Introduction to Fluid Mechanics and Fluid Engineering
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Lecture - 29
Dynamics of Viscous Flows: Navier Stokes Equation (Contd.)

We were discussing about the derivations of the equation of motion for viscous flows last time. And let us just briefly see that what are the important conclusions that we had out of our previous derivations. So, first we started with the general equation of motion and we started with the integral form, from the integral form we derived the corresponding differential form with certain assumptions.

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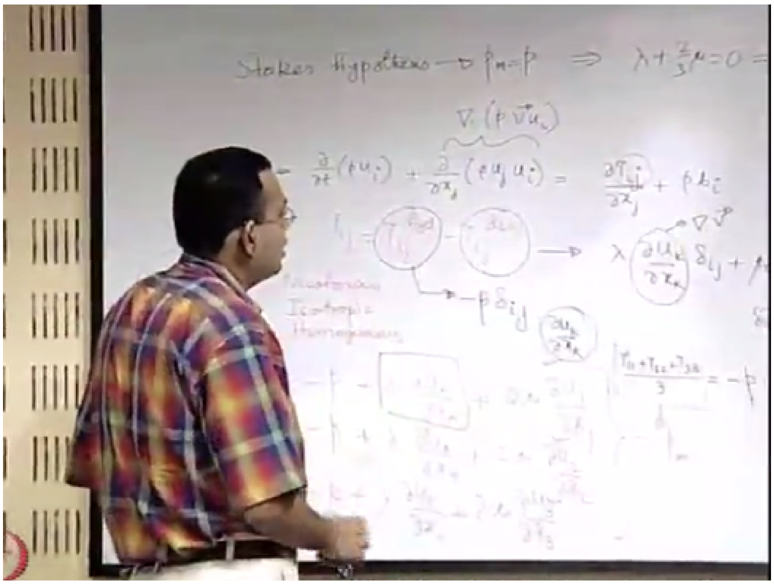

And the differential form for linear momentum conservation along the direction i was written in this way, in the Cartesian index notation. So, this is the first thing that we did. Of course we have the corresponding vector forms and we also discussed. We also discussed what is going to be the vector form. Now with the vector form you can clearly see that you may write this as the divergence of rho V into the velocity component that you are talking about.

So, you may use interchangeable forms. Our next objective was to figure out that how we may express tau ij in terms of the radiance of velocity and the fluid property may be one fluid property or 2 fluid properties or whatever. We figure out that in a most general case you might have required 81 independent constants to express the tau ij, the deviatoric component of the tau ij with the rate of deformation.

But at the end we found that if you have a homogeneous and isotropic fluid it boils down to 2 scalars and we could identify what are the important scalars for expressing the deviatoric component of tau ij. So, the first important thing is that what are the assumptions under which this equation is valid? Yes, what are the assumptions? Well, continuum hypothesis is valid of course that is otherwise you cannot write continuum conservation equations.

But most important is that you have stationery reference frame with respect to which you are writing the equations. So, if it is not a stationery reference frame and a very general case of moving reference frame with rotation and acceleration then you would have correction terms. And those correction terms may easily be derived by starting from the Reynolds transport theorem with the non-accelerating reference frame as the basis.

So, if you convert that to a differential form you will get some extra terms and those extra terms are because of the transformation of the reference frame. Now, when we wanted to express the tau ij the 2 components of the tau ij were like one was the hydrostatic component and the other one was the deviatoric component. The whole idea was that you have a component of the stress tensor which is a function of the deformation.

And a component of the stress tensor which is not a function of the deformation and whatever is not a function of deformation should also be prevalent in a case when the fluid is at rest. Because when the fluid is at rest then it is a state of stress is something which is denoted by a simple normal stress condition which is because of the pressure distribution and that prevails or that manifest in this hydrostatic component.

It does not mean that when the fluid is at motion it is not there, it is definitely there but something additional is there which is attributed to the deformation of the fluid and that we figured out that how we relate that with the rate of deformation. So we related this deviatoric stress component with the rate of deformation by this one, where lambda and mu are 2 material property dependent constants.

We figured out that out of our basic understanding of the viscosity and the effect of viscosity that this mu is same as their viscosity coefficient that we learned sometimes back in the elementary discussions in our course. And lambda is a constant which is sort of known as the second coefficient of viscosity is related to the volumetric deformation of the fluid element. Because this term is nothing but the divergence of the velocity vector.

See here there is an index k which is a dummy index and it is repeated 3 times. So it is like first partial derivative of U1 with respect to X1 then + U2 with respect to x2 and U3 with respect to X3. So, that is the divergence of the velocity vector. So, since divergence of the velocity vector gives the indication of the rate of volumetric deformation, so this part is related to the rate of volumetric strength and this part is related to the angular deformation.

An angular deformation we have seen that it is the shear deformation that matters for the stress because the rotational component is a sort of effect that does not give rise to any viscous stress. So only the symmetric part of the deformation is what is going to give rise to the viscous stresses. What are the assumptions under which this is valid? See, Newtonian fluid is one important thing, so you have Newtonian fluid.

The most important property that we have used is isotropic fluid. Homogeneous fluid will imply that this lambda and mu are not dependent on x, xi that is position. So, if lambda and mu are dependent on position still you may make a bit of more general form of this one which is a form of a non-homogeneous fluid element of material. But if you assume that lambda and mu are position independent then you say that it is also homogeneous.

So, let us for the time being assume that it is also homogeneous that is lambda and mu are position independent and so with these assumptions coupled with the assumption for stationery reference frame it will now be possible to substitute the tau ij in terms of this one. So, we have to keep in mind that the tau ij also had a hydrostatic component.

And this hydrostatic component is something which we realize that is because of the pressure distribution. And it is a normal component so we use the delta ij notation when i = j only this is 1 otherwise it is 0. So, that means when you are talking about a normal component of stress then only this term comes into the picture. For all other cases it is 0. So, remember that delta ij = 1, if j = i and = 0, if j is not = i.

Now, we will try to see or we will try to find out that what is this relation that is trying to give us. See, when we were talking about the equation of motion this equation of motion had several unknowns. So, one of the unknowns or rather some of the unknowns are the velocity components. So, U1, U2, U3 and the other unknown was the component any component of the stress tensor. So, 6 independent components of the stress tensor tau ij.

Now, we have expressed the stress tensor in terms of the velocity gradients or no new unknowns arise because of these. But because of pressure there is a new unknown that has come into the picture. So, if you substitute this form in the equation of motion you will get how many unknowns? You will get the velocity components, independent velocity components, say U1, U2, U3 and pressure as the unknowns.

We will try to see that how these equations and unknowns maybe closed. Because one important understanding is when you utilize this equation you have to, your understanding is you have to solve for velocity and pressure, these are the unknowns. And at the most you get 3 independent linear momentum conservation equations. For angular momentum conservation you do not get any extra equation.

Because tau ij = tau ji that has already been incorporated in some of the derivations. You do not get anything new out of that and we will see that still with 4 unknowns and sort of 3 independent equations apparently how we can close the system. But before that let us try to look into some of the implications of these terms. So, we will first concentrate on the term P, the pressure.

Sometime back we were discussing about a concept called as the distinction between thermodynamic pressure and mechanical pressure and we will revisit that again here. So, when we are talking about the pressure here this is what how do you feel the pressure or how do you relate the pressure with other parameters at a point. So pressure at a point may be related with any other parameters at the same point through the equation of state.

So the equation of state may not be as straight forward as that of an ideal guess for any general substance. But any general substance will have some form of equation of state. For ideal guess it is like pv is equal to that type of equation of state is there. So, thermodynamically what it means is that if you know at a point what is say density and temperature you find out what is pressure by having a functional relationship with those.

So, if you have 2 independent intensive thermodynamic properties by that you are able to describe pressure as a thermodynamic property. So, it is a thermodynamic property because it is a function of 2 other independent thermodynamic properties. From the ideal guess it is like you are expressing it directly in a very simple relationship with pressure and temperature for other substance density and temperature.

For other substances it may be much more complicated and not as straight forward. So, this pressure that we are having here we call it a thermodynamic pressure. We will see that how we distinguish it from a mechanical pressure but to do that we will first write the normal components of the stress. So, we will write tau 11. So what is tau 11? Tau 11 is –P, why we are only writing the normal component is because if we write the shear component this P will not come into the picture.

So, if – P then + lambda, so even with lambda that also does not come into the picture for a shear component and that is quite obvious because a volumetric deformation should not be confused with a shear and then + 2 mu, right. Similarly, you can write tau 22, what is that – P + lambda the same thing. So, here i = 1 right and tau 33.

So, if you now add this together, all these 3 expressions what you get is something which is tau 11 + tau 22 + tau 33 let us say we are interested to find out the arithmetic average or arithmetic average mean of that. So, we divide it by 3. So if we add and divide by 3 right hand side the first term becomes – P. Then if you see, if you add the last 3 terms together this this and this you will get this term, right.

So, you can take here 3 lambda and here 2 mu as common with the del Uk del xk term and that when divided by 3 will become lambda + 2/3 mu**,** okay. This is very simple and straight forward. Now the term which is there in the left hand side this we call as – of mechanical pressure. So, what is mechanical pressure? Mechanical pressure is a sort of arithmetic average of the normal components substrates.

The sign is adjusted because the positive normal component of stress is sort of taken as tensile and pressure by definition is compressive. That is why this with a – sign it is adjusted. So, this is a definition of mechanical pressure. We will now try to understand what physical meaning does it convey, but first the definition. So, you can clearly see that we have a relationship that relates the mechanical pressure with a thermodynamic pressure.

And the first and foremost observation is that in general they are not equal because you have a correction term here. The question will be that when they are equal, when they are not equal and what is the consequence of that equality or inequality? But to simplify things out Stokes came up with hypothesis which is known as Stokes hypothesis. Stokes hypothesis was that mechanical pressure = thermodynamic pressure.

We will see that under what conditions this hypothesis works and under what condition this hypothesis may not work. So, Stokes hypothesis says mechanical pressure = thermodynamic pressure that for a general substance will worth only if you have lambda + 2/3 mu = 0. That means lambda = - 2/3 mu. Since this is a hypothesis obviously there was no proof of that but there was some expectation based on which this hypothesis was postulated.

And we will try to understand that expectation. See, mechanical pressure what does it represent? It sorts of represents the effect of the translational mode of energy of molecules. Whereas, a thermodynamic pressure, it sorts of represents the translational, rotational, vibrational all modes of energy of a system of molecules. So, when you see the distinction, the distinction will be because of the rotational and vibrational modes of energy.

So, one is representing only the translational modes of energy and one is representing the translational, rotational and vibrational modes of energy. But there are certain cases when you will not have a rotational and vibrational mode of energy. You have purely translational mode of energy. Think of a dilute monoatomic gas, so there you have a translational mode of energy is the sort of is the mode of energy even when you look into it thermodynamically.

So, in that case you do not have any distinction between mechanical pressure and thermodynamic pressure and then Stokes hypothesis is not an hypothesis it is exact reality. There is no deviation. But all substances even if you think of gases or compressible substances all substances are not dilute monoatomic gases. And therefore for all substances you will not have mechanical pressure = thermodynamic pressure.

But we have to understand that what is the expectation. The expectation is let us say that you have a bubble. Inside the bubble you have some gas and let us say that the bubble is changing its state. So, when it is changing its state, what is happening? Say there is some change in temperature. So, because of the change in temperature may be it is changing its volume and maybe there is also a change in pressure.

Now, how this pressure volume and temperature will be related? It depends on the mannering which it is changing its state or the thermodynamic process. So, we are not going into the specific thermodynamic process but we are just keeping it a bit abstract that by some thermodynamic process it is changing its state from say state 1 to state 2 to state 3 like that. So, when it is changing its state you have initially a pressure P, say P1.

Now, in the new state you have a pressure P2. So, when you change the state from pressure state from 1 to 2 and there is a change in pressure from P1 to P2, now if you want to have the entire effect manifested in the form of mechanical pressure then you should allow a time which will make it locally equilibrate. That means the sudden change in pressure because of a change in thermodynamic state will give rise to a change in vibrational mode of energy, change in rotational mode of energy, everything.

And eventually all these changes in equilibrium will be manifested in form of a change in translational mode of energy it is only manifested by mechanical pressure. So, you should allow a time over which there is a conversion of all those modes of energy into the translational mode of energy that is captured by the mechanical pressure and that requires the time. So this is called as a relaxation time.

So, the relaxation time is very important. What is the relaxation time? So, you are having a system, you are imposing a change to the system and the system must have sufficient time to absorb that change and have changes in properties by local equilibrium by having a new state of local equilibrium because of that change. So, it should have a time over which it changes that state of equilibrium. It cannot be instantaneous.

Because it cannot be instantaneous there is a time scale which is called as relaxation time scale. Now, if the time change that is taking place is so fast that the time scale of the change is faster than the relaxation time. Then what will happen? Then the material will not be able to come to that equilibrium before a new change of state has taken place. So, it is like the material is trying its best to come to a new equilibrium state at each and every instant.

But till it is successful again a new change has been imposed on it. So, that is a very rapid process. Let us say, that you have such a process in which a bubble is rapidly expanding and contracting. So, if the bubble is rapidly expanding and contracting and it is doing with a very high frequency then that high frequency means it has a very short time scale over which the change has been imposed and this change what is being imposed is externally imposed.

Maybe because of a change in thermal environment maybe the temperature is fluctuating very faster or whatever. And inside that the poor material inside the bubble is trying its best to adjust to that. But it is not possible to adjust to that if the change time scale is very, very rapid and then it cannot equilibrate locally to come to a state where is mechanical pressure becomes = thermodynamic pressure.

So, mechanical pressure may not be = to the thermodynamic pressure if the time scale of change is faster than the relaxation time scale. If that time scale of change is slower than the relaxation time scale, then yes. Then whatever is the change in thermodynamic state the system adjusts to that eventually that mode is entirely transformed into a translational mode of energy reflected in form of a mechanical pressure.

So, the entire vibrational and rotational modes of energy should also be done in the form of a sole translational mode of energy. And that will be manifested in form of mechanical pressure. So, it is a sort of like inter conversion of one mode to another mode and you have to give it that enough time. For which, during which it will do that inter conversion.

So the summary of this understanding is that if you have a very rapid change or a very rapid time scale or very fast time scale over which a change is imposed a thermodynamic change is imposed on the system then the system may not be able to attain equilibrium locally based on its relaxation time which may be slower than that rate of change that is imposed. And then it will not be able to achieve a state where mechanical pressure is = thermodynamic pressure.

And then Stokes hypothesis will be violated. But for most of the practical applications that we are concerned about the Stokes hypothesis. Because usually the relaxation time scales for fluids these are very fast. So, usually the changes which are imposed on a system are not faster than that. So, whatever change is imposed on a system at least locally at a point it tends to attain equilibrium and that is call as local equilibrium.

It is not a global equilibrium that means it is not at throughout the system at each and every point property is as same, it is not like that. But at each and every point at least it sorts of attains an equilibrium where its properties are related by equation of state and eventually the inter conversion from one mode to the other is possible that time is available. And that is why stokes hypothesis works for most of the problems.

Only for a few rare cases where the change is so rapid in terms of the time scale there it may not work. Otherwise, it is working and therefore when it works for the general case see, you do not have the divergence of the velocity vector to be 0 in a general case. Only for incompressible flow it is 0. So, for a general case seems the divergence of the velocity is not 0 in a general case. Therefore, the other term must be 0.

So, this lambda is = - 2/3 mu is something which is obeyed for all fluids. There are certain fluids for which we do not care whether this is obeyed or not. What are those special fluids? One, we discussed is the mono atomic gas. So, we have a mono atomic gas you see that it has to be obeyed not that we do not care whether it is obeyed or not. We do not care whether it is a hypothesis or not. Then it is exact reality. So, it is not a hypothesis any more.

So, for a mono atomic gas this exactly will be valid and therefore no question of approximating mechanical pressure with thermodynamic pressure. They are like exactly equal for a dilute mono atomic gas. But for other gases this is not an exact relationship but this works because of the time scales that we talked about. For incompressible flows we do not care about these.

That is the only substance for which we do not care about this because its multiplier is 0. So, for an incompressible flow since the divergence of the velocity vector is 0 the consequence is that mechanical pressure will be exactly = thermo dynamic pressure. And the reason is that for incompressible flow you do not have to satisfy a new equilibrium state through such an equation of state. Because equation of state is not very explicit for a incompressible flow.

You do not have pressure density temperature relationships of the kinds, explicit kinds that you have for compressible fluids. So, for incompressible flows you have mechanical pressure exactly = thermodynamic pressure. So, even if you say that I do not care whether the Stokes hypothesis is valid or not, still it is okay for incompressible so far as the analysis goes.

Now, the other important observation is if you look into the lambda see the viscosity of our fluid is positive that we have seen earlier. So, it clearly shows that lambda which is the second coefficient of viscosity will be negative. What is the physical consequence of that? So, look into say maybe one of the expressions say tau 11. So, in the expression for tau 11 if lambda is negative then you just look into or concentrate on the term that contains lambda.

It is lambda into the divergence of the velocity vector. So, let us say that the divergence of the velocity that is positive. That means what? That means volumetrically the material is expanding because it is the rate of volumetric deformation and if lambda is negative that means the corresponding stress actually is reduced. So, more is the value of the divergence of the velocity vector, the corresponding proportionate change is actually less.

Because it is being multiplied with a negative term. What it means? It means that if a material if a fluid element is already expanding then the proportional enhancement in stress to expand it further is less. If it is implicit already expanding. So, if you have a divergence of the velocity vector positive that means it has a sort of tendency to expand. And therefore the proportional enhancement in the stress that is necessary to expand it further is actually not an enhancement but a deduction.

So, but if it not already expanding, so if divergence of the velocity vector is negative that means the stress that is associated with the further expansion is more if the divergence of the velocity vector magnitude wise is more. So, that is the important consequence of the negative sign of lambda. Not only that it may be shown we will not come into that here because it requires a bit of thermodynamics.

And that is beyond the scope of this course that not only you have mu > 0 you should also have lambda + 2/3 mu > 0. So, = 0 is the special case which is the stokes hypothesis. If the stokes hypothesis does not work still it will be > 0 and the reason is that otherwise it would highlight the second law of thermodynamics. So, because there is not enough scope to discuss about that we will not go into that how this comes.

It is very easy ones you go through the thermodynamic course you will be able to appreciate that it is very straight forward to figure out that why but important thing is to understand at least at this stage that second law of thermodynamics give you a restriction on certain natural processes or on the ranges of certain properties. So, you cannot have all properties in all possible ranges.

Because the second law of thermodynamics restricts the cases that at the end of the process the total entropy of the system and surrounding should increase together. Entropy of the system and entropy of the surroundings together should increase at the end of a process and if this is not satisfied then that is violated. And therefore, it is possible to find out also the possible ranges of parameters.

See, that is where there will be a difference between doing abstract mathematics and the physical reality. When we are talking about an abstract mathematics and if the second law of thermodynamics you do not care you may see like up to these no derivation has restricted really that this should be >= 0. You could use any value. Say you may say that well; I will use a positive mu.

Till now we have discussed that we should have a negative lambda but with many combinations of positive mu and negative lambda this may not be satisfied. But we have to satisfy these otherwise it will violate the spontaneous change in a natural process. That should be according to the second law of thermodynamics.

Now, with this bit of physical understanding we will try to simplify the equation of motion based on the derivation that we have made till now. So, the first simplification is we will put back the tau ij expression in the equation of motion. So, if you put back the tau ij expression in the equation of motion then it will boil down to, okay.

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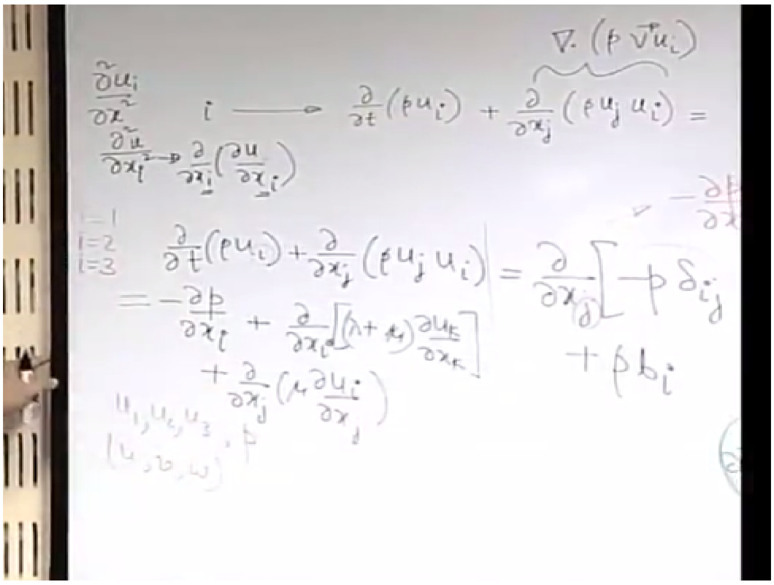

So, we have done nothing special we have just substituted the tau ij in the equation of motion. So, we can make certain simplifications. Let us say we are bothered about the first term. See, delta ij = 1 when j = i. So, the first term will become, so you replace the index j here with i then only this is 1. Then next, for the next term also it is the same. For this term replace the index j with i because of the multiplication with delta ij.

For the term in the last bracket you have basically 2 terms. The first term is this and the next term is, okay. See, when you have a second derivative and you write in an index notation you have to be very careful say you are writing this term, okay. So, if you write it in an index notation say this is ui. So, if you write it in this way it will give you a false impression that the index i is only ones but fundamentally it is like this one.

So, actually it is a repeated index. So, you may always use this notation of del square but you have to be careful that that is just a notation. It is basically a first order derivative repeated ones. So, you actually have a repeated index. This will not give an appearance that it is a repeated index. So, here that is why we have just kept it the form which is like a first order derivative operating on another first order derivative.

So, that the sense of the repeated index is preserved. Now, if you see that this is, okay first of all let us say that we are dealing with constant properties. So, if you are dealing with constant properties maybe the mu 1 you can take out the derivatives. Now, if you take the mu out of the derivatives see here say for this term you take mu out of the derivative if it is a constant it may not be an absolute constant mu may be function of time say as an example.

But important thing it is position independence because these are derivatives with respect to position and for a homogeneous fluid it has to be like that. So, our earlier assumption of homogeneous fluid is not very inconsistent with this one. Now, then it is possible to switch over with these derivatives. That is, you may have i here and you may have j here, why? Because it is a continuity of the partial derivatives, second order partial derivative.

So, if the second order partial derivative is continuous say you have this you can write as this one. If the second order partial derivative is continuous, so you may either do derivative with respect to x first and then y or with respect y first and then x still they will give you the same result if it is a continuous second order partial derivative. So, here we are assuming that it is a continuous second order partial derivatives.

So, there is no discontinuity that we are assuming and in a continuum hypothesis that is something what we should presume anyway. So, then what does this term become? Maybe again you may put mu back inside because mu is a constant, so you can play with it either inside or outside. So, then it will become, right, okay. This is it possible to write it in this way? mu * you replace j with k, yes or no? Yes, because j is a dummy index.

So, there is a summation over that from 1 to 3. So, basically you are substituting this first 1, 2 and 3 and adding. It does not matter what name you give it to this j, k, l, m whatever, right. So, this is called as a dummy index. That means what you get in the right hand side let us therefore, the right hand side, so this is = -. So, you may club 2 terms, which terms? You may club one term like this with another term like this, right.

So, you can write this into lambda + mu and then, okay. So, when we have come to this stage let us recall what are the assumptions which are valid which are there for which it is valid. What are the assumptions? So, first we started with a stationary reference frame then the next assumptions Newtonian fluid and homogeneous isotropic fluid. Stokes hypothesis not yet imposed, right.

Before imposing the Stokes hypothesis let us make another simplification by now looking into the number of equations and number of unknowns. See, whenever we are solving a problem mathematically we are always bothered about the number of equations and number of unknowns which somehow we try to match and when we try to match we have an important understanding that number of independent equations should be same as the number of unknowns.

So that you can solve for the unknowns in principle from the equations. It is not so important first to know how you solve but at least it should be a well posed system which is solvable. So, to understand that if you see here now let us consider a Cartesian system. So, when you are writing ui see i is a free index. So, you have 3 equations for i = 1, i = 2 and i = 3, which are like linear momentum conservations along x1, x2, x3 or xyz.

3 mutually orthogonal coordinate axis. The number of unknowns you have U1, U2, U3 which is like UVW in the other notation and pressure. So, we have a 4th unknown. And that is one of the big troubles you do not have an explicit governing equation for pressure. And to have that we only have a resort that we have 1 equation extra that we can always use that is continuity equation.

But that does not explicitly contain pressure but still it contains the unknown U1, U2 and U3. So, at least, if you couple these with the continuity equation that will close your number of equations and number of unknowns. They will match. So, we are always going to use these equations in conjunction with the continuity equation. That must be valid. So, let us now try to simplify this equation by utilizing the continuity equation.

Till now we have not simplified it using the continuity equation. So, this form of the equation is known as the conservative form of the equation because see have not disturbed the form which has come from the control volume conservation. So, we stared with the control volume conservation by using the Reynolds transport theorem and we just convert it to its corresponding differential form. But we have not disturbed the form.

So, this still is something which you can recollect that has come from a conservation principle directly. We have not simplified it further. But now utilizing the continuity equation we will simplify these further in the left hand side and that will bring us to some form known as non-conservative form. So, to do that let us right the left hand side.

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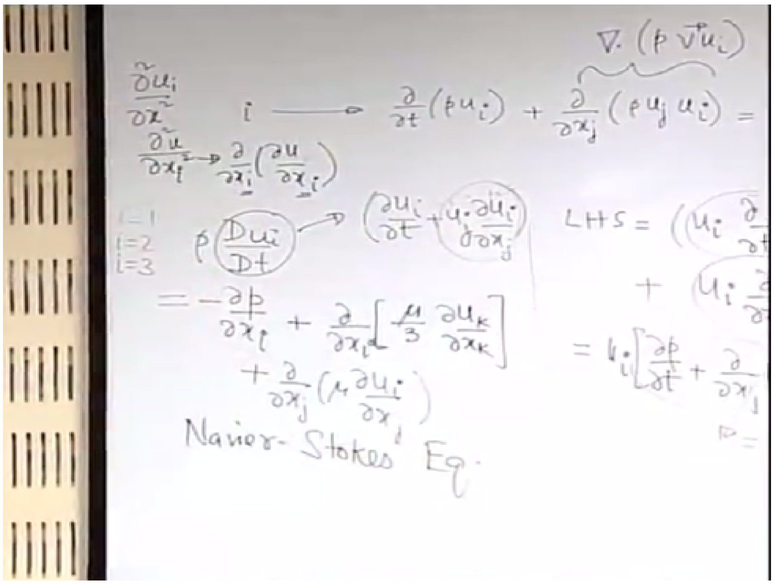

So, we will use the product rule of the derivatives and simplify it. So, you can write this as ui that is the first term. Then for the next term you club up rho uj as one term and ui as another term and use the product rule of the derivatives. So, you have, okay. Now let us combine these 2 terms where you have ui as common. So, if you combine these 2 terms you have ui and then let us write the remaining term, okay.

Clearly, what we can see that the first term in the square bracket is identically = 0 by the continuity equation. So, this is 0, exactly = 0 by the continuity equation, okay. So, you come up with a simplified form of the left hand side which is nothing but just the second term which is remaining. And this term you can write in terms of the total derivative capital DDT of what ui by the notation of capital DDT that we have discussed earlier.

So, the equation of motion maybe simplified now to this form, what is that form? So, we just change the left hand side with this simplification which is as good as, okay. Sometimes when this form is used, it gives an illusion. What is that illusion? As if it is valid for a constant density fluid. So, this is where knowing the derivations are important.

If you do not go through the derivations and just look into it, use of common sense might lead you to a conclusion that as if rho is a constant that is why it is out of the derivatives and intuition wise that is correct if you do not know the derivation. But if you know the derivations you will realize that this rho has come out not because that it is a constant it may be still being a variable.

It has come out by use of the continuity equation and most general form of the continuity equation not for rho = constant, okay. So, then is this valid for compressible flow? Yes, it is very much valid for compressible flow. Because we have not simplified any further by using the concept of incompressible flow or the consideration for incompressible flow.

So, the assumptions over which these are valid are the same as the assumptions for which this was valid in its previous form that is a conservative form. So, these are just 2 different forms and one may interchange one form with the other. Usually, we will see that, see these equations are not very straight forward. See, these equations are some of the very complicated equations to solve applied physics and applied mathematics.

So why? See these are coupled non-linear partial differential equations. So, these are not linear partial differential equations. These are coupled because you have these at least these equations + the continuity equations. So, at least the 4 equations which are there which are coupled with each other because each equation contains some of the terms which are related to the other highly non-linear pressure is residing on one some of the equation but you do not have separate governing equations for pressure.

So these are some of the challenging aspects of this equation. So, in most practical cases you cannot solve these equations analytically and you have to use computations, numerical computations to solve this equation and that leads to our entire specialized branch of fluid mechanics known as computational fluid dynamics or CFD.

So, that is, CFD is basically all about the numerical solutions of the partial differential equations of like momentum conservation continuity and then these may be again coupled with heat transfer, mass transfer if those are also involved. So, all these couple partial differential equations, the other transport equations that is the heat transfer equation or the mass transfer equation at least they are not non-linear.

But the momentum equations they are non-linear because you have terms like if you look into this term like look into uj into the partial derivative of ui with respect to xj. So, this is actually the non-linear term. So, if somehow this term is not there we will see under what conditions these term not there. Then it simplifies the situation considerably and there are certain special cases in which analytical solutions are available.

And we will work out some of those analytical solutions as a part of this course which are known as like exact solutions of the equations of motion. Now next thing is that we have not used the Stokes hypothesis and let us use the Stokes hypothesis at this stage. So, you substitute lambda = - 2/3mu. So, then lambda + mu, lambda + mu is known as bulk viscosity of a fluid, it is just a name.

So, the bulk viscosity then it will become mu/3. So, this term will become mu/3, okay. So, we have utilized now the Stokes hypothesis in the equations of motion. So, the corresponding equation of motion will now become in place of this we will substitute the Stokes hypothesis. Let us write it a bit cleanly. So, mu/3, so we started with the Navier's equation of equilibrium which is basically this one substituted the constitutive form of tau ij for a Newtonian fluid for homogeneous and isotropic fluid material.

And then now we have substituted Stokes hypothesis and then this is known as Navier Stokes equations. So, Navier's equation for a Newtonian fluid + homogeneous isotropic special type of Newtonian fluid and most important your fluid that obeys Stokes hypothesis. That is known as the Stokesian fluid. So, just like a fluid that obeys Newton's law of viscosity is known as a Newtonian fluid.

A fluid that obeys stokes hypothesis is known as Stokesian fluid. So, the fluid is the very special fluid for which we are writing this equation Newtonian and Stokesian fluid and with homogeneous and isotropic fluid properties. So, that is known as Navier Stokes equation. When you have, so this form of the Navier Stokes equation is valid both for compressible as well as incompressible flow.

When you come down to the case of an incompressible flow as we have discussed that it really does not matter whether stokes hypothesis is there or not whether it is violated or not.

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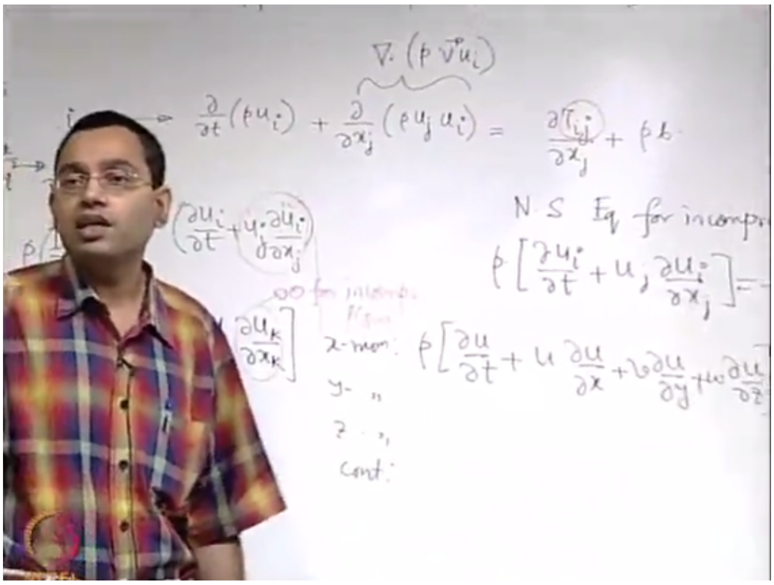

So for incompressible flow we can clearly see that this term is not there because this is 0 for incompressible flow because that is the rate of volumetric deformation. The most fundamental definition of incompressible flow is that you have 0 rate of volumetric deformation. So, when you have that incompressible flow it does not matter whether lambda is = - 2 third mu. This term is identically = 0.

So, then if you write the Navier Stokes equation for incompressible flow which we will be dealing with in our subsequent exercises, so that will be rho, okay. So, although Navier Stokes equation it actually does not requires Stokes hypothesis. It is just good enough to have a Newtonian fluid with homogenous and isotopic fluid properties. And we will be often using this form of equation for our calculations or for the exact solutions that will be deriving in the subsequent classes.

So, it is possible to write these in a vector form also. So, if you want to write this in a vector form or let us first write it in a well-known U, V, W component form. So, if you want to write it in a U, V, W component form you have. So, this is write if you want to write by using the notation U1 = U, U2 = V and U3 = W and write it in an expanded form. This expanded form is often used if you solving it analytically because you are interested to get a full feel of the terms.

So, if you see that what do these terms indicate. Left hand side is like what when you are writing rho into capital DDT of ui. So, rho is mass per unit volume and this is acceleration of flow. So, this is like mass * acceleration of flow per unit volume and by Newton's second law the right hand side therefore should be force per unit volume. So, these, so it is just like Newton’s second law of motion written for fluids in this frame.

Nothing more than that. So, the right hand side what are the forces which are there? This is the force due to pressure gradient. This is the force because of viscous effects and this is the body force. If, so the body force may be anything like it may be because of gravity or whatever but it is important to recognize that it is something which is a very general force that we have kept here.

So, there may be effects of electrical field, magnetic field or many other forces may influence the fluid flow and all those extra effects will be appearing in this form of the body force. So, what we say at the end is that if you have the Navier Stokes equation, the Navier Stokes equation should be coupled with the continuity equation. So, this is, this we say that this is the x component of the momentum conservation.

Similarly, you can write the y component and the z component. So, y z component and the continuity equation. So, that will be the coupled equations which will involve your all equations and unknowns and we should be in a position to simplify those for specific problems to solve the equations to get the velocity and the pressure fields. So, that will be our objective from the next class onwards.

We have to keep one thing in mind that these Navier Stokes equations whatever we have derived here, see there are certain things which we tend to take as rituals. So, Navier Stokes equations in fluid mechanics have been taken as rituals for long and it was thought that these equations are giving the correct physical meaning for almost all cases.

But there are certain cases where it was found that this equation is not giving the right picture although continuum hypothesis is not totally violated, still it is not giving a very consistent picture. I will give you 1 or 2 very simple examples. Let us say that you have a molecule, gas molecule which is subjected to a very high density or temperature gradient.

So, when it is subjected to a very high temperature gradient because of the temperature gradient the gas molecule may start moving and this is sort of called as self-diffusion. So, it moves within its own medium because of strong local gradients of temperature or density.

Now, that is nowhere manifested in this equations. So, if you just use these equations you will get a 0 velocity.

Because you do not have any term in this equation which is representing a sort of local density or local temperature gradient driven motion of flow. And this is an example which is known as that. So, or sometimes it is called as diffuse because of the self-diffusion. So, these types of motions which are achievable in highly redefied systems this is not manifested in these equations.

So, what we try to do is that we try to correct the Navier Stokes equation and try to incorporate those terms. So, extra terms, so that it may be able to physically incorporate the effects of those like thermo types of motion and that is how we extended the Navier Stokes equations.

So, this beyond the scope of these study or these class to discuss that how we did it and what are the important remarks but only one observation that is one of the thought processes is that I would like to share with you is that whatever you are learning, see utmost times as teachers we tend to believe or we tend to tell you that as if we are Gods we know everything and it is your responsibility to like digest that.

But most of the things actually we do not know and I think our greatest objective should be to tell you that what we do not know rather than what we know. But first we tell you what we know because it is easier for you to digest that. So, the Navier Stokes equation in the very classical form was something which has a mystery which did not contain something. We try to correct that I am not saying that we corrected it fully because we could do it only for ideal guesses.

But at least it was an extended form. So, the long history of how the Navier Stokes equations were there at least over the last 2-3 years we could change it and extend it for more general types of flows. So, with this we stop this lecture today and will continue in the next class. Thank you.