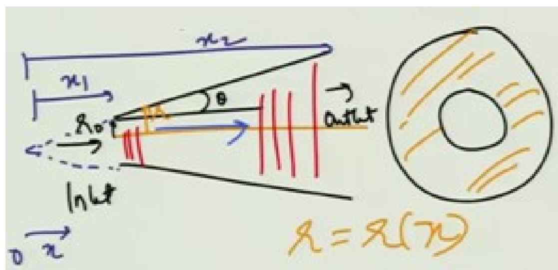
Muffler Acoustics - Application to Automotive Exhaust Noise Control
Prof. Akhilesh Mimani
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur

Lecture - 28
Solution of Webster's horn equation for Conical Ducts

Welcome to lecture 3 of week 6 of our NPTEL course on Muffler Acoustics. For this class we are going to discuss about wave propagation; one-dimensional wave propagation in conical duct. So, we have our **1-D waves in conical ducts**.



So, you see here this is a schematic of a conical duct and the wave propagation. Let us say this is the inlet and this is the outlet. So, what we are going to do is basically use our Webster's horn equation to study acoustic wave propagation in such a system.

So, as you see its given by the following x being the; let us denote the direction by *x*. So, as we know its given by and we have your Webster's horn equation or a one-dimensional equation.

$$\frac{d^2\tilde{p}}{dx} + \frac{1}{S}\frac{dS}{dx}\frac{d\tilde{p}}{dx} + k_0^2\,\tilde{p} = 0$$

Now, for a conical duct we can let us look at this schematic a bit carefully. Now, Sx could be varying in any form using any functional form, but for a conical duct how does it vary? So, if we consider the radius to be given from the section r a function. So, the question here is that, how is radius varying?

So, for a conical duct the radius is linearly increasing. So, r then is basically something

$$r = r_0 + \alpha x$$

where, r_0 is some constant. So, basically, we can consider r naught to be something like this and plus α could be the slope. It is basically related to tan θ.

So, if you consider this theta angle like I said θ is constant for a conical duct; that is the rate of the flare angle is constant throughout the cross section that is a slope is constant. So, basically $\alpha = tan\theta$, theta is the flare angle ok.

Now, the thing is that how do we go about solving for this thing. So, a better way perhaps a little more simplified way of doing this stuff is to assume some kind of a hypothetical or some kind of a point here where r_0 is 0. So, let us actually do the following instead of measuring x right from here. Let us say we measure *x* from here; where x is 0 here and it has certain values say x_1 here and x_2 here ok, x_1 here and x_2 at the other side.

So, in that case I would write r or the radius to be directly proportional to x; that is radius is increasing with the axial distance *x*. So, since area is πr^2 that would mean that S is proportional to x^2 also, x is a cross sectional area is proportional to the distance from the hypothetical epics let us.

$$r \propto x \quad =>$$

$$S = \pi r^2 \quad S \propto x^2 \qquad => \frac{dS}{dx} \propto 2x$$

So, what will this imply? This implies

$$\frac{1}{S}\frac{dS}{dx} = \frac{2}{x}$$

So, we probably can substitute this guy in this equation the Webster’s equation. So, what do we get?

$$\frac{d^2\tilde{p}}{dx^2} + \frac{2}{x}\frac{d\tilde{p}}{dx} + k_0^2\tilde{p} = 0$$

So, how do we go about solving it? It might look a little difficult to solve, but remember we have the transformation. So, if we put p.

$$\tilde{p} = \frac{q}{x} \quad => \frac{d\tilde{p}}{dx} = -\frac{q}{x^2} + \frac{1}{x}\frac{dq}{dx}$$

Let us see what we get, let me box this equation. So, what this would mean? So, we get the following. With this particular thing we; obviously, can put all the stuff here and other than that we probably have to differentiate this guy again to get the following.

$$=> \frac{d^2\tilde{p}}{dx^2} + \frac{2}{x^3}q - \frac{2}{x^2}\frac{dq}{dx}$$

Let us do a term by term differentiation. So, basically once we do that its 2 by x cube times q, then we will get this thing plus we have contributions due to the other term. So, because of the lack of space I am just directly

$$+\frac{1}{2}\frac{d^2q}{dx^2} - \frac{1}{x^2}\frac{dq}{dx}$$

So, what we get here is that we probably can put this as your rid of this guy and put it - 2 by x^2. We get this now what we will do is that; we will put this term here and this term here. So, and then we will simplify.

$$=> \quad \cancel{\frac{2}{x^3}}q - \cancel{\frac{2}{x^2}\frac{dq}{dx}} + \frac{1}{x}\frac{d^2q}{dx^2}$$

We write this and then we will multiply this by 2 divided by x. So, what we will get? We will get

$$+\cancel{\frac{2}{x}\left(-\frac{q}{x^2} + \frac{1}{x}\frac{dq}{dx}\right)} + k_0^2\frac{q}{x}$$

With this substitutions what can we expect? We can expect that well this term and this term cancel away right. And here you have minus sign sitting here is not it and you multiply this thing. So, this sign and this will sort of go away and 1 by x we can take common to get the following.

$$=> \frac{1}{x}\left(\frac{d^2q}{dx^2} + k_0^2\,q\right) = 0$$

So, basically this term the circle term that must be equal to 0 because we are talking about general x. So, I would write this thing in a nice clean slide.

$$\frac{d^2q}{dx^2} + k_0^2\, q = 0 \qquad\qquad where \quad q = q(x)$$

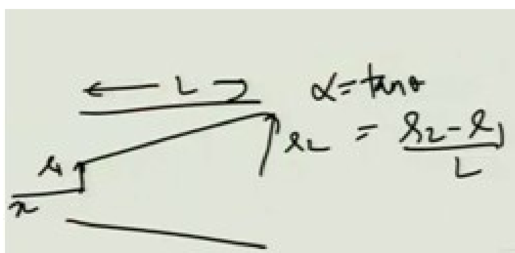
Where q understandably is a function of x. So, now the solution of this is should be fairly straightforward;

$$=> \qquad q\ (x) = C_1\, e^{-jk_0x} + C_2\, e^{jk_0x}$$

Now, remember q was given by

$$=> \quad \tilde{p}(x) = \frac{1}{x}\ \left(C_1\, e^{-jk_0x} + C_2\, e^{jk_0x}\right)$$

So, basically what it would mean. So, we get the solution for the one-dimensional wave propagation in a conical duct obviously with a constant flare. And that flare angle we can decide basically once you obtain the thing like transfer matrix. So, in that thing we can specify the radius at 2 sections.



So, for example, if we have a conical duct like this. So, this is in terms of general x where x is measured in here. So, once we specify r_1 and r_2 and all; obviously, x or L the distance.

We have our three parameters to decide the slope. The radius at the 2 section and how far apart the sections are which is sort of a kind of a common knowledge.

So, this is the expression for the acoustic pressure field and you see any resemblance with the spherical wave. In fact this particular equation that we got this thing, this equation itself is basically the same as the equation for the spherical wave one-dimensional spherical wave equation.

Let me write down the generalized form of the spherical wave. So, here note that we are making a small, but important digression its important to understand certain things conceptually. **Spherical Wave Equation** or basically wave equation in spherical coordinates. So, basically that is done 1 by

$$\frac{1}{r^2}(r^2 pr)_2 + \frac{1}{r^2 \sin\theta}\ (sin\theta\ \phi_0)_\theta$$

$$-\frac{\phi_{tt}}{C_0^2} = 0$$

Where ϕ can be any function here let us actually replace ϕ by $\tilde{p}$ where $\tilde{p}$ is now the acoustic pressure.

$$\frac{1}{r^2}(r^2 \tilde{p}r)_r + \frac{1}{r^2 \sin\theta}\ (sin\theta\ \tilde{p}_0)_\theta$$

So, here theta suffix means derivative with respect to θ. And actually what we have done here? We have purposely sort of we have kind of omitted your the phi term. So, its spherical coordinates comprises of r radius and theta that is your polar angle theta and phi which is your azimuthal angle.

So, we have not really considered the phi coordinate let us be happy with only *r* and *r* and theta coordinate. So, let us be happy with that and you know go about solving this equation.

$$\tilde{p} = R(r)(\Theta)(0)$$

$$+ k_0^2\,\tilde{p} = 0$$

So, once we substitute this particular part in the above equation what do we hope to get?

$$=> \frac{(r^2 R)_r}{r^2 R} + \frac{(sin\theta\ \Theta_\theta)_\theta}{r^2 \sin\theta\ \Theta} = k_0^2$$

You know after simplifying things we will get general thing that the solution for this particular part, this will separate out and we will get the legendry function. So, the solution for that.

$$(1-z^2)\,\frac{d^2\Theta}{dz^2} - \; 2z\frac{d\Theta}{dz} + C\,\Theta = 0$$

So, readers are urged to or students are basically viewers are urge to have a look at the book by David Black stock on fundamental of acoustics or possibly canceller and fray for a detailed discussion and development of these equations and here of course,

$$C = n(n+1), \qquad z = cos\theta$$

If basically separate out this form evaluate it to be a certain constant and substitute z is equal to cos theta and carry out the algebra to arrive at this thing. The solution of above equation this is for theta and the reason that, I am telling because this will have a dependence on n. So, here n is integer changing from 0, 1, 2 and so on. So, the solution of this thing is gain by the legendary function of integer order which is basically a polynomial

$$\Theta = P_n(z) = \; P_n(\cos\theta)$$

$$-1 \leq z \leq 1$$

$$0 \leq \; \theta \; \leq \; \pi$$

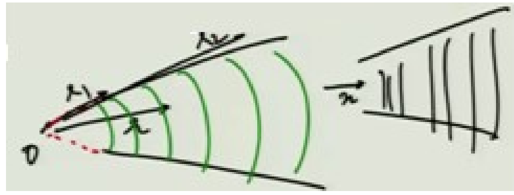
So, we have this kind of a thing or theta alternatively ranges from 0 to π. Now, it so happens that once we decide on a particular n value, the spherical wave equation can then or the radial part of the spherical wave equation turns out to be after obviously some simplifications the following. So, we get the following.

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k_0^2 - \frac{n(n+1)}{r^2}\right)R = 0$$

$$n = 0$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + k_0^2\,R = 0$$

When we substitute n is equal to 0 in the above equation we will get the following. Note that this particular form, this particular equation is identical with this part, with this equation where the only difference is that x is replaced with r. And here p was a function $\tilde{p}$ was a function of the axial coordinate from measured from the hypothetical epics x.



And here it is measured also from the well hypothetical epics o and this is the radial coordinate. So, r then is measured from the hypothetical epics of the cone o and r can be anything here if it is at this point it is r_1 at this point it is r_2. So, basically what it would mean? It would mean spherical waves that go like this.

But axial waves how do they go? They go like this, but the form remains the same. And here we need to specify the radius from this thing and here we basically will specify the axial distance. So, there is a difference x is measured along the axis here r is the radius it can have any angle, but we specify to be r_1 r_2.

So, basically the Webster's horn equation for one-dimensional waves and conical ducts is identical with a spherical wave equation for the zeroth order mode that is basically there is no dependence on the polar angle and of course, no dependence on the azimuthal angle as well.

So, where do we go from here? We its now I guess time to derive the expression for the velocity. So, we again resort back to the linearized Euler equation remember. Momentum equation for the conical duct does not change or basically for any duct to the gradually varying cross section area where the flare is not varying very rapidly.So, your momentum equation is pretty much the same as

$$\rho_0 \frac{\partial \widetilde{U}}{\partial t} = -\frac{\partial \tilde{p}}{\partial x}$$

Now, that we know it is of this form,

$$\rho_0 \ j\omega\widetilde{U} = -\frac{\partial}{\partial x}\left\{\frac{1}{x} C_1 \ e^{-jk_0 x} + \frac{1}{x} \ C_2 \ e^{-jk_0 x}\right\}$$

So, after some algebra basically once we need to take a partial derivative of this part C_1 times this particular thing.

So, after some algebra we will figure out that u tilde x is given by $j\omega\rho_0 x$ and in the bracket. So, this is what I already worked out is given by

$$\rho_0\,\widetilde{U}(x) = \frac{j}{\omega\rho_0 x}\left\{\left(-jk_0 - \frac{1}{z}\right)C_1 e^{-jk_0 x} + \left(jk_0 - \frac{1}{z}\right)C_2 e^{jk_0 x}\right\}$$

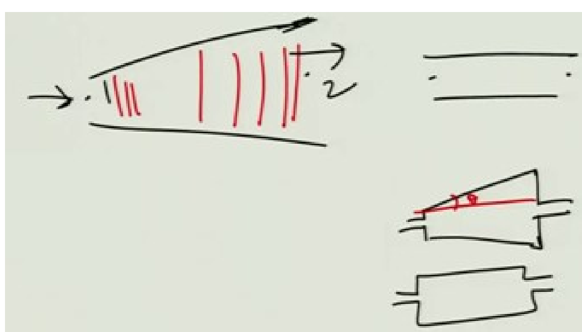
So, we have the expression for the particle velocity. Now, clearly if we multiply both the sides by ρ_0 and get rid of the term here and multiply here by C_0 and then C_0. So, here you will get we will get k_0. And actually there will be in x sitting here not this thing will be an x here.

So, we will get this thing. So, this will give you the and actually when you multiply this by the cross sectional area at any point this multiplied by

$$\rho_0\,\widetilde{U}S(x) = \frac{jS(x)}{k_0 C_0 x}\left\{\left(-jk_0 - \frac{1}{z}\right)C_1 e^{-jk_0 x} + \left(jk_0 - \frac{1}{z}\right)C_2 e^{jk_0 x}\right\}$$

So, now what do we need to do for a conical duct is that; we need to fix our axial coordinates that is where are we heading at. So, here at x_1 and x_2 like I have been talking about.

So, once we get those values or once we decide what is our x_1 or x_2 value or the length l and obviously, the radius at both the points. So, then we can formulate the transfer matrix. To do so we will probably take up the transfer matrix derivation in the next class lecture 4. And after that for a generalized element and what we will do is that our objective will be to get the transfer matrix for a conical duct from point this point to this point exactly like we got for a uniform duct.



So, we will get the transfer matrix you know in this form where we will get the 4 parameters between point 2 and 1 like this. And then you will be good learning exercise if you go to the following things in next class.

$$\begin{Bmatrix} P_1 \\ V_1 \end{Bmatrix} = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} \begin{Bmatrix} P_2 \\ V_1 \end{Bmatrix}$$

Firstly, derive show that the transmission loss you know for a conical duct. Basically a diffuser kind of an element even when there is no sudden expansion, remember these are progressive discontinuities. So, you know the waves are going like this. So, the idea is that because the area changes gradually it is a finite some finite area changes there.

So, transmission loss will never be 0 even if you do not have any sudden area discontinuity is like you know just like we have a system here. But if you have a system like this thing instead of a diffuser kind of an element if you have a muffler with a sudden expansion. But, only difference is that now we do not have uniform section; we have the section with gradually varying flare a conical muffler the muffler which is conical.

In such a case we will also derive that one-dimensional a transmission was based upon one-dimensional theory without mean flow effects of course. So, that will be our goal for the next class next lecture. And in the final lecture of this week, we will probably talk about some advanced techniques like segmentation approach and all that; to and how they can be used to handle complicated duct more general theories.

So, thanks a lot, thanks for attending stay tuned.