you convert the partial differential equation to ordinary differential equation and you can easily solve ordinary differential equations with given boundary conditions. Today we learn one new method, which is an approximate method known as integral method.

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There are many situations where it is desirable to obtain the approximate analytical solutions. When can we have this approximate analytical solution? When exact solution is not available or cannot be easily obtained and when solutions are too complex implicit or require numerical integration. So, the advantage of these approximate solutions is; the integral method is simple and it can deal with complicated factors.

The integral method is used extensively in fluid flow heat transfer and mass transfer. The mathematical simplifications are there in approximate solutions because number of independent variables are reduced.

When you consider 2 dimensional situation, you can see that using integral method you can convert the partial differential equation to ordinary differential equation. So, you can see that there is reduction in order of differential equation.

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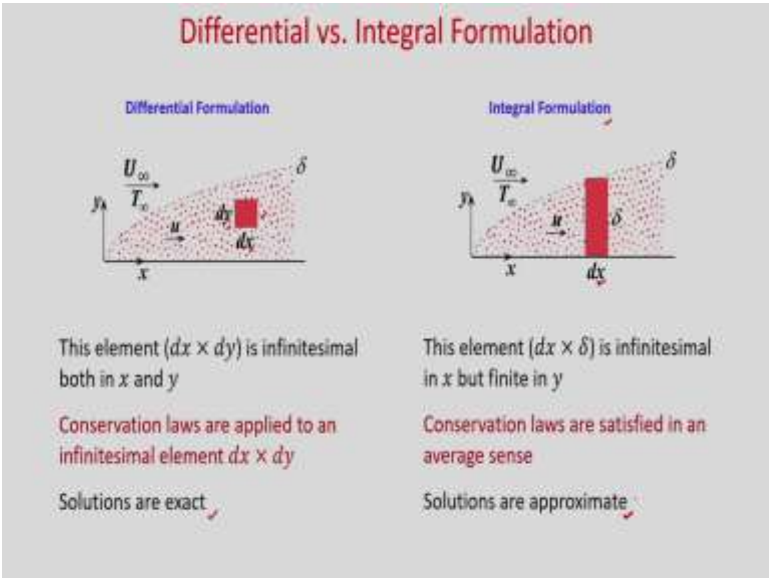

So in differential formulation we know that we consider one infinitesimal element of length dx and dy and the conservations laws are applied to this infinitesimal element. So, the solutions are exact whereas, in integral formulation the element is infinitesimal in x, but finite in y.

In this particular case when you consider boundary layer equations you can see for a flow over flat plate δ is the boundary layer thickness and your infinitesimal element is dx. So, the element is (dx X δ). So; obviously, we apply the conservation laws in an average sense and hence, solutions are approximate in integral formulation.

Now, let us discuss what is the procedure we will follow when we use this approximate method or integral solution.

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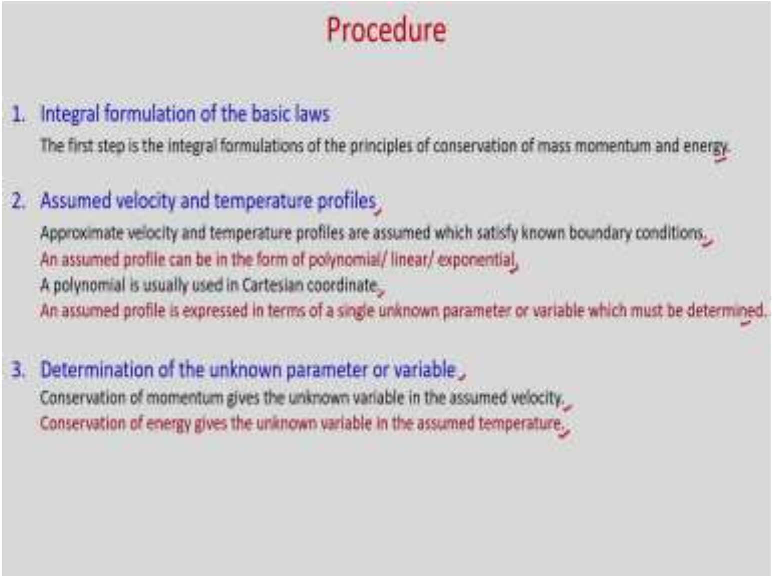

First, we have this integral formulation of the basic laws. The first step is the integral formulation of the principles of conservation of mass, momentum and energy. So, first you see, what are the governing equations for that fluid flow and heat transfer phenomena, then you write it in integral form.

Next, you assume the velocity and temperature profiles. So, approximate velocity and temperature profiles are assumed, which satisfy known boundary conditions. An assumed profile can be in the form of exponential. A polynomial is usually used in Cartesian coordinate. An assumed profile is expressed in terms of a single unknown parameter or variable which must be determined.

So, in this boundary layer equation for flow over flat plate, we will see that this unknown variable is your boundary layer thickness δ. And finally, you determine the unknown parameter or variable.

So, conservation of momentum gives the unknown variable in the assumed profile and conservation of energy gives the unknown variable in the assumed temperature. So, if you follow these 3 steps, then you can use this integral approach to solve the boundary layer equations and we can find the velocity and temperature profile, as well as we can find the heat transfer coefficient and the Nusselt number.

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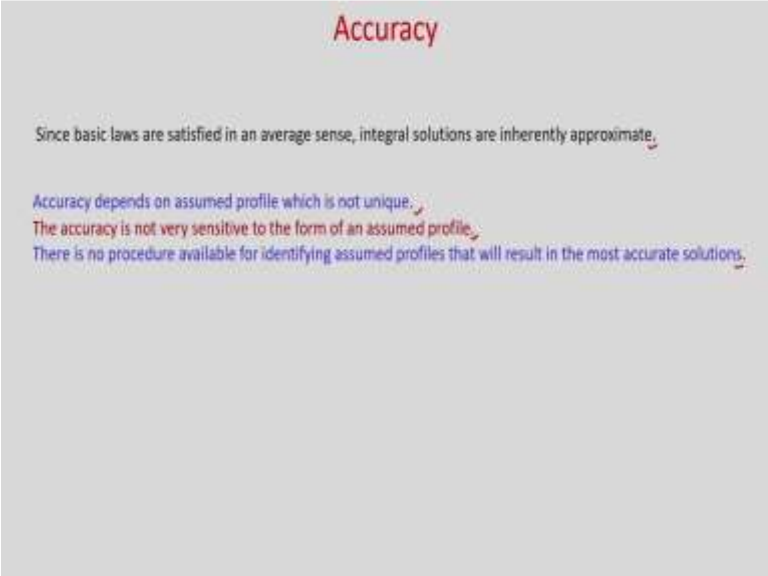

So, what is the accuracy of this integral approach? Since basic laws are satisfied in an average sense, integral solutions are inherently approximate. So, accuracy depends on assumed profile which is not unique, because you can use third order polynomial or second order polynomial. So accordingly, you will get the velocity profile or the temperature profile.

The accuracy is not very sensitive to the form of an assumed profile. So you can see there will be little variation when you use different degree of polynomial in the final solution of temperature profile or velocity profile or the boundary layer thickness. There is no procedure available for identifying assumed profiles that will result in the most accurate solutions. We do not know the optimum temperature or velocity profile for which you will get the results which is closer to the exact solutions.

Today, we will consider only fluid flow, because you know that in convective heat transfer we need to solve the fluid flow equation as well as the energy equation. In today’s class, we will solve the fluid flow equation using the integral method and we will find what is the boundary layer thickness, then we will find what is the shear stress acting on the flat plate and then we will find the coefficient of friction.

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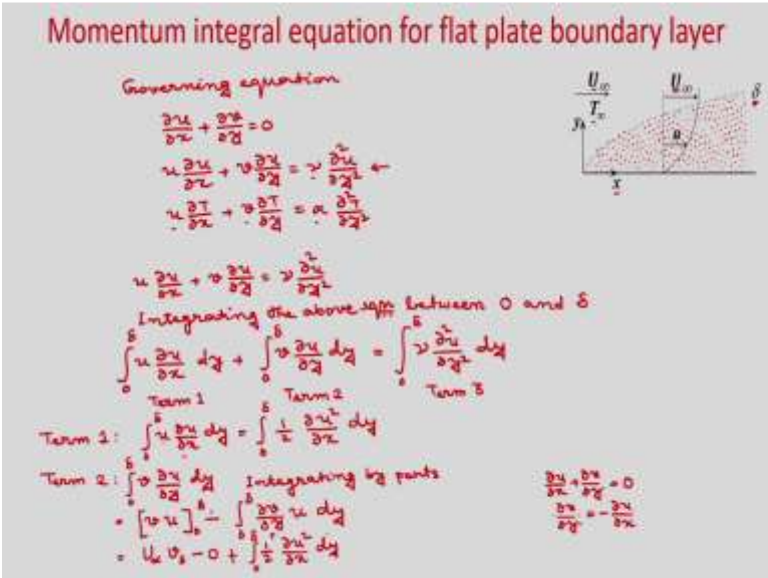

So now, let us consider flow over a heated flat plate. First, we will solve the fluid flow equation, then we will solve the energy equation for two different boundary conditions; with constant wall temperature and with constant heat flux.

So you can see, this is your heated flat plate; this is the x direction; perpendicular to the plate is y direction; your free stream velocity U_∞; and free stream temperature T_∞; and this is your age of the boundary layer. And this is the boundary layer thickness δ; obviously, it is hydrodynamic boundary layer thickness and the velocity profile will vary like this where U is function of x and y and here U_∞ is the free stream velocity.

So, let us write the governing equations. First is continuity equation; that is your $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. So; obviously, you are considering a 2 dimensional case, then you have boundary layer equation for flat plate, you know, $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$.

And; obviously, energy equation you can write as $u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$. So; obviously, you can see ν is your kinematic viscosity and α is your thermal diffusivity.

So now you can see, once to solve the energy equation we need to know the velocity profile, because in the energy equation you have the velocities. So first, let us solve this equation for the case flow over flat plate using this integral approach.

So, already we discussed the first approach is that you have to integrate the governing equation. So, first we are considering the momentum equation, which is your $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$ right.

So now, let us integrate this equation in the boundary layer because 0 to δ. So, as you are using integral approach; obviously, in y direction we are using δ and in x direction it is infinitesimal distance dx. So, if you integrate it; integrating the above equation between 0 and δ, where δ is your hydrodynamic boundary layer thickness.

So you can see, you can write $\int_0^\delta u\frac{\partial u}{\partial x}dy+\int_0^\delta v\frac{\partial u}{\partial y}dy=\int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy$. Now, we will consider this each term separately and we will integrate step by step. So, first let us give the term as 1, this is the first term; the second term is this one in the left hand side and in the right hand side let us name as term 3.

So, term 1; let us write as $\int_0^\delta u\frac{\partial u}{\partial x}dy$. We will take this u inside this derivatives. So, if you put it inside the derivative then you can write $\int_0^\delta \frac{1}{2}\frac{\partial u^2}{\partial x}dy$. So, you can see; so, if you take the derivative; obviously, you will get 2 u and you will get back this term. Now term 2; so, this is the term 2. So, this is your $\int_0^\delta v\frac{\partial u}{\partial y}dy$.

So now, we will use integration by parts. And, you because there are 2 variables, v and $\frac{\partial u}{\partial y}$. So use integration by parts. So, if you write it, so integrating by parts what you will get? So, $[vu]_0^\delta-\int_0^\delta \frac{\partial v}{\partial y}udy$.

So now, let us see the continuity equation. So, what is your continuity equation? Continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. So, this $\frac{\partial v}{\partial y}$ term, you can write as $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$.

So you see, this we put the limit. So if you put y = δ, so at y =δ, what is the velocity u? u will be U_∞, because that is the age of the boundary. So you will have U_∞. So, you can write $U_\infty v_\delta$, this is unknown and if you put the lower limit so velocity is at 0. So, it will be 0 only.

So, if you put $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}$ and if you multiply with u and this u if you take inside this derivative then you can write $\int_0^\delta \frac{1}{2}\frac{\partial u^2}{\partial x}dy$.

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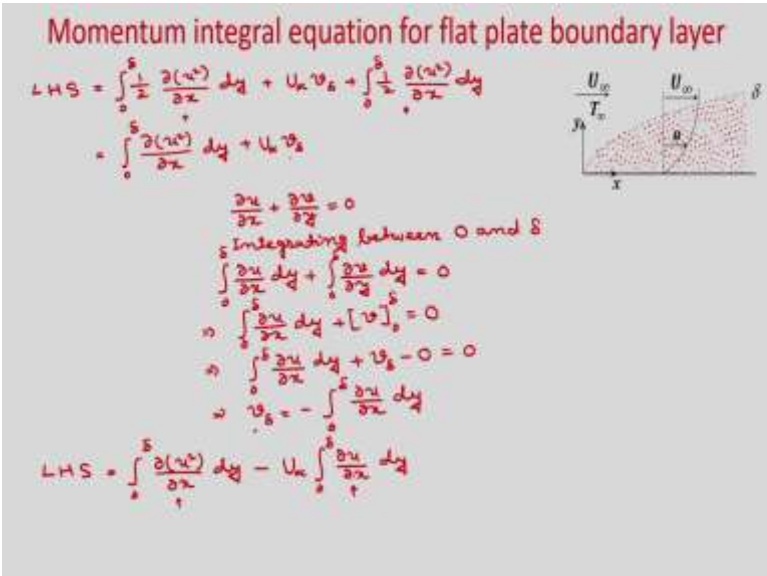

So, the left hand side, let us write all the terms. So, in the left hand side we have all the terms as $\int_0^\delta \frac{1}{2}\frac{\partial(u^2)}{\partial x}dy + U_\infty v_\delta + \int_0^\delta \frac{1}{2}\frac{\partial(u^2)}{\partial x}dy$.

Now, in this equation you can see v_δ is unknown right. So, what will do now, we will again use the continuity equation and we will integrate between 0 and δ. So, these two terms if you put together, it is 1/2 and 1/2, so it will be 1. So it will be just $\int_0^{\delta} \frac{\partial(u^2)}{\partial x} dy + U_\infty v_\delta$. And, this we need to determine, v_δ.

So, let us consider the continuity equation. So continuity equation is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. And, integrating between 0 and δ; within the boundary layer thickness.

So, we can see it will be $\int_0^{\delta} \frac{\partial u}{\partial x} dy + \int_0^{\delta} \frac{\partial u}{\partial y} dy = 0$. So now you can see this term will remain as it is, $\int_0^{\delta} \frac{\partial u}{\partial x} dy$ at this term. So, it will be integral 0 to δ dv. So it will be just $[v]_0^{\delta} = 0$.

So you can see, you can write as $\int_0^{\delta} \frac{\partial u}{\partial x} dy + v_\delta - 0 = 0$. So you can find the $v_\delta = -\int_0^{\delta} \frac{\partial u}{\partial x} dy$.

So this v_δ now you put here. So now, left hand side you can see; you can write as $\int_0^{\delta} \frac{\partial(u^2)}{\partial x} dy - U_\infty \int_0^{\delta} \frac{\partial u}{\partial x} dy$.

So you can see, in the first term, so in the first term you have, $\frac{\partial(u^2)}{\partial x} dy$. So, this now; we will use the Leibniz theorem. So that we can integrate this term which is having derivative and you can take this derivative outside the integral.

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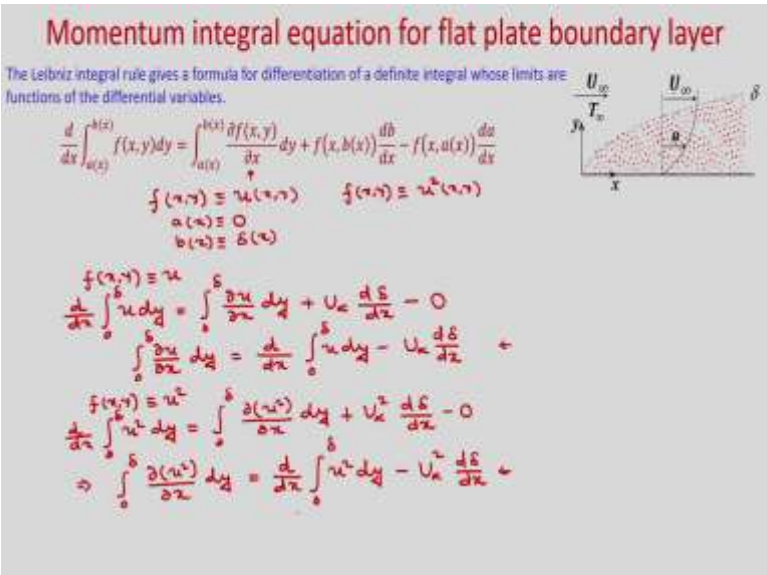

So, you can see the Leibniz integral rule gives a formula for differentiation of a definite integral whose limits are functions of the differential variable. So, if you have, $\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)dy = \int_{a(x)}^{b(x)} \frac{\partial f(x,y)}{\partial x} + f(x,b(x))\frac{db}{dx} - f(x,a(x))\frac{da}{dx}$. So, a and b are the limits and function of x.

So now, you see in our left hand side the first term, so u^2 you can take as f right. So you can see, that this f (x,y) so this you can take as u and this limits; obviously, you can see a x. So this is your lower limit, so it is 0.

And, the upper limit in our case, it is δ and which is function of x, and; obviously, u is function of x and y. So now, you can see that in the left hand side we have this term. So, this we want to write this derivative with respect to a, we can take outside the integral using this Leibniz theorem.

So you can see now, so you can see we have this term as well as this term; $\frac{\partial u}{\partial x}$. So, f will be one time u and another time will be u^2 and we will take the derivative outside the integral and, another time f (x,y) will take u^2.

So now you can see, if you put this f (x,y) as u then you can write $\frac{d}{dx}\int_0^\delta u dy = \int_0^\delta \frac{\partial u}{\partial x} dy$. now, f at x = b. So now, u at y = δ. So, you have to see that f at y = b, so in this case u at y = δ.

So, what is that? That is your free stream velocity u. So, you can write plus U_∞ and what is b? b is your δ. So, $\frac{d\delta}{dx}$ and minus, so this is your at y = 0. So that is your 0 right. So a is also 0. So this term will become 0.

So, you can see, you can write $\int_0^\delta \frac{\partial u}{\partial x} dy = \frac{d}{dx}\int_0^\delta u dy - U_\infty \frac{d\delta}{dx}$. And, if you use f (x,y) = u^2, then what you can? Write $\frac{d}{dx}\int_0^\delta u^2 dy = \int_0^\delta \frac{\partial (u^2)}{\partial x} dy + U_\infty^2 \frac{d\delta}{dx} - 0$.

So, you can see, you can write $\int_0^{\delta} \frac{\partial(u^2)}{\partial x} dy = \frac{d}{dx}\int_0^{\delta} u^2 dy - U_\infty^2 \frac{d\delta}{dx}$. So you notice these two equations. So now, inside the integral we had partial derivative, but using this Leibniz integral rule you can see you have used ordinary derivative $\frac{d}{dx}$ and $\frac{d\delta}{dx}$, similarly for this equations.

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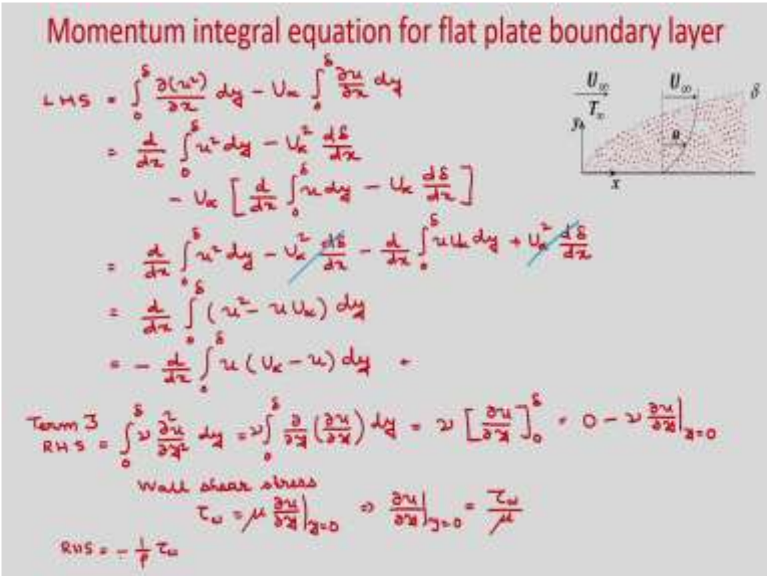

Now, you put this expression in the left hand side terms. So, we can write finally, so left hand side we had $\int_0^{\delta} \frac{\partial(u^2)}{\partial x} dy - U_\infty \int_0^{\delta} \frac{\partial u}{\partial x} dy$. So now, this using Leibniz rule whatever you have got, so that you just write it.

So, this we have written, $\frac{d}{dx}\int_0^{\delta} u^2 dy - U_\infty^2 \frac{d\delta}{dx} - U_\infty[\frac{d}{dx}\int_0^{\delta} u dy - U_\infty \frac{d\delta}{dx}]$.

So, from Leibniz integral rule whatever we got, the partial differentiation inside the integral that we have written in terms of the ordinary derivative outside the integral; so that we have just put the terms in this expression. So, now, if you rearrange you can see; so, it will be, $\frac{d}{dx}\int_0^{\delta} u^2 dy - U_\infty^2 \frac{d\delta}{dx} - \frac{d}{dx}\int_0^{\delta} u U_\infty dy + U_\infty^2 \frac{d\delta}{dx}$

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So, you cancel these two terms. So, you will get $\frac{d}{dx}\int_0^{\delta}(u^2 - uU_\infty)dy$. And you can write $-\frac{d}{dx}\int_0^{\delta}u(U_\infty - u)dy$. Now, let us consider the third term, which was in right hand side of the boundary layer equation. So, if you see the third term. So that is in the right hand side. So you have $\int_0^{\delta}\nu\frac{\partial^2 u}{\partial y^2}dy$.

So, what you can write this? 0 to δ; ν we can take it outside because this is your fluid property and constant. So, you can write $\nu\int_0^{\delta}\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})dy$. So you can see. So this del y del y will get cancelled, so you can write this as $\nu[\frac{\partial u}{\partial y}]_0^{\delta}$. Now you see, that at y= δ which is your edge of the boundary layer.

So, what is the velocity gradient? $\frac{\partial u}{\partial y}$,because at the edge of the boundary layer and outside the age of the boundary layer you have a free stream velocity U_∞. So, velocity gradient becomes 0. So that means, $\frac{\partial u}{\partial y}$ will be 0 at y = δ. So you can see, at y = δ it will be 0 and $-\nu\frac{\partial u}{\partial y}|_{y=0}$ at wall will have some value right. So that is y = 0.

Now, this term we can represent in terms of shear stress; wall shear stress right. In terms of wall shear stress, τ_w. So, you can write wall shear stress. So, $\tau_w = \mu\frac{\partial u}{\partial y}|_{y=0}$. So you can see, you can write $\frac{\partial u}{\partial y}|_{y=0} = \frac{\tau_w}{\mu}$.

So now you see, the left hand side term is this one; which has one negative sign and right hand side this is the term and if you put the value of $\frac{\partial u}{\partial y}|_{y=0}$ you will get as minus, so ν is your, what is ν? So $\nu = \frac{\mu}{\rho}$. So you will get $\frac{1}{\rho}\tau_w$.

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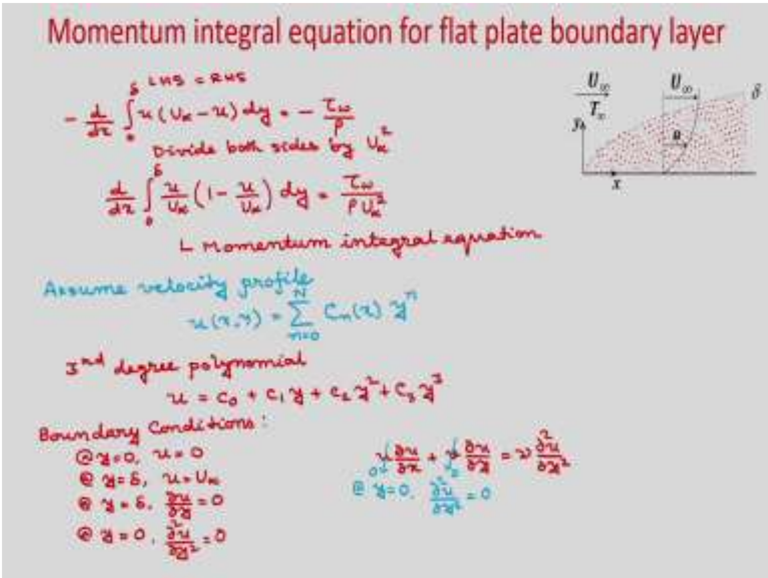

So you can see, you have minus. So, now if you write left hand side is equal to right hand side, so what you are going to get? So, if you put left hand side is equal to right hand side, so we will get $-\frac{d}{dx}\int_0^\delta u(U_\infty - u)dy = -\frac{\tau_w}{\rho}$. So, you divide both side by U_∞^2.

So, and this minus and right hand side minus we will cancel, so you can write $\frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy = \frac{\tau_w}{\rho U_\infty^2}$. So this equation is known as momentum integral equation.

So, we started with the momentum equation. We integrated between 0 and δ and finally, we have arrived in this expression. So, you can see this is known as momentum integral equation and you can see that this is your ordinary derivative $\frac{d}{dx}$. Now, we need to assume the velocity profile. Now, we need to go to the second step.

So, we need to approximate the velocity profile. Now, let us assume the velocity profile. So, assume velocity profile. So, we will use polynomial expression. So, $u(x,y) = \sum_{n=0}^{N} C_n(x)y^n$. So today we will consider 3rd degree polynomial. So if you consider 3rd degree polynomial, then you can express this velocity profile as 3rd degree polynomial.

You will get $u = c_o + c_1 y + c_2 y^2 + c_3 y^3$. So, considering 3rd degree polynomial we got 4 coefficients. So we need to determine c_0 ,c_1 ,c_2 and c_3. So how many boundary condition do you require to find this 4 coefficients? Obviously, you need 4 boundary conditions.

So, now you can see, easily you can find 2 boundary conditions at y = 0, no slip boundary condition. So u = 0 and y = δ, you have u = ∞. Another boundary condition at y = δ, you can easily find that velocity gradient is 0, so; that means, that y = δ, $\frac{\partial u}{\partial y}$ =0. So this 3 boundary conditions you got easily.

So we need another boundary conditions. Now the fourth boundary condition will derive, satisfying the Navier-Stokes equation or satisfying the boundary layer equation at the wall. So at y = 0, let us see whatever boundary layer equation you have, what expression you get and from there we will get the boundary condition that is why it is known as derived boundary condition.

So let us write the boundary conditions. So, at y = 0 you have u = 0 at y = δ, you have u = ∞ that is your free stream velocity, at y = δ, your velocity gradient is 0. So $\frac{\partial u}{\partial y}$ = 0 and at y = 0. Now, we will see the boundary layer equation is $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$.

So this is your boundary layer equation. Now, you put at y = 0. What happens? So we will derive this boundary condition. So at y = 0, u is 0; at y = 0, v is 0. So we can see, left hand side 2 terms are 0. So, we will get $\frac{\partial^2 u}{\partial y^2}$= 0. So, this is your derived boundary condition. So, $\frac{\partial^2 u}{\partial y^2}$= 0. So now, you have 4 boundary conditions, so find these 4 coefficients.

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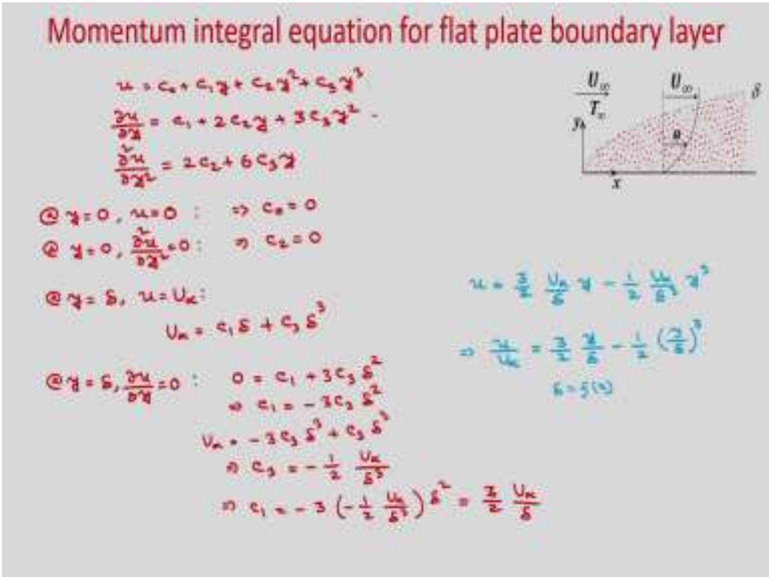

So, we have $u = c_o + c_1 y + c_2 y^2 + c_3 y^3$. So $\frac{\partial u}{\partial y}$, if you take the derivative with respect to y.

So you find the, $\frac{\partial u}{\partial y} = c_1 + 2c_2 y + 3c_3 y^2$ and $\frac{\partial^2 u}{\partial y^2} = 2c_2 + 6c_3 y$.

Now, apply those boundary conditions. So, at y = 0, you have u = 0. So, if you see this equation at y = 0. So these last 3 terms will become 0 and left hand side u = 0, so that will give c_0 is 0.

Then at y = 0, you have $\frac{\partial^2 u}{\partial y^2}$=0. So, if you put it here, y = 0 left hand side this is 0, so it will give c_2 = 0. And now at y = δ, you have u = U_∞ right. So, if you put it in this expression, so you will get $U_\infty = c_1\delta + c_3\delta^3$.

And at y = δ, you have $\frac{\partial u}{\partial y}$ = 0. So, if you put here, so left hand side is $0 = c_1 + 3c_3\delta^2$. So, you can see, you can find $c_1 = -3c_3\delta^2$. And if you put it here, so you will get $U_\infty = -3c_3\delta^3 + c_3\delta^3$.

So, what you will get? So, you will get you see here,- $2c_3\delta^3$. So, $c_3 = -\frac{1}{2}\frac{U_\infty}{\delta^3}$.

And now, $c_1 = -3(-\frac{1}{2}\frac{U_\infty}{\delta^3})\delta^2$. So what you will get? So, it will get plus so it will be $\frac{3}{2}\frac{U_\infty}{\delta}$.

So now, you can write the velocity profile as $u = \frac{3}{2}\frac{U_\infty}{\delta}y - \frac{1}{2}\frac{U_\infty}{\delta^3}y^3$. So, you can see your velocity profile. You can write $\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}(\frac{y}{\delta})^3$. So you can see, that assumed velocity is in terms of a single unknown variables δ and $\delta = f(x)$.

So, now, we will go to the next step. So, what is that step? Determination of this unknown variable. So, we have express the velocity profile in terms of hydrodynamic boundary layer thickness δ, which is function of x. So now this unknown variable δ, we need to find. So that will determine.

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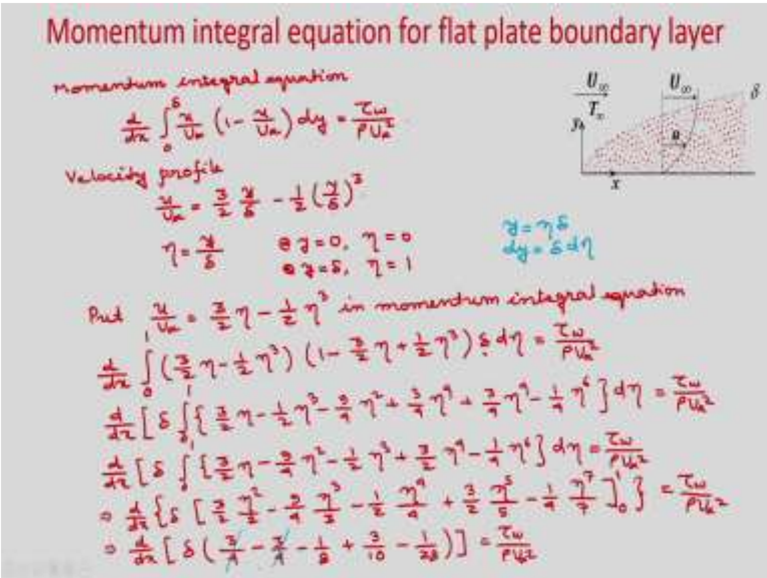

So, for this we will use this momentum integral equation. So, we have momentum integral equation, which we have already derived. This is $\frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}(1-\frac{u}{U_\infty})dy = \frac{\tau_w}{\rho U_\infty^2}$.

And we have velocity profile; $\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}(\frac{y}{\delta})^3$.

So now, let us put this velocity profile in this momentum integral equation and find the unknown variable δ. So, we will just use $\eta = \frac{y}{\delta}$. So you can see, at y = 0 η = 0 and at y=δ you will have η =1. So, this limit if you put it here, so all this expression you put, so put $\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$ in momentum integral equation.

So, what we will get? So, we will get $\frac{d}{dx}\int_0^1 (\frac{3}{2}\eta - \frac{1}{2}\eta^3)(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3)\delta d\eta = \frac{\tau_w}{\rho U_\infty^2}$.

So you can see here, your δ is function of x only right. So, this δ you can take it outside this integral, because this you are integrating with respect to η which is your y. So; obviously, you can take it outside the integrals.

So, you can write, $\frac{d}{dx}[\delta\int_0^1 \{\frac{3}{2}\eta - \frac{1}{2}\eta^3 - \frac{9}{4}\eta^2 + \frac{3}{4}\eta^4 + \frac{3}{4}\eta^4 - \frac{1}{4}\eta^6\}]d\eta = \frac{\tau_w}{\rho U_\infty^2}$.

So if you rearrange it, what you will get?

$\frac{d}{dx}[\delta\int_0^1 \{\frac{3}{2}\eta - \frac{9}{4}\eta^2 - \frac{1}{2}\eta^3 + \frac{3}{2}\eta^4 - \frac{1}{4}\eta^6\}]d\eta = \frac{\tau_w}{\rho U_\infty^2}$.

So if you integrate now, so you will get

$\frac{d}{dx}\{\delta[\frac{3}{2}\frac{\eta^2}{2} - \frac{9}{4}\frac{\eta^3}{3} - \frac{1}{2}\frac{\eta^4}{4} + \frac{3}{2}\frac{\eta^5}{5} - \frac{1}{4}\frac{\eta^7}{7}]_0^1\} = \frac{\tau_w}{\rho U_\infty^2}$.

So now, at η = 0 anyway all these terms will become 0, at η = 1 you just put the value, then what you will get? $\frac{d}{dx}[\delta(\frac{3}{4} - \frac{3}{4} - \frac{1}{8} + \frac{3}{10} - \frac{1}{28})] = \frac{\tau_w}{\rho U_\infty^2}$. So these 3/4 and 3/4 you can cancel.

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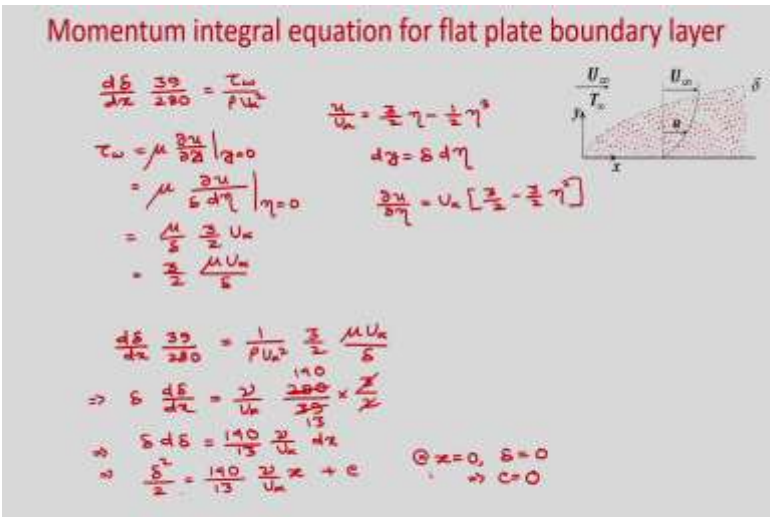

So, you can get, so now if you rearrange this and you can get finally, in the left hand side as $\frac{d\delta}{dx}\frac{39}{280}$ and in the right hand side it is $\frac{\tau_w}{\rho U_\infty^2}$. So, now let us find what is the wall shear stress, because wall shear stress you can express in terms of the velocity gradient at y = 0 and velocity profile already you have derived. So you will be able to find what is the shear stress.

So, you can write $\tau_w = \mu \frac{\partial u}{\partial x}|_{y=0}$ and you can see you can have velocity profile $\frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$. And also we have dy = δ dη right, because $\eta = \frac{y}{\delta}$. So, you can write $\mu \frac{\partial u}{\delta d\eta}|_{\eta=0}$.

Then you can write, $\frac{\partial u}{\partial \eta} = U_\infty[\frac{3}{2} - \frac{3}{2}\eta^2]$; so, you can see $\frac{\partial u}{\partial \eta}|_{\eta=0}$. So you will get $\frac{\mu}{\delta}\frac{3}{2}U_\infty$.

So, finally $\tau_w = \frac{3}{2}\frac{\mu U_\infty}{\delta}$.

So, now, if you put τ_w here, so you will get so you can write $\frac{d\delta}{dx}\frac{39}{280} = \frac{1}{\rho U_\infty^2}\frac{3}{2}\frac{\mu U_\infty}{\delta}$.

So you can see, you can write $\delta \frac{d\delta}{dx} = \frac{\nu}{U_\infty}\frac{280}{39}\frac{3}{2}$. So, this will be $\delta d\delta = \frac{140}{13}\frac{\nu}{U_\infty}dx$.

So if you integrate this equation, so you will be able to find the unknown variable δ. And you know that x = 0, you have the hydrodynamic boundary layer thickness as 0 so; that means, δ =0. So, if you integrate it so you will get $\frac{\delta^2}{2} = \frac{140}{13}\frac{\nu}{U_\infty}x + c$. So now, you put at x = 0, δ = 0 right so; that means, it will give c = 0 ok. So if your c = 0.

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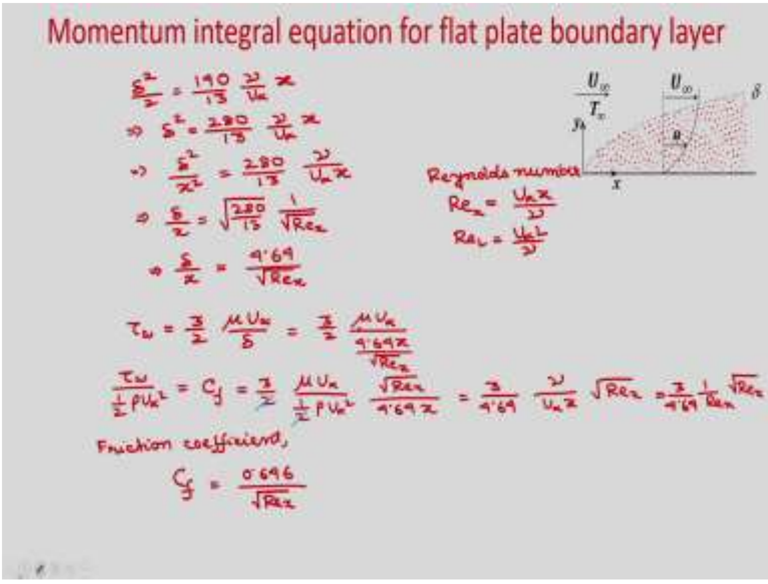

So, you can write $\frac{\delta^2}{2} = \frac{140}{13} \frac{\nu}{U_\infty} x$. So if you rearrange, you will get $\delta^2 = \frac{280}{13} \frac{\nu}{U_\infty} x$ or you can write $\frac{\delta^2}{x^2} = \frac{280}{13} \frac{\nu}{U_\infty x}$.

So now you see, in the last term you have $\frac{\nu}{U_\infty x}$. So that you can express in terms of Reynolds number. So, if you define the Reynolds number. $\mathrm{Re}_x = \frac{U_\infty x}{\nu}$, which is x is your axial direction. So, at x = l for the plate length l you can write $\mathrm{Re}_L = \frac{U_\infty L}{\nu}$. So, if you put it here, so we can see you can write $\frac{\delta}{x} = \sqrt{\frac{280}{13}} \frac{1}{\sqrt{\mathrm{Re}_x}}$. Or you can write $\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}}$.

So now, we have found the unknown variable; the hydrodynamic boundary layer thickness δ. So now, once δ is known now you will be able to calculate the velocity profile ok, because velocity profile we have expressed in terms of δ. So, now, δ you have found, so you will be able to find what is the velocity profile using the integral method.

Now, let us find the coefficient of friction. So, we have found the shear stress $\tau_w = \frac{3}{2}\frac{\mu U_\infty}{\delta}$. So now you can put the value of δ here. So, what you will get? $\frac{3}{2}\frac{\mu U_\infty}{\delta}$ value you can see here. So, δ you can write as, $\delta = \frac{4.64x}{\sqrt{\mathrm{Re}_x}}$.

So if you put this, so you can see you can write as; so now, you can write $\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = C_f = \frac{3}{2}\frac{\mu U_\infty}{\frac{1}{2}\rho U_\infty^2}\frac{\sqrt{\mathrm{Re}_x}}{4.64x}$. So, you can see here, these 2 and 2 you can cancel.

Now, you will get $\frac{3}{4.64}$.

And here you see, it is; so it will be just $\frac{3}{4.64}\frac{\nu}{U_\infty x}\sqrt{\mathrm{Re}_x}$.

So the friction coefficient you will get as $C_f = \frac{0.646}{\sqrt{\mathrm{Re}_x}}$. So today, we considered the flow over flat plate and used approximate solution method to find the boundary layer thickness as well as the friction factor using assumed velocity profile.

So, in integral approach there are three steps. First step is that you have to integrate the governing equation, then you assume the velocity profile and next you find the unknown variable.

So, using the integral approach we have found $\frac{\delta}{x}$ as well as the friction coefficient C_f in terms of the Reynolds number.

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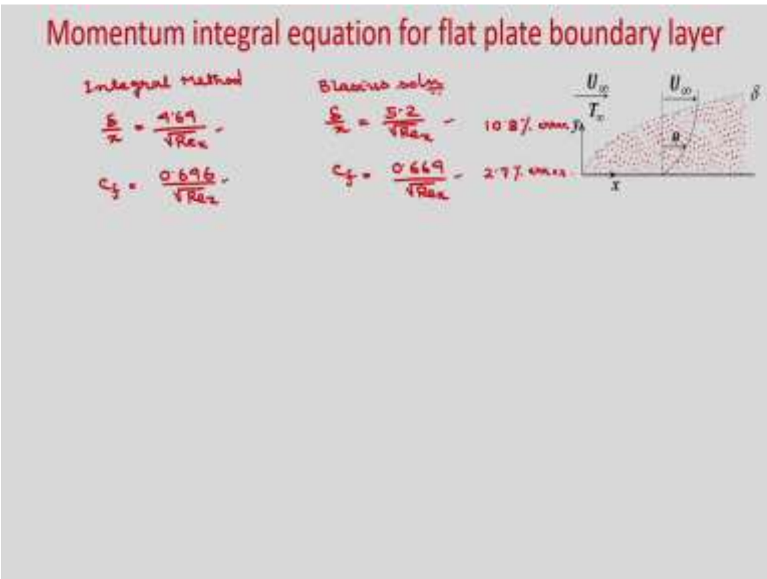

And finally, you can see that if you compare the results you got from the integral method with the exact solution, so there is; if you see this is your almost 10.8 % error and if you compare the C_f it is 2.7 % error.

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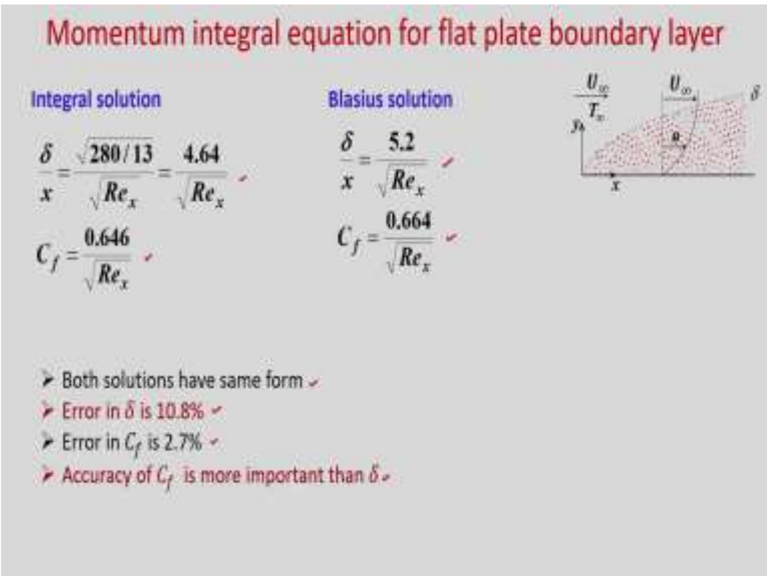

So you can see; finally, we have derived the hydrodynamic boundary layer thickness using integral solution $\frac{\delta}{x} = \frac{4.64}{\sqrt{\mathrm{Re}_x}}$. Similarly, coefficient of friction we have derived

$C_f = \frac{0.646}{\sqrt{\mathrm{Re}_x}}$. So you can see these are approximate solution, because we have used some approximate velocity profile.

Whereas, you have the exact solution from Blasius solution; you can see $\frac{\delta}{x} = \frac{5.2}{\sqrt{\mathrm{Re}_x}}$ and $C_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$. So, you can see both solutions have same form.

If you compare δ by x with this integral solution and Blasius solution, you will get error in δ as 10.8 % and error in C_f is 2.7 %. Obviously, error in C_f is less than the error in δ and accuracy of C_f is more important than δ in design point of view.

Thank you.