Fundamentals of Convective Heat Transfer
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Module - 07
Convection in Internal Flows - III
Lecture - 24
Heat transfer in plane Couette flow

Hello everyone. Today, we will consider Heat transfer in plane Couette flow. Already we have derived the velocity distribution in plane Couette flow, where upper plate is moving and bottom plate is stationary and you have seen that velocity varies linearly from bottom wall to top wall. So, today, we will assume fully developed laminar flow. There is no internal energy generation. However, we will consider the viscous heat dissipation effect.

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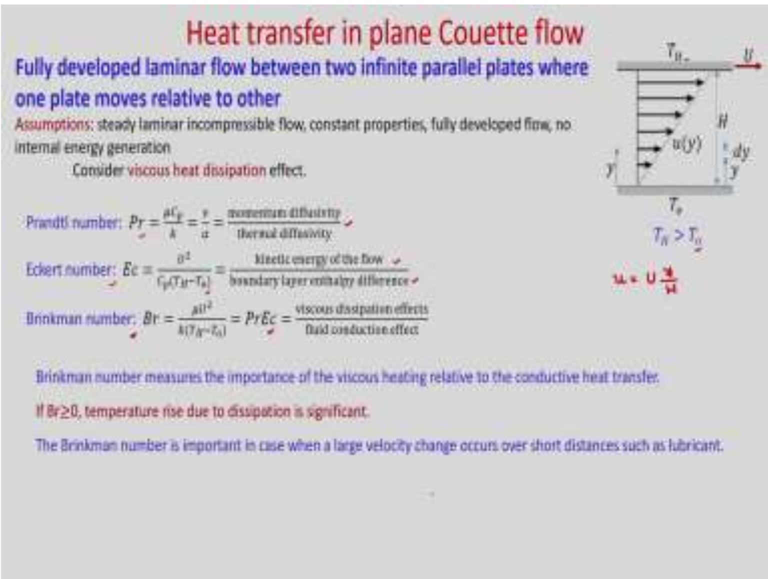

So, let us consider the flow between two parallel plates; the bottom plate here velocity is 0, so it is stationary plate and upper plate, you can see it is moving in the x direction with a constant velocity u. The bottom plate is maintained at temperature T_0 and upper plate is maintained at temperature T_H, where T_H we are considering is greater than T_0 and y is measured from the bottom wall and the distance between two plates is H.

So, you can see that already we have derived the velocity profile u; u is linear and $u = U\frac{y}{H}$. So, before going to that let us revisit the non-dimensional numbers, Prandtl number. What is Prandtl number? Prandtl number is the ratio of momentum diffusivity to thermal diffusivity right.

So, you can see that $\Pr = \frac{\mu C_p}{k} = \frac{\nu}{\alpha}$ and it is the ratio of momentum diffusivity to the thermal diffusivity. We also define Eckert number earlier. $Ec = \frac{U^2}{C_p(T_H - T_0)}$.

So, Eckert number is the ratio of kinetic energy of the flow to the boundary layer enthalpy difference. Now, we will define one new non-dimensional number which is Brinkman number. So, Brinkman number is the product of Prandtl number and Eckert number and it is given as $Br = \frac{\mu U^2}{k(T_H - T_0)}$.

So, Brinkman number measures the importance of the viscous dissipation effects to the fluid conductive heat transfer. If Brinkman number ≥ 0, then temperature rise due to dissipation is significant. The Brinkman number is important in case when a large velocity change occurs over a short distance such as lubricant.

So, here you know that we are considering plane Couette flow; that means, it is a shear defined flow, there is no imposed pressure difference right. So, velocity profile is generated due to the shear of the upper plate because upper plate is moving at a constant velocity u. So, let us write the energy equation and invoking the assumptions, let us simplify it.

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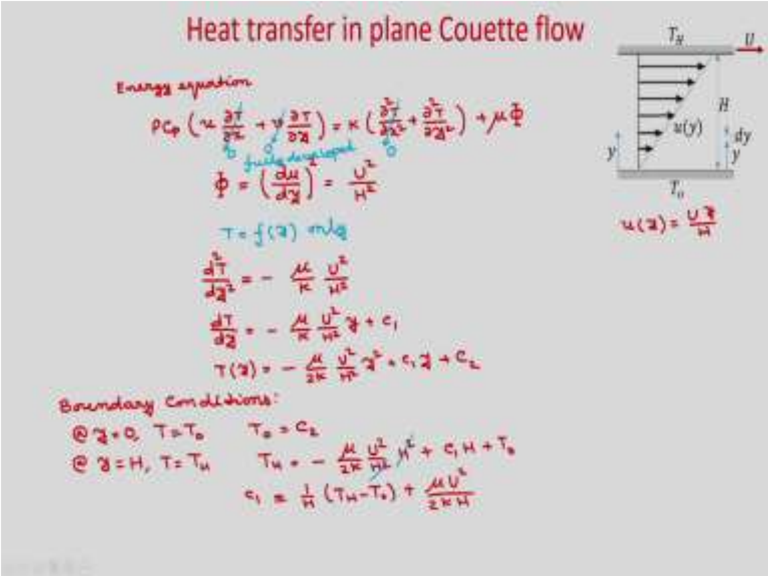

So, our energy equation is $\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=K\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\phi$. Here, Φ is viscous dissipation coefficient and it is for this particular case, you can see it will give $\phi=\left(\frac{du}{dy}\right)^2$. So, you have a expression of Φ right. So, if you see all other terms become 0 because your velocity profile u which is function of y only and it is $u(y)=U\frac{y}{H}$.

So, in this particular case, you can see the velocity profile is linear. So, it will be just $\frac{U^2}{H^2}$ because distance between two parallel plates is H and it is linearly valuing. So, $\frac{du}{dy}=\frac{U}{H}$. So, $\left(\frac{du}{dy}\right)^2=\frac{U^2}{H^2}$.

Now, let us simplify it. So, we are telling it is a fully developed flow. So, obviously, your v = 0. In this particular case, as there is no imposed pressure difference, so velocity is generated due to the shears shear force on the exerted on the upper plate. So, your fully developed temperature profile also is constant.

So, in axial direction, there will be no variation of temperature if $\frac{\partial T}{\partial x}$=0. So, it is a fully developed condition. So, in this particular case, your $\frac{\partial T}{\partial x}$=0 because it is fully developed. So, you can see that essentially you can neglect the axial heat conduction. So, $\frac{\partial^2 T}{\partial x^2}$=0 because $\frac{\partial T}{\partial x}$=0 due to fully developed condition; so obviously, $\frac{\partial^2 T}{\partial x^2}$=0. So that means, axial heat conduction is also 0.

So, in this particular case for Couette flow, your temperature is function of y only. So, your fully developed temperature profile, you get as $\frac{\partial T}{\partial x}$=0. So, there is no variation of temperature profile in the axial direction. So, $\frac{\partial T}{\partial x}$=0. Hence, your temperature is only function of y. So, you can write the governing equations as so T is function of y only.

So, your simplified energy equation is $\frac{d^2T}{dy^2} = \frac{\mu}{K}\frac{U^2}{H^2}$. So, this is our governing equation and we have two boundary conditions at y = 0; T =T_0 and at y = H; T = T_H. So, you integrate this equation twice and find the two constants applying the boundary conditions.

So, if you integrate twice, what you will get? $\frac{dT}{dy} = -\frac{\mu}{K}\frac{U^2}{H^2}y + C_1$. If you integrate again, $T(y) = -\frac{\mu}{2K}\frac{U^2}{H^2}y^2 + C_1 y + C_2$.

So, now apply the boundary conditions. So, at y = 0, you have T =T_0. So, if you put that, then T_0 = C_2. Because y = 0, so this right hand side, first two terms will become 0 and at y = H, you have T = T_H.

So, if you put this one, so you will get $T_H = -\frac{\mu}{2K}\frac{U^2}{H^2}H^2 + C_1 H + T_0$. So, you can now write this C_2 = T_0 and $C_1 = \frac{1}{H}(T_H - T_0) + \frac{\mu U^2}{2KH}$. So, now, we have found the two constants. So, let us put it in the temperature profile and find the final temperature distribution.

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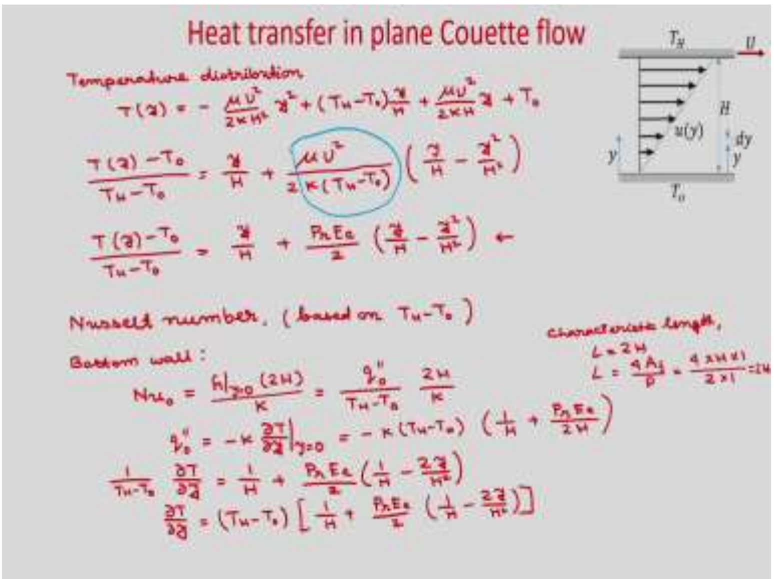

So, temperature distribution . So, you can see

$$T(y) = -\frac{\mu}{2K}\frac{U^2}{H^2}y^2 + (T_H - T_0)\frac{y}{H} + \frac{\mu U^2}{2KH}y + T_0 .$$

So, if you rearrange it, you can write as $\frac{T(y)-T_0}{T_H - T_0} = \frac{y}{H} + \frac{\mu U^2}{2K(T_H - T_0)}\left(\frac{y}{H} - \frac{y^2}{H^2}\right)$.

So, you can see that this term what is this term? So, if you see, so it is obviously, the product of Prandtl number and Eckert number which is your Brinkman number, already we have defined . So, you can write it $\frac{T(y)-T_0}{T_H - T_0} = \frac{y}{H} + \frac{\text{PrEc}}{2}\left(\frac{y}{H} - \frac{y^2}{H^2}\right)$. So, this is the temperature profile, we will plot it later after finding the Nusselt number.

So, in this particular case, we will calculate the Nusselt number based on the temperature difference T_H -T_0 . Here imposed temperature difference is T_H -T_0 because T_H >T_0 and we will define the Nusselt number based on T_H -T_0.

You can also find the mean temperature T_m and also find the Nusselt number based on the temperature difference T_H –T_H or T_m -T_0, but it is convenient for this particular case to calculate the Nusselt number based on the temperature difference T_H -T_0. So, for this particular case, now find the Nusselt number.

So, Nusselt number we will calculate. So, what is Nusselt number? So, Nusselt number based on temperature difference T_H -T_0, this you should remember because in earlier cases, all we have considered the temperature difference T_w -T_m, where T_m is the mean temperature ok.

But in this particular case, we are calculating the Nusselt number based on T_H -T_0. Obviously, if you calculate the Nusselt number based on that temperature difference T T_H –T_m, then expression will be different.

So, now Nusselt number on bottom wall. So, we can calculate Nu_0. So, it is $q_0^{"}$. So, this is your 0 means at y = 0 divided by or we can write this.

So, under bottom wall, we have Nusselt number at bottom wall that is your we are denoting with 0. So, $Nu_0 = \frac{h|_{y=0}(2H)}{K}$.

So, in this particular case, now if you see what is h at y = 0? So, this is $\frac{q_0^{"}}{T_H - T_0}\frac{2H}{K}$. So, characteristic length, here we are considering as 2 H. So, now what is $q_0^{"}$? So, this is your $q_0^{"} = -K\frac{\partial T}{\partial y}|_{y=0}$.

So, you see, we are calculating the heat flux at bottom wall. So, y is perpendicular to the bottom wall. So, $q_0^{"} = -K\frac{\partial T}{\partial y}|_{y=0}$. So, you can calculate $-K\frac{\partial T}{\partial y}|_{y=0}$. So, you can see from previous expression; $\frac{\partial T}{\partial y} = -\frac{\mu}{K}\frac{U^2}{H^2}y + C_1$.

So, from this temperature distribution, first let us calculate the temperature gradient. So, if you take the derivative of this equation with respect to y, you will get $\frac{1}{T_H - T_0}\frac{\partial T}{\partial y} = \frac{1}{H} + \frac{\text{PrEc}}{2}\left(\frac{1}{H} - \frac{2y}{H^2}\right)$. So, this is the temperature gradient. So, $\frac{\partial T}{\partial y} = \left(T_H - T_0\right)\left[\frac{1}{H} + \frac{\text{PrEc}}{2}\left(\frac{1}{H} - \frac{2y}{H^2}\right)\right]$.

So, you can see $\frac{\partial T}{\partial y}\Big|_{y=0}$, what will be the value? So, y = 0. So, you will get $q_0^{"} = -K\left(T_H - T_0\right)\left(\frac{1}{H} + \frac{\text{PrEc}}{2H}\right)$.

So, now, in this particular case characteristic length twice H, how you did you get? So, you see the characteristic length L, you can find as $L = \frac{4A_f}{P}$. So, in this particular case, you can see flow area is 4 $\times$H$\times$1. So, that is the H is the distance between 2 parallel plates. So, it is your H and in perpendicular direction per unit width, if you consider then 1.

So, this is your flow area divided by the perimeter. What is the perimeter in this particular case? It is 2$\times$1 because in per unit width you are considering right in perpendicular direction. So, you have one on the top plate, one at the bottom plate. So, it will be 2. So, hence, in this particular case, characteristic length becomes 2 H. So, from here we are just calculating the Nusselt number based on the characteristic length 2 H.

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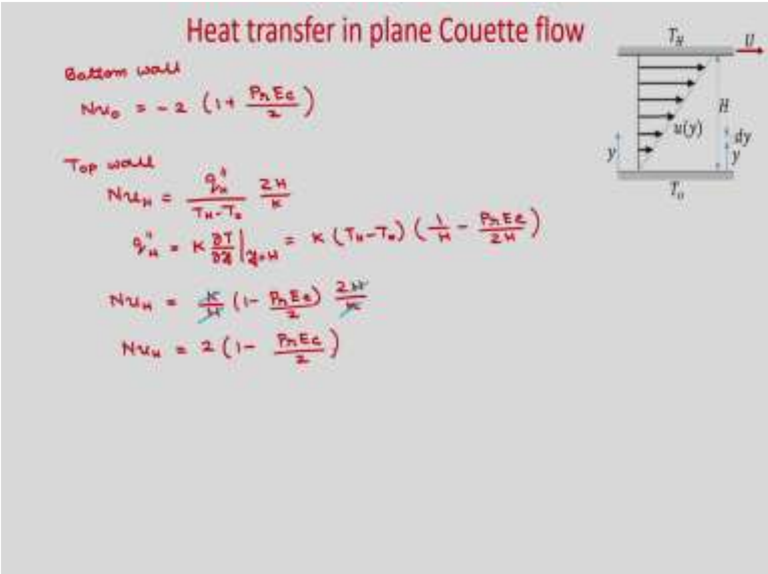

So, now in bottom wall if you calculate the Nusselt number, then you will get, $Nu_0 = -2\left(1 + \frac{\Pr Ec}{2}\right)$. So, if you put all these values and simplify it, you will get this one. Now, you calculate on the top wall. So, Nusselt number at top wall. So, it will be $Nu_H = \frac{q''_H}{T_H - T_0}$. So, this is your heat transfer coefficient H.

Now, you see in this expression $\frac{\partial T}{\partial y}$, if you want to find the heat flux at top wall. So, it will be $q''_H = K\frac{\partial T}{\partial y}|_{y=H}$ and y = H, if you put it here. So, you are going to get here. So, if you write it, so you will get q''_H.

So, it will be now y is in upward direction. Now, when you are calculating the heat flux on the top wall. So, we are calculating it is coming in the negative y direction, so obviously, $q''_H = K\frac{\partial T}{\partial y}|_{y=H}$.

So, it will be $q''_H = K\frac{\partial T}{\partial y}|_{y=H}$. So, now, if you put the value, so it will be $q''_H = K\left(T_H - T_0\right)\left(\frac{1}{H} - \frac{\Pr Ec}{2H}\right)$. So, if it is so, then you can write Nusselt number H as $Nu_H = \frac{K}{H}\left(1 - \frac{\Pr Ec}{2}\right)\frac{2H}{K}$ and it will be $Nu_H = 2\left(1 - \frac{\Pr Ec}{2}\right)$

So, now, we have found the Nusselt number on top wall and bottom wall based on the characteristic length 2 H and the temperature difference T_H - T_0. Now, let us try to plot the temperature profile and see the effect of Brinkman number. That means, the product of Prandtl number and Eckert number.

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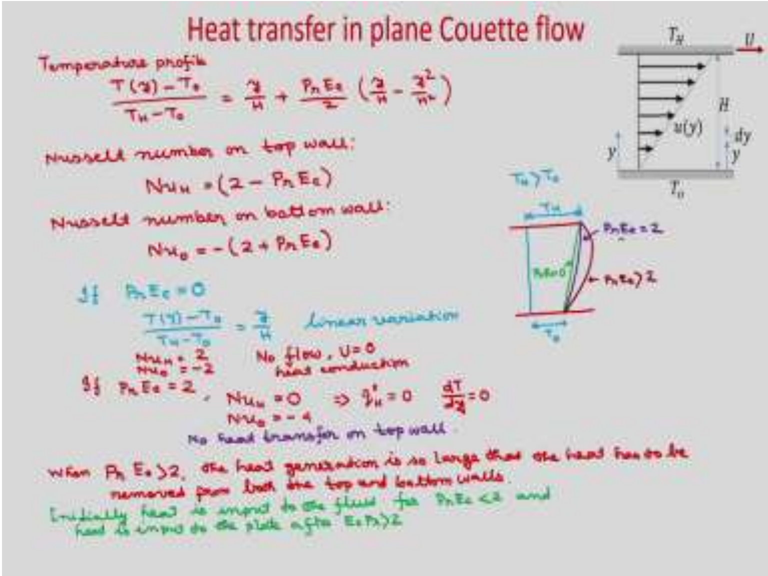

So, the temperature distribution we got $\frac{T(y)-T_0}{T_H-T_0}=\frac{y}{H}+\frac{\Pr Ec}{2}\left(\frac{y}{H}-\frac{y^2}{H^2}\right)$.

So, this is your temperature profile and you have Nusselt number on top wall. $Nu_H=\left(2-\Pr Ec\right)$ and Nusselt number on bottom wall, we got $Nu_0=-\left(2+\Pr Ec\right)$.

So, now, let us try to see what happens when your Brinkman number is 0 . So that means, there is no viscous heat dissipation. So, only the temperature at bottom wall is T_0, upper wall is T_H and there is no viscosity. So, you will get a linear profile of temperature distribution.

So, you see this is your top wall, this is your bottom wall separated by distance H. Now, if you see the temperature profile, we have seen that $T_H > T_0$. So, if this is your T_0 on bottom wall, so obviously, T_H will be higher; so, this is let us say T_H on the top wall.

Now, we are seeing the if your Pr $\times$ Ec = 0, so if Pr $\times$ Ec = 0, so you see what is the temperature profile? So, in this case, you can see this Prandtl number Eckert number will be 0. So, you will get $\frac{T(y)-T_0}{T_H-T_0}=\frac{y}{H}$.

So, you see it is a linear profile right; linear variation of temperature. So, if you plot it ok, so it will be just linear variation. So, this is your Pr $\times$ Ec = 0. Now, you see when Pr$\times$Ec = 2. So, if Pr$\times$Ec = 2, then you see Nusselt number H what you will get?

if Prandtl number Eckert number is 0, then you are getting $Nu_H = 2$ and $Nu_0 = -2$ and what will be the velocity in this particular case, when Eckert number is 0?

Can you tell me, if Pr $\times$ Ec = 0, what will be the velocity profile? Obviously, U = 0; that means, there is no relative motion between 2 plates that means, plates are stationary, U=0. Hence, you are getting only the heat conduction from bottom wall to top wall.

So, that is why you are getting a linear profile of temperature variation. So, in this particular case U is 0; that means, no flow; no flow, U = 0 in this particular case because Pr $\times$ Ec = 0 and Eckert number velocity is there. So, u is 0 and only heat conduction takes place.

But when you are considering Pr $\times$ Ec = 2, then on the top wall Nusselt number is 0. What does it mean? That there is no heat transfer from the top plate right because if Nusselt number is 0, then your $q_w^{"}$ on top wall is 0 that means, your temperature gradient on the top wall will be 0 .

So that means, here your $q_H^{"}$ on the top wall is 0 and hence $\frac{dT}{dy}$ will be 0 . So that means, your temperature profile will cut the top wall perpendicularly. So, this will cut perpendicularly ok. So, this is the profile for Pr $\times$ Ec = 0.

Now, if Pr $\times$ Ec >2 , then your viscous heat dissipation effect will dominate and there will be more heat generation and your maximum temperature will occur in between top plate and bottom plate.

So, in this case, you will get no heat transfer on top wall. Now, when Pr $\times$ Ec >2, the heat generation is so large that the heat has to be removed from both the top and bottom walls and what will be the temperature profile in this particular case?

So, your temperature profile will be; so you can see in case of Brinkman number 0 and Brinkman number 2, your maximum temperature occurs on the top wall itself. But when Br > 2, your maximum temperature is occurring in between the domain.

So, this is the case for Pr × Ec >2, this is the plot for Pr × Ec = 2 and in this particular case, Pr × Ec = 0; that means, there is no flow. So, this is the linear profile. Here, your heat transfer is 0 on top wall and for Pr × Ec >2, you will get the maximum temperature inside the domain.

So, if you see so initially heat is input to the fluid, for Pr × Ec < 2 and heat is input to the plate after Ec × Pr >2 because when Ec × Pr >2, then your maximum temperature is occurring inside the domain, so heat transfer will occur from the fluid to the top wall and when you have Pr × Ec < 2, then heat transfer will occur from the top wall to the fluid.

So, when Pr × Ec < 2, heat generation within the fluid is small and therefore, heat stills comes in from water top plate. So, how do we find the location of the maximum temperature? So, simply you just put $\frac{\partial T}{\partial y}$ = 0.

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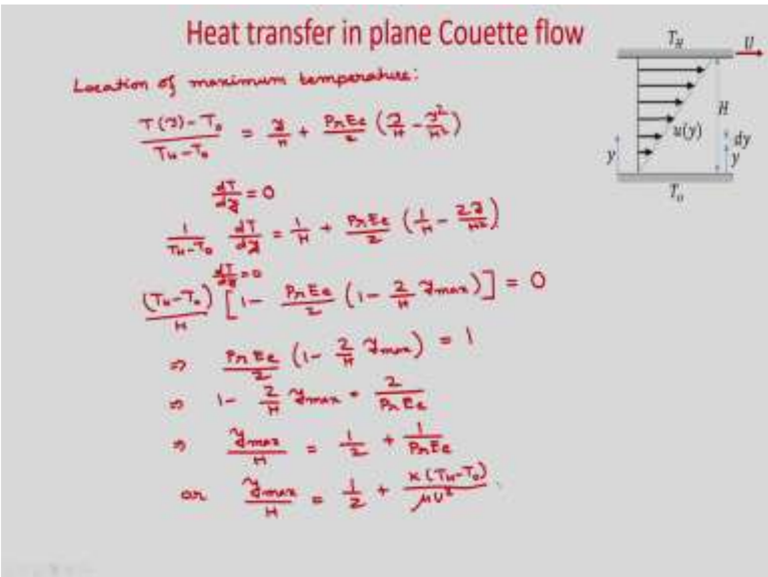

So, your temperature distribution, so location of maximum temperature. So, how we will find? So, let us see the temperature distribution $\frac{T(y)-T_0}{T_H-T_0}=\frac{y}{H}+\frac{\Pr Ec}{2}\left(\frac{y}{H}-\frac{y^2}{H^2}\right)$.

So, you will get so from maximum temperature $\frac{dT}{dy}=0$. If $\frac{dT}{dy}=0$, so that means, $\frac{1}{T_H-T_0}\frac{dT}{dy}=\frac{1}{H}+\frac{\Pr Ec}{2}\left(\frac{1}{H}-\frac{2y}{H^2}\right)$.

So, now if you put $\frac{dT}{dy}=0$, so you will get if you simplify it. So, it will be, $\frac{T_H-T_0}{H}\left[1-\frac{\Pr Ec}{2}\left(1-\frac{2}{H}y_{max}\right)\right]=0$.

So, if you simplify it, so it will be $\frac{\Pr Ec}{2}\left(1-\frac{2}{H}y_{max}\right)=1$ or you will get $1-\frac{2}{H}y_{max}=\frac{2}{\Pr Ec}$ or you will get $\frac{y_{max}}{H}=\frac{1}{2}+\frac{1}{\Pr Ec}$ or you can write $\frac{y_{max}}{H}=\frac{1}{2}+\frac{K\left(T_H-T_0\right)}{\mu U^2}$.

So, you see at this location, you will get the maximum temperature inside the domain. So, now let us consider another case, where top plate is having temperature T_H and bottom plate is adiabatic. What does it mean adiabatic? Adiabatic means there will be no heat loss from the bottom wall.

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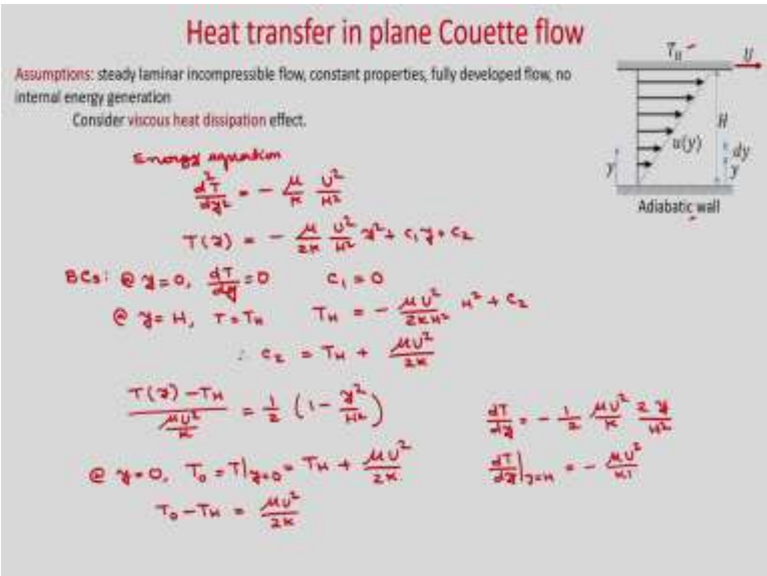

So, here we are considering T_H on the top wall and adiabatic wall on the bottom. So that means, there will be no heat transfer. So, in this particular case, if you see already we have derived the energy equation and your energy equation is energy equation already we have derived. So, $\frac{d^2T}{dy^2} = -\frac{\mu}{K}\frac{U^2}{H^2}$, we have consider the viscous dissipation effect.

So, if you see, $T(y) = -\frac{\mu}{2K}\frac{U^2}{H^2}y^2 + C_1 y + C_2$. So, now, apply the boundary condition, in this particular case at y = H, you have temperature T_H; but bottom wall is adiabatic. So, heat flux is 0, hence $\frac{dT}{dy}$ will be 0 on the bottom wall.

So, boundary conditions at y = 0 , $\frac{dT}{dy} = 0$. So, if $\frac{dT}{dy} = 0$ on the bottom wall, so you can see that you will get your C_1=0; C_1=0 and at y = H, you will get T as T_H. So, you will get T_H, C_1=0 and you will get, $T_H = -\frac{\mu U^2}{2KH^2}H^2 + C_2$.

So, $C_2 = T_H + \frac{\mu U^2}{2K}$. Now, if you put, then you will get the temperature profile $\frac{T(y) - T_H}{\frac{\mu U^2}{K}} = \frac{1}{2}\left(1 - \frac{y^2}{H^2}\right)$. So, this is a parabolic profile, temperature distribution. Let us find the Nusselt number based on the characteristic length 2H and the temperature difference T_H -T_0. So, you have to find what is the temperature at the bottom wall.

So, if you do. So, so you see at y = 0 , so what will be the temperature ? So, you see if you have a temperature at the bottom wall, if you find it as T_0, then it will be, $T_0 = T\big|_{y=0} = T_H + \frac{\mu U^2}{2K}$. So, you can see that $T_0 - T_H = \frac{\mu U^2}{2K}$. So, now we will find the Nusselt number based on this temperature difference $T_0 - T_H$.

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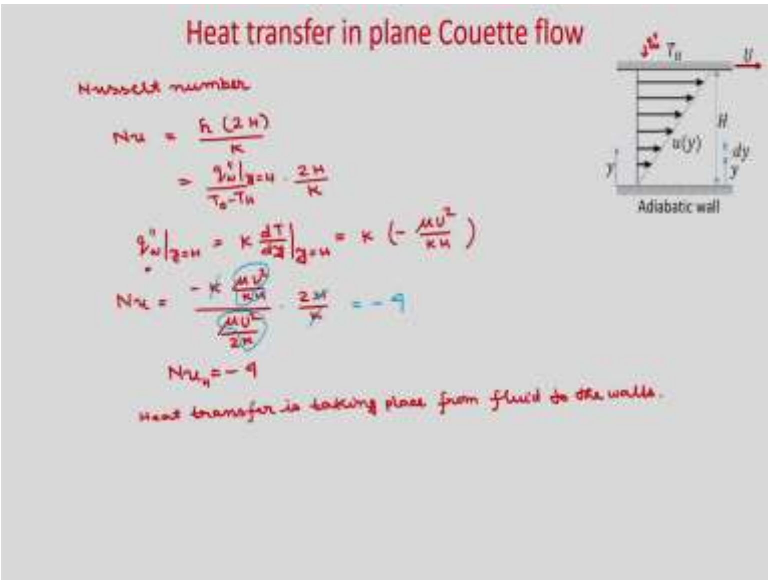

So, Nusselt number, so we will find Nusselt number based on this temperature difference. So, it will be, $Nu = \frac{h(2H)}{K}$, where $H = \frac{q_w^{"}|_{y=H}}{T_0 - T_H}$.

So, this is the temperature difference, and Nusselt number is Nu=$\frac{q_w^{"}|_{y=H}}{T_0 - T_H}\frac{2H}{K}$, and $q_w^{"}|_{y=H}$ you can find it is $q_w^{"}|_{y=H} = K\frac{dT}{dy}|_{y=H}$. So, what will be that? So, your temperature profile is this one right.

So, you can see you can find $\frac{dT}{dy}$. It will be $-\frac{\mu U^2}{KH}$. So, this is your $\frac{dT}{dy} = -\frac{\mu U^2}{KH}$.

So, it will be just $Nu = \frac{-K\frac{\mu U^2}{KH}}{\frac{\mu U^2}{2K}}\frac{2H}{K}$. So, you can see this simplify it, this H, this H will get cancelled; this $\frac{\mu U^2}{K}$, $\frac{\mu U^2}{K}$ will get cancelled; this K, this K will get cancelled. So, you will get 2. So, it will be -4.

So, Nusselt number you are getting as - 4. So, you can see that bottom wall is adiabatic, so there will be no heat transfer through the bottom wall. However, top wall is maintained at constant wall temperature T_H. Hence, the heat transfer will take place from the fluid to the top wall and temperature of bottom wall anyway it will increase.

As it is adiabatic, so there will be no heat transfer through the bottom wall. However, its temperature will increase. So, the Nusselt number whatever we defined here, you can see the heat flux on the top wall right. So, Nusselt number whatever we have found here is for your top wall.

So, we can write $Nu_H = -4$ and negative sign represents that heat transfer is taking place from the fluid to the top wall because while calculating the heat flux $q_w''|_{y=H}$, we took it as negative y direction. So, we assumed that heat transfer is taking place from the top wall to the fluid. However, we have seen that due to the viscous heat dissipation, the temperature of the fluid will increase and the heat transfer will take place from the fluid to the top wall.

So, let us summarize. Today, we considered fully developed laminar flow between 2 parallel plates, where top plate is moving in the positive x direction with a constant velocity U; whereas, your bottom wall is stationary.

We considered two different types of problem, where in first case we considered the temperature on the bottom wall as T_0 and on the top wall as T_H, where $T_H > T_0$ and in other case, we considered bottom wall as adiabatic and top wall is T_H. In both the cases, we consider the viscous heat dissipation effect. First, we found the temperature profile for both the cases, then we found the Nusselt number.

In first case, we considered the Nusselt number for the bottom wall and top wall separately. And this Nusselt number, we have calculated based on the characteristic length 2 H and the temperature difference T_H - T_0 and we have seen that if your Pr × Ec = 0, then there is no flow at all.

Hence, your heat conduction will take place from top wall to bottom wall linearly. Next when Pr × Ec = 2, then Nusselt number on the top wall becomes 0; that means, there will be no heat flux on the top wall and $\frac{dT}{dy} = 0$.

In these two cases or in between cases, you can see that when your Pr $\times$ Ec ≤ 2, your maximum temperature occurs on the top wall and your heat transfer is taking place from the walls to the fluid.

However, when your Pr $\times$ Ec >2, then your viscous heat dissipation effect comes into picture and your temperature becomes higher in the fluid domain than the top wall. Hence, your heat transfer takes place from the fluid region to the top wall.

Next case, we considered the Nusselt number based on the temperature difference between bottom wall and top wall and the characteristic length 2 H. In this particular case, we got the Nusselt number as -4 because we considered that heat transfer is taking place from the top wall to the fluid.

But as it is negative that means, your viscous heat dissipation effect is there and temperature in the fluid zone is higher than the top wall. Hence, your heat transfer is taking place from the fluid to the walls.

Thank you.