Introduction to Fluid Mechanics and Fluid Engineering
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Lecture - 30
Some Exact Solutions of Navier Stokes Equation

In our previous lecture, we were discussing about the derivations of the Navier-Stokes equation. Now what we will do? In this lecture, we will work out some exact solutions of the Navier-Stokes equation. As we discussed, the Navier-Stokes equation is not a very simple equation to solve in principle because these are nonlinear, coupled partial differential equations.

But in certain simple cases, exact solutions are there that means one may analytically solve the Navier-Stokes equation and it is possible to work out some of those solutions using very simple mathematics and that we will do in this elementary course. We will not go into the details of all possible cases where exact solutions of Navier-Stokes equations exist but certain very simple elementary cases which fall within the domain of an introductory course in fluid mechanics.

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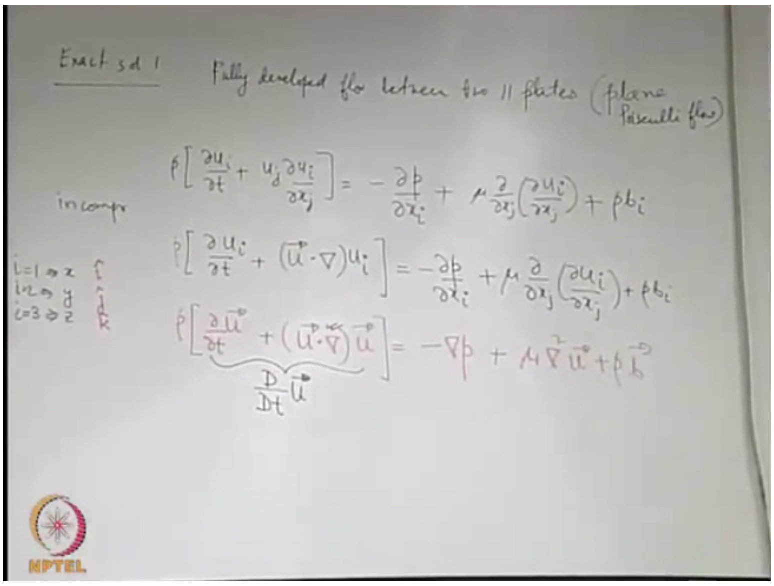

So if we try to revisit that what was the form that we derived for the Navier-Stokes equation, so we derived this particular form? So this particular form was for a special case. Can you remember what is the special case? Incompressible flow and what are the other assumptions?

Okay let us put a body force maybe to make it a bit more general. So the assumptions are incompressible flow.

Because if it was a compressible flow, there is a term which is associated with a divergence of a velocity vector. That particular term would appear. Then, Newtonian and Stokesian fluid homogenous and isotropic, so they are for the original derivations and with the additional incompressibility constraint, this is the form of the equation that we derived. Up to this we did in the previous lecture.

And why this incompressible form is important is because whatever exact solutions will be deriving in this course will be only for incompressible flows. So from now onwards whatever exact solutions will be deriving we will not explicitly mention time and again that it is incompressible flow but you just keep in mind that that is the assumption that we are following for the remaining derivations.

Now it is also not a bad idea to write this in terms of a vector form because this form when you write in terms of the indices i and j, this is an elegant form but this we wrote in terms of Cartesian index notations, but if you have a curvilinear coordinate system or the most general case of maybe a non-orthogonal curvilinear coordinate system, then for all those cases these types of index notation in this form will not work.

You require a different form because these are Cartesian index notations, but you may have different coordinate systems but vector form is general. So if we write its equivalent vector form then that vector form will be valid independent of the coordinate system. So if you want to write it in a vector form, so let us say we are interested to write it for a velocity components ui.

So the next term if you want to write in terms of the vector say this is the velocity vector so we are just writing one component, we will then write the other component. So we have only written one of the terms in vector form. The other terms we have still retained in the original index notation. So this is the ith component. So we should remember that i=1 will imply x component, i=2 will imply y component, i=3 will imply the z component.

So what we will do? That for i=1 this is x component, so we will take the x component and then multiply that x component with i cap, then i=2 that will be y component, multiply that with j cap, the corresponding unit vector. Similarly, i=3 will multiple with k cap and add those together. So if we add those together then what will be the corresponding form? So rho this will be partial derivative with respect to time u1i+u2j+u3k.

So that will become the velocity vector u right. Similarly, this will become -gradient of p okay, just like taking the individual components and finding the resultant. So this also as we have seen in the previous notation in terms of the total derivative this maybe written in terms of capital D/Dt of u. So this vector form is important and we will use this vector form for some of the cases.

And where one coordinate system differs from the other is the dell operator is differently expressed in different coordinate systems and that is how expressions were written in terms of the coordinates of the coordinate system will differ from one system to the other. Somewhere x, y, z, somewhere r theta z, somewhere r theta phi depends on the coordinate system that you are using but where all those coordinates appear will be in terms of the del operator.

Because del operator has different expressions for different coordinate systems but when you write it generically in terms of del operator it is independent of the coordinate system that you are writing and that is one of the good things about this form. Now we will start with the exact solutions for a case where we will come down to a much more simplified version of this particular form and that first exact solution.

So exact solution 1, we will consider something called as fully developed flow between 2 parallel plates. This has a technical name; it is known as plane Poiseuille flow. So we will look into this special case, first will draw a sketch to understand that what is the specific form that we are looking for of the Navier-Stokes equation in this case. Here we can very elegantly use the Cartesian index notation.

Because the coordinate system inherent to this description is the Cartesian one, so the understanding is that you have 2 parallel plates.

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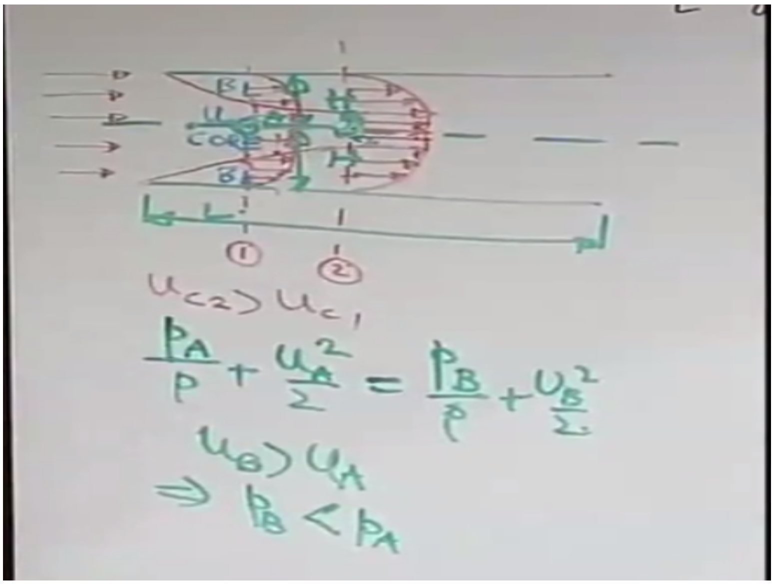

So let us say that you have a top plate and you have a bottom plate. These plates are separated by a gap, gap maybe large or small or whatever and these plates are such that the dimensions in this plane are much smaller than the dimension perpendicular to this plane. That means you have a plate like this, so one plate like this, another plate parallel to this where the third dimension is like infinity, very large.

So the dimension perpendicular to the plane of the board is much, much larger in comparison to the dimensions in the plane of the board. So it makes the problem equivalently a 2-dimensional problem. Why? Because if the third dimension is very, very large then the gradients in the third dimension are very, very small because gradients are basically change in variable/the length over which the change takes place.

If the length scale in the third dimension is very, very large then the characteristic changes in terms of the gradients are very, very small and then the effect of the third dimension may be neglected. So if we have let us say that we give certain dimensions, let us say this is the center line of the channel so this height is say H. This height is H, let us say that the length of each plate is L and the width is W such that W is much, much greater than 2H.

So when you come in terms of the cross section the effect of the H in terms of creating the gradient is much, much more important than the effect of the dimension w that is the width okay. Now let us try to first understand that physically what happens because we have a keyword fully developed flow which we have not yet explained. So we will try to explain that keyword or collection of keywords through understanding that physically what goes on.

So if you have say a fluid flow entering the channel say with a uniform velocity. Somehow let us say that there is a uniform free stream and that enters the channel. So just think that the channel constitutes of 2 parallel plates and for the time being forget about 2 plates, just consider that you have 1 plate. So if you consider that you have 1 plate then what happens, you will see that because of the effect of the plate the boundary layer starts growing.

We have discussed about this earlier, so let us say that from this point the boundary layer starts growing and it grows like this. Why does the boundary layer grow? Let us just recapitulate very briefly that you have the no-slip boundary condition at the wall, so the effect of the wall is that it brings the fluid to rest at that point and because of the viscosity of the fluid, this effect is propagated towards the outer fluid.

So the fluid elements which are very close to the wall fill the effect of the wall the strongest and then the fluid elements which are further and further away from the wall will feel some effect of the wall but not as strong because they are also being pulled by a fluid element on the other side, which is moving faster and further and further away from the wall you go, you feel that the fluid elements which are more and more away from the wall will feel the effect of the wall much, much less.

And we will come to a height beyond to which the effect of the wall is not felt at all, that means viscosity is not capable enough of creating further velocity gradient and that means we have come to a state where beyond at the velocity profile is virtually uniform and that location at each and every point at each and every section is given by a height which is some distance from the wall.

Depending on how speedy the incoming flow is this distance will be small or large. If it is a high speed flow, this distance will be very small. If it is a low speed flow, this distance will be large and we will come into the details of the largeness or smallness of estimation of this when we study in details the boundary layer theory in one of our subsequent chapters but like here we are interested more in terms of qualitative terms that what is happening.

So if you draw a velocity profile, let us try to draw a velocity profile at 2 different sections. Let us say that we draw it at one of the sections which is relatively closer to the inlet, let us say this is section 1 and we draw it at a section which is section 2 which is bit further away from the inlet. So at the section 1 if we draw the velocity profile see the velocity will have a gradient till you come to the edge of the boundary layer then the velocity will be uniform.

You have to keep in mind that on the other side, there is a plate which has perfect geometrical similarity and physical similarity that means boundary condition here is no-slip, here also it is no slip. So whatever will happen at the bottom plate, there is nothing which makes us believe that same thing will not happen at the top plate. So same thing will happen at the top plate and there will be a symmetrical development of the boundary layer.

So the understanding of the boundary layer is that within the boundary layer, so the red line is the age of the boundary layer, below that you are having velocity gradients because of the viscous effects, outside that velocity gradients are not there. It does not mean that the viscosity is 0 but velocity gradients are simply not there that means effect of the wall has not propagated beyond that point.

Now if you draw the velocity profile at the other end also it will be very, very symmetrical to what it was at the bottom end and then it will be uniform outside the boundary layer. So say this is the velocity profile. Now let us try to draw the velocity profile at a section 2, so when you draw the velocity profile at a section 2, so the effect of the wall will be felt now to a greater distance from the wall.

Because the fluid has now suffered greater effect of the wall by virtue of propagating deeper and deeper into the channel. Before drawing the velocity profile of the section 2 if we focus our attention on the velocity profile for the section 1, we see that it is divided into 2 parts, one is the boundary layer region and another is outside the boundary layer region, which we call as a core region okay.

So this is what you have for a core region and the boundary layer region. Now similarly at section 2, there will be some core region, core region is now thinner and boundary layer region is now thicker. So the first question that we would like to ask ourself is whatever is the velocity at the core region at section 1 say it is uc1 and let us say that the velocity at the core region for the section 2 is uc2.

Question is, is uc1>uc2? Is it < uc2 or is it = uc2? So we will try to answer this logically not by a guess work. So let us try to do that. To do that you have to understand that let us consider that we are assuming that it is a steady flow. So over each cross section, the same flow rate is there. That means flow rate at section 1 is same as flow rate at section 2. When we say flow rate it is usually mass flow rate but if the density is taken as a constant, it is as good as volume flow rate.

So volume flow rate how it is obtained? By integrating the velocity profile over the cross section that gives the volume flow rate. So if you integrate the velocity profile over the cross section and it remains constant for sections 1 and 2 then what is the feature of section 2? See at section 2 you have now a greater distance from the wall that is suffering the effect of the wall.

So the effect of the wall is now felt to a greater distance both top and bottom because they are symmetrical, that means if the boundary layer was not growing in terms of its size or thickness then whatever was the extent to which the fluid is slowed down now the extent to which the fluid is slowed down is more and to compensate for that the fluid in the core region should move faster than the whatever it used to move for the section 1 to keep the integral of the velocity same for both sections 1 and 2.

So here simply you have a greater distance over which the fluid has suffered greater resistance. It has to compensate for that with the higher velocity in the core to have the integral of the velocity over the section concept. That is the flow rate concept okay. So that means the first understanding is you have uc2 > uc1 okay. Now the question is if you have uc2 > uc1 then what about the pressures at sections 1 and 2?

So when you are having a velocity and when you are trying to relate that with pressure you are very much tempted with use of Bernoulli’s equation right. So we have to see that whether the Bernoulli’s equation here is applicable or not. So let us say that we consider a stream line along the center line and the stream line is connecting say 2 points A and B located on the center line.

A is a point on section 1 and B is a point on section 2, both along the center line and that is one of the stream lines because of the symmetry. Now can you apply the Bernoulli’s equation between A and B? See there are many assumptions of Bernoulli’s equation but the first assumption that you should look for that is it a inviscid flow or a viscous flow? Then look for all other things.

Because if the first question is not satisfied then you should not look for any other thing because you cannot apply it for a case where viscous effects are there. The interesting thing is that here the flow is what? Flow is viscous flow because it has viscosity but effect of viscosity is not felt here because you do not have a velocity gradient so you have no rate of deformation and the shear stress is the viscosity times the rate of deformation.

So if the rate of deformation is 0, it does not matter whether viscosity exist or not. So in terms of feeling the shear stress, there is no shear stress here and that means it is like an inviscid flow. So effect of the wall has not propagated into the center line. So that means you may apply Bernoulli’s equation between A and B provided other assumptions of Bernoulli’s equation are also justified.

So like we use the steady version by assuming there is a steady flow and special case of incompressible here we are considering a constant density flow. So when you write the Bernoulli’s equation between A and B you have pA/rho+uA square/2. Let us say that A and B are so close that that difference in height between these 2 is not important. So just it is same as pB/rho+uB square/2.

And we know from this analysis that uB>uA therefore pB is<pA. If you are not happy by assuming that A and B are of the same height and you have all the right not to feel happy because you might have a vertical parallel plate like this.

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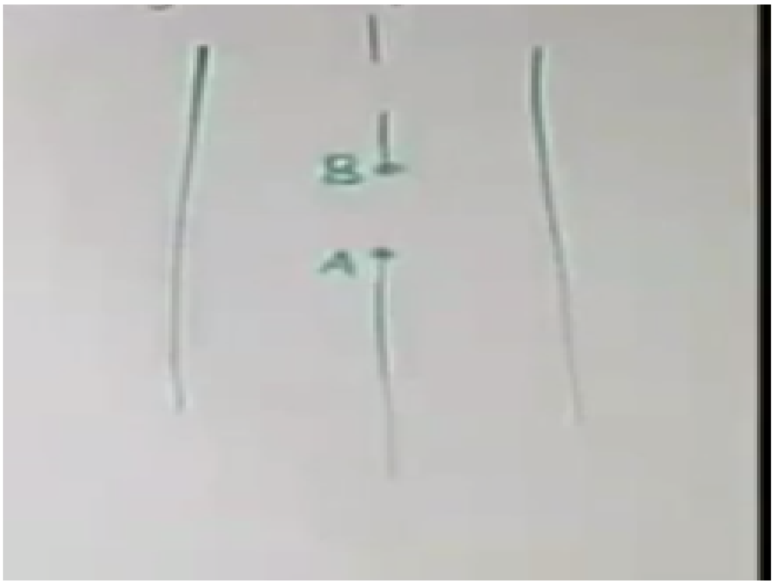

So if you have vertical parallel plate and you are considering 2 points say A and B on the center line, yes there is every chance that no matter like whatever distance you go it is a vertical change, so it is a total vertical change and it may be quite substantial depending on where are the points A and B. So if you still consider that well I am considering A and B which may be located at different heights.

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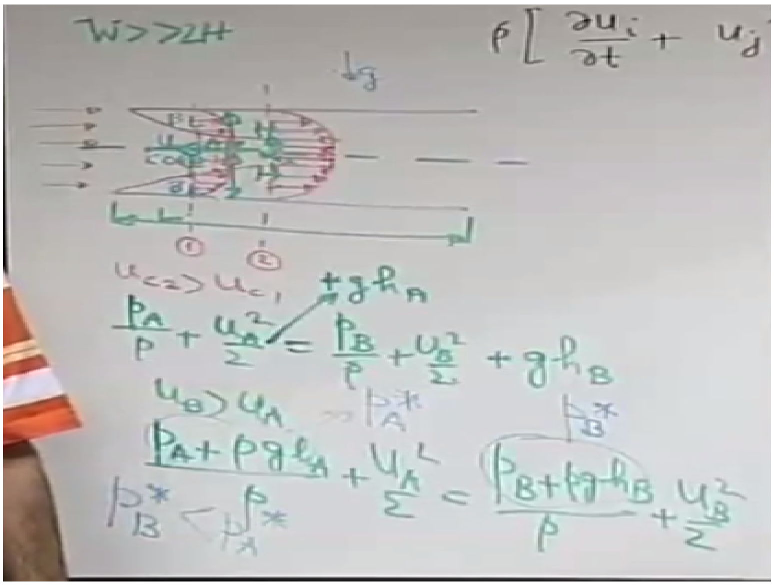

So you add even g into we use a coordinate system like h because x, y, z we are preserved for different coordinate. So this plus so we may say that in terms of writing it not just by pressure but if we write it as pA+rho ghA/rho+uA square/2 that we can write pB+rho ghB/rho+uB square/2. We have discussed earlier that the equivalent pressure that we have as pressure+the effect of the gravitational head together is known as the piezometric pressure.

So here what we have written is the piezometric pressure, so we call it, will give it a symbol pA star okay. We give it a symbol pA star and we give it a symbol pB star. So when we write it in terms of piezometric pressure, the good thing is that it becomes independent of the orientation of the channel that we come to a conclusion that yes in terms of piezometric pressure, we should have pB star < pA star right.

And when you are writing this, see for parallel plate channel which is horizontal or maybe having an inclination with the vertical which is slight then this rho gh effect is not important and then it is straightaway giving pB < pA what if the height effect the difference in height between 2 points that is important then it is p star or the piezometric effect that is going to be important.

So from here we may conclude one important thing that the flow is taking from higher piezometric pressure to lower piezometric pressure right, from A to B. So question is that means there is a driving pressure gradient from the point where the fluid is flowing to the point where the fluid is flowing, which is giving this pressure gradient.

So you must have a driving device which is allowing the fluid to have this pressure gradient so that it can maintain the flow because why it requires a pressure gradient, see it has to accelerate along the core. Its velocity should increase along the core, so there must be a driving force which should make it accelerate and that is a driving piezometric pressure gradient from a higher pressure gradient to lower pressure gradient.

Physically, what may provide that you may have a pump that provides that pressure gradient. So you must have an energizing mechanism, it need not necessarily be a pump but you must have something which energizes the fluid so that it may have an acceleration along the core and that is also one of the important technical or technological understanding that what is the consequence of what happens in the core.

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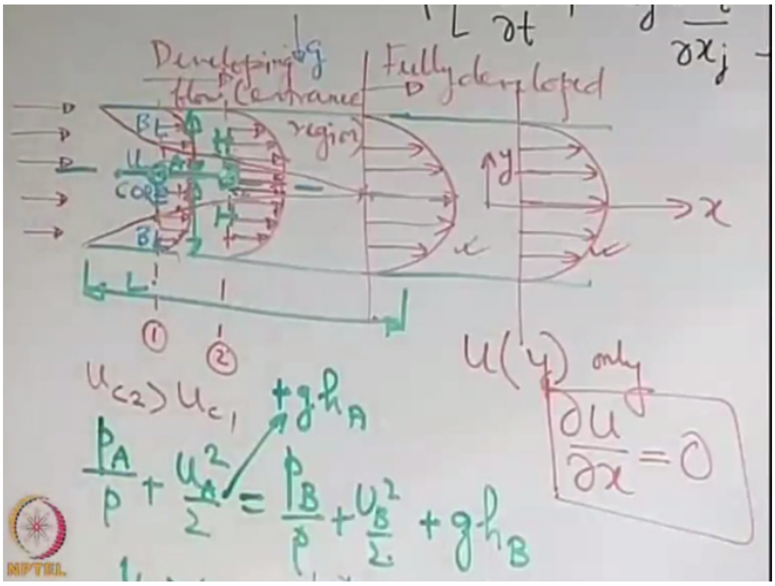

Then the important thing for our exact solution of the Navier-Stokes equation is that the fluid in or the boundary layers which are there in the region where the boundary layers are developing the edges of the boundary layer because they are symmetric when they meet. They meet at the center line somewhere, what happens beyond this is something which is interesting.

So if you draw a velocity profile here, from our previous discussions we may conclude that whatever is the velocity of the center line here that is the maximum of all because the fluid is continuously accelerating. So it comes to a maximum velocity here and at the wall the velocity is 0 and you get a sort of velocity profile. The question is will this velocity profile change any further as you move away from this.

See what makes the velocity profile change is as follows. In the region which is towards the left of this section, the boundary layer is continuously growing. So there is an adjustment at each and every section to keep the flow rate unaltered and that depends on the extent to which the wall effect has propagated into the fluid. Here the wall effect has propagated fully into the fluid.

That means the center line also now knows yes there is a wall and I am feeling the effect of the wall because like the edge of boundary layer is now limitingly on the center line but beyond this what happens beyond this there is no change because now the entire fluid knows that what is the effect of the wall and it response to that in the same way. There is no further development of the boundary layer.

So we say that the flow has become fully developed. When there is fully developed, there is something which is not fully developed and that is known as a developing flow. So the region which is here is known as the developing flow basically the boundary layers are developing here and it is also known as an entrance region. These are just different names. So you have an entrance region and you have a fully developed region.

And the hallmark of the fully developed region is that now if you draw velocity profile at different sections, let us say that you draw velocity profile at a section somewhere else then these velocity profiles would be just identical. These are the same. What it means? It means that if you are considering that u as the velocity component along x, now u becomes function of y only.

What are our x and y coordinates? Let us set up x and y coordinates as this as x and this as y or you may write partial derivative of u with respect to x=0. Because only variation of u is with respect to y. See fundamentally u could be function of in this coordinate system functions of x, y, z and time.

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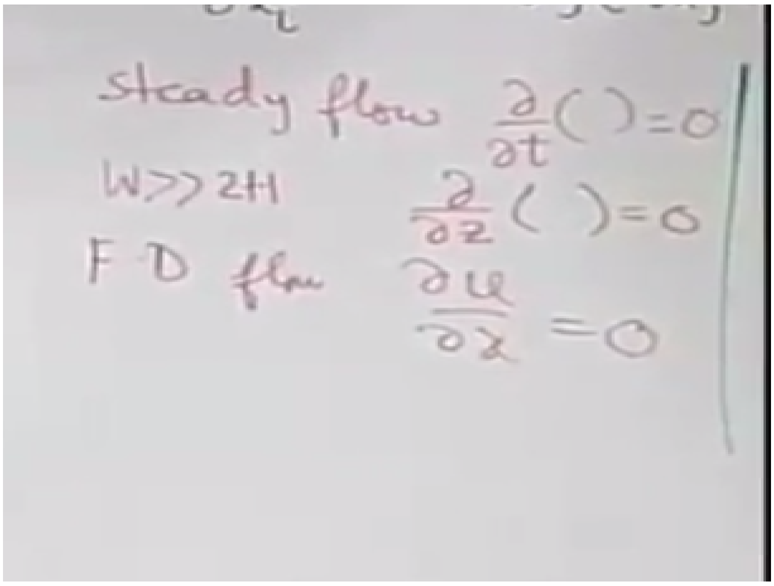

We have considered first assumption is steady flow, so you have any partial derivative with respect to time of any flow property or velocity that is 0. So it could be function of x, y and z. The gradients with respect to z are approximately 0 because you are considering infinitely large width in the z direction. So you have so width is much, much larger than say 2H and that means z direction is along the width.

So you have this=0, so when you have to use these assumptions to begin with, you had only 2 choices possible or rather 2 dependences possible u could be dependent on x and y. Now because of treating it as a fully developed flow, so we are now going to analysis only that is valid in the fully developed region. So u not a function of x, so fully developed flow will mean it is only for u.

For example, p varies with x so here we have written for any variable but here only for u not for p that you have to keep in mind. So with these assumptions you come to a conclusion that yes u therefore is a function of y only. So the partial derivatives all with respect to u with respect to y will now soon boil down to ordinary derivatives of u with respect to y because it is just a function of y.

Now to understand how these equations are simplified, we will first come to our objective. Our first objective is to find out the mathematical form of this velocity profile. How u varies with y okay? So that is our first objective. To satisfy that objective what we will do? We will solve the fluid flow equations. When you consider the fluid flow equations remember that you have the continuity equation that has to be satisfied always.

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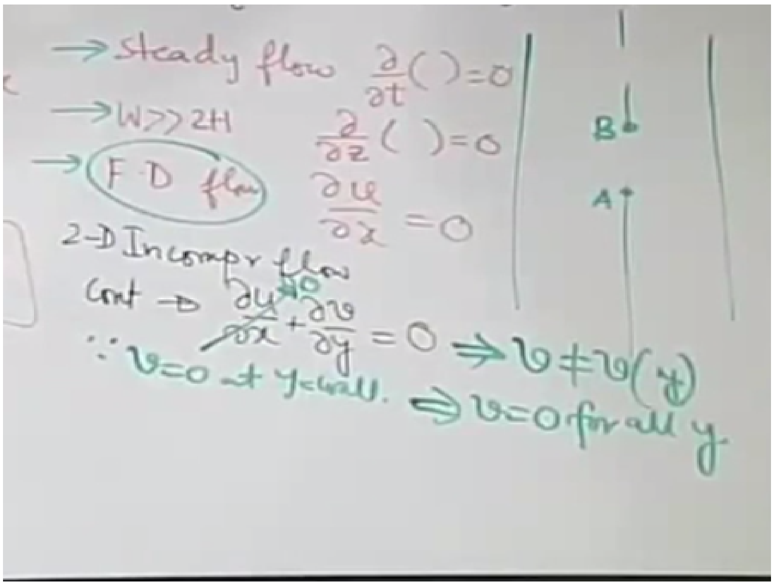

And what is the form of the continuity equation that we are looking for? Remember we are talking about incompressible flow and 2-dimensional incompressible flow because effectively the flow is 2-dimensional in the xy plane. So 2-dimensional incompressible flow you have the continuity equation, partial derivative of u with respect to x+v with respect to y that is 0 okay.

Now you consider the assumptions one after the other, come to fully developed flow. See these are the important assumptions that we are going to follow for the problem that we are considering. So when you have fully developed flow, then really this term will become 0. That means by imposing the constraint of fully developed flow what does the continuity equation give us? v is not a function of y right.

So v is not a function of y. Since v is not a function of y if we may find out that what is the value of v at some particular y that v should be true for all values of y right. So solution for v is not that difficult if we just have to find out that what is the value of v at some point which is known to us say at the boundary for example.

So if you look at the boundary, at these points what is the value of v? 0 because of no penetration boundary condition. So because of no penetration, the v is 0 at the wall, so since v is 0 at y=wall, so there are 2 walls one is +i channel that is –H that means what we can say that v is 0 for all values of y because v is not a function of y so at some particular y once we have found out the value of v that should be true for all values of y.

So this means that v=0 for all y and when we say that v=0 this is not an approximation that v is approximately=0, this is v is identically=0. Later on, we will come to certain situations where we will have v is much, much < u that is v is not 0 but we may consider it to be 0 as compared to u, those are different cases. This is a case where it is not an approximation. It is just an exactness.

So when v=0, then basically you are having to deal with only one velocity component that is u. Now keeping that in mind let us write the momentum equations of the Navier-Stokes equation. So we have the Navier-Stokes equation in terms of the coordinates Cartesian index written so let us write it for the x and y components. For convenience, we will write it in terms of first the y component.

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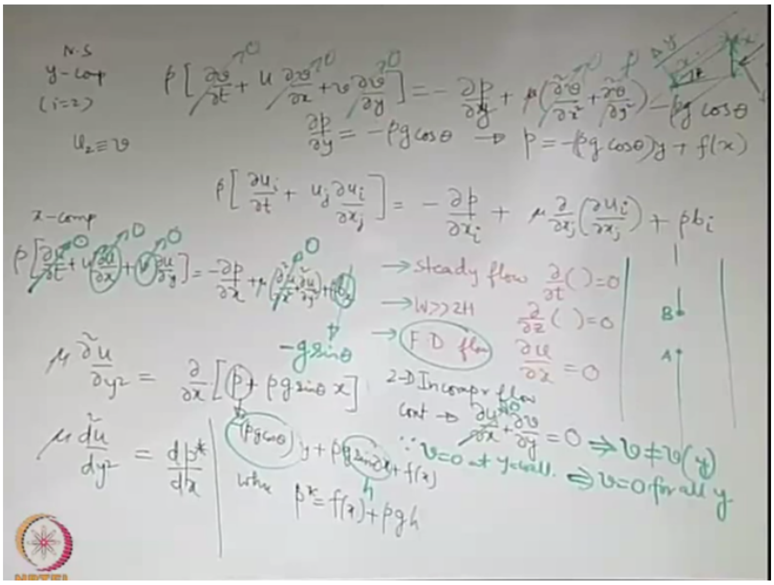

Because it will nullify many effects so let us write the y component. Y component means i=2 Navier-Stokes equation. So you have rho, so u2 is the velocity component along x2 which we give an identical name as v, u, v, w at the velocity components which we use as simple notations. We are also assuming constant properties and what is the body force along y direction? Let us consider an extreme case as the vertical and extreme case as horizontal.

The more common case is neither horizontal or vertical but inclined. So I mean arbitrarily inclined. So let us consider that we are talking about a channel which is like this, x is the coordinate along the channel, y is the transverse coordinate and the acceleration due to gravity is acting along this. So what is the body force that you will have? So you should look for the body force component along y direction. What is that?

So you have this g you have its own components. Let us say that this angle is theta that is the angle made by the horizontal with the axis of the channel is theta. So then this angle is theta, so it is –rho g cos theta okay. Now let us look into the different terms. First see look into the assumptions steady flow. So this will be 0, see always this is a systematic way of looking into the problems that we have with the use of the Navier-Stokes equation.

First write the full equation then see what are your assumptions, accordingly you make a simplification of the equation. So do not remember any equation like a formula. So we will always start with the Navier-Stokes equation and come up with the simplified form based on the assumptions okay. So this is steady flow, then next term see v is 0 therefore whatever is involved with v there is no gradient of v as well so these terms are 0. Therefore, these terms are all 0.

At the end, this gives you=-rho g cos theta. So if you integrate it you have p as –rho g cos theta*y so +some function of x right. Next, let us look into the x component of the momentum equation, which will be the important equation for getting the velocity profile but the background is important otherwise you cannot simplify that equation. So let us write the x component.

So let us substitute different terms for steady flow, so the first term is 0. Then, next term, fully developed flow so this is 0 because of fully developed flow v is 0 so the entire left hand side becomes 0 and that gives a lot of advantage because now the nonlinear partial differential equation has become a linear equation okay.

Now come to the right hand side because u is not a function of x, it is first order derivative with respect to x is 0 and therefore second order derivative is also 0 and what is the body force? Just look into the figure and say what should be the body force along x okay just for dx we will substitute –g sin theta right. So it is rho g-rho g sin theta. So what you have that mu*okay this is what we can write.

So p has what? p is –rho g cos theta y +rho g sin theta x+fx right. That is why substituting p from there. **"Professor - student conversation starts."** Which one? This one, yes, right, this is + okay. **"Professor - student conversation ends."** Then, whatever in the bracket it is there. So substituting p from what we got from the y momentum equation+the rho g sin theta x term.

Now if you see that anyway partial derivative of this with respect to x will be 0 because this only contains y so the remaining terms see this is a function of x only and this is the function of x only. Therefore, this term is the function of x only and if you see the effect what is this term actually? This term is just like a pressure which is varying with x+the gravitational head because what is x sin theta?

So if you go from say this point to this point which you traverse along x then this height H is x sin theta if this distance is x. So that is like rho gh okay so you have some form of pressure which is varying with x on the top of that rho gh so it is giving you a piezometric pressure effect fx+rho gh and the important thing is since this is varying with x, we can just write this as dp star/dx where p star is fx+rho gh right and y d/dx not partial derivative?

Reason is clear. These are only functions of x and this become 0 when partially differentiated with respect to x. Left hand side, what do you get in the left hand side? See u is a function of y only therefore the left hand side is mu d2u/dy2 okay. Because u is the function of y only so partial derivative and ordinary derivative are the same.

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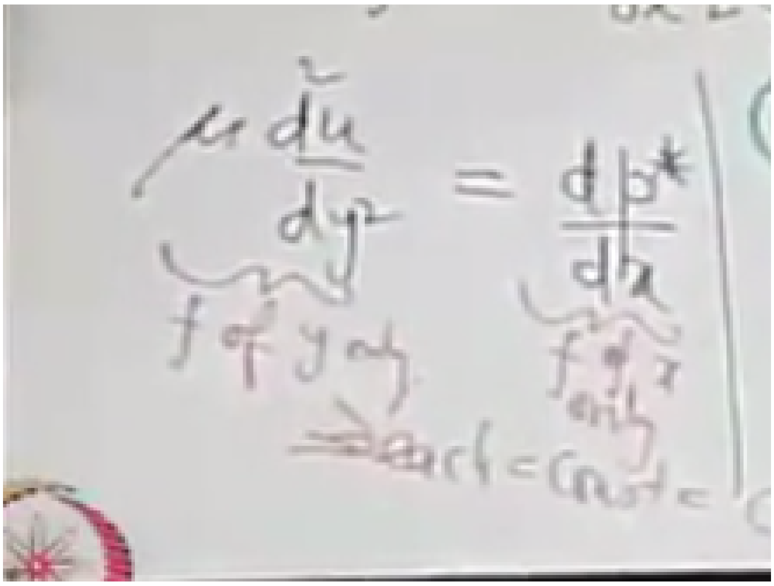

So now you have an equation where the left hand side is a function of y only, right hand side is a function of x only right. It is possible only when each is a constant, otherwise you cannot have an effect where some function of x will nullify some function of y so this implies that each equal to constant, say the constant is equal to c. So once you have each equal to constant the remaining work will be easy.

Because you may integrate it twice to get the velocity of the function of y so let us quickly do that.

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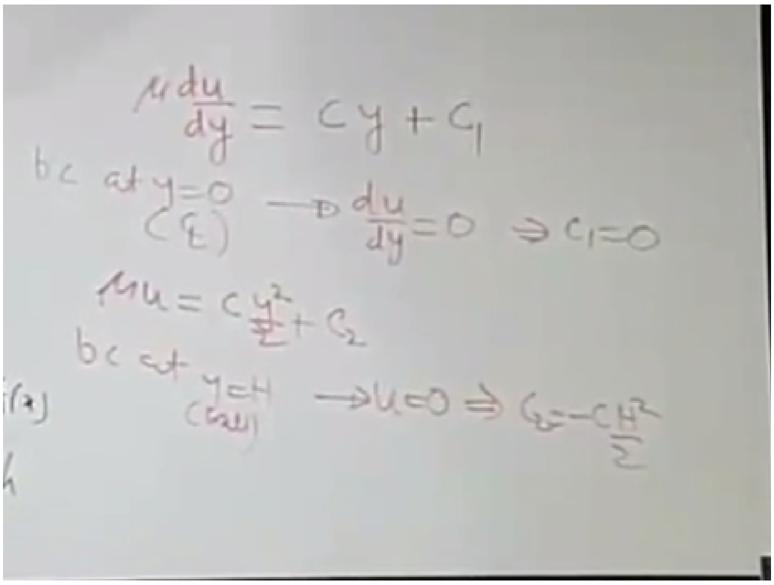

So if we integrate it with respect to y, you have mu du/dy=cy+c1 right. Now let us fix up a domain in which we want to solve our equation. See the domain is symmetrical, the top half and the bottom half they are symmetrical. So we may just solve it for the top half, the bottom half will be just the mirror image of that.

So for analytical calculation that is not necessary always but if you want to solve it numerically reducing the size of your effective domain by utilizing the symmetry saves your computational time. So it is something which one should always utilize if you find symmetry. Remember symmetry should be in terms of 3 important things, symmetry in geometry, symmetry in boundary condition and symmetry in physics.

So we have to make sure that all these 3 things are maintained and here all those things are maintained in the top half and the bottom half. Now next is using the boundary condition. So our domain is now from y=0 to y=H, so what is the boundary condition at y=0? That is the center line du/dy is 0 because u is the maximum here or otherwise from symmetry. In some case, it could even be a minimum but whatever it is it should have an extremum at the center line.

So its derivative with respect to y is 0. So that means you have c1=0. Next let us integrate it once more so you have mu u=c y square/2+c2. What is the boundary condition at the other end? Boundary condition at y=H that is the wall, u=0 that is the no-slip boundary condition. So that means you have c2=-cH square/2. So we have got a form of the velocity profile and let us write that form.

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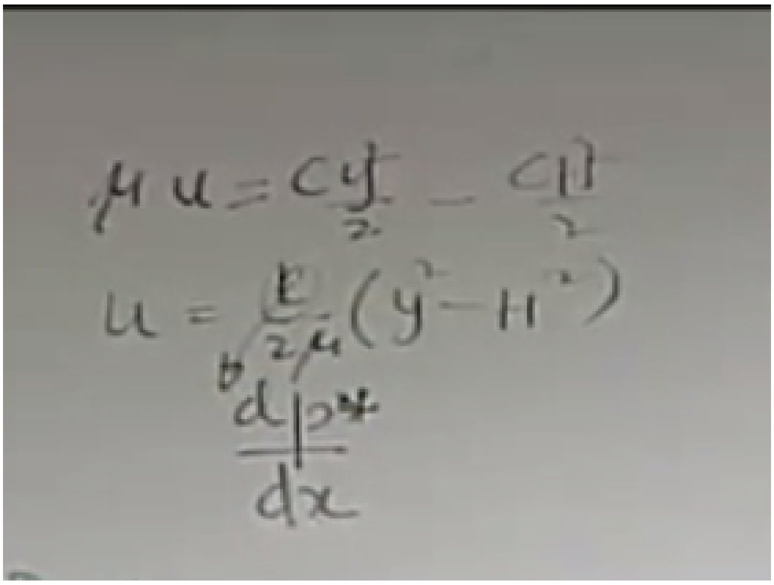

So what is the form? mu u=c y square/2-cH square/2. So u=c/2 mu*y square-H square. See it is symmetrical with respect to y=0, so a positive y or negative y has no consequence because it is y square. The other important thing is y square-H square is always<=0 because your maximum value of y is H, but you are having u along the positive x direction that means c must be negative.

And that is what we have seen that you must have a negative gradient of the piezometric pressure, it should be from higher piezometric pressure to lower piezometric pressure and so this dp star/dx which is=c is negative. So you know it in terms of the pressure gradient and so this c is=dp star/dx. Sometimes it is not a bad idea to write it in terms of the average velocity. So what is the average velocity?

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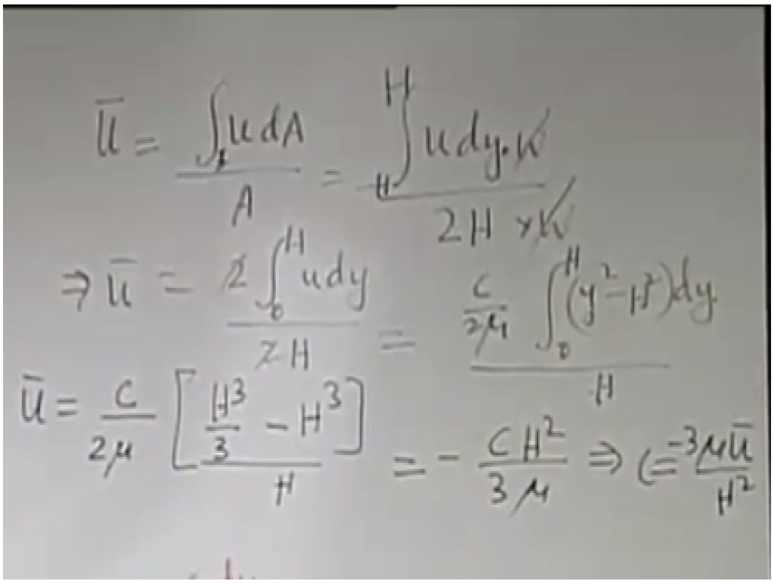

How do you find out what is the average velocity? So average velocity u average is what? Integral of u over the area of cross section/area of cross section right. So here the area of cross dA is what you take at a distance y so let us say that this is the channel at a distance y we take a small strip of width dy and dA is dy*the width of the channel. So it is integral of u dy*the width of the channel/2H*width of the channel and the effect of the width of the channel gets canceled out.

And this is actually from –H to H because of the symmetry this is as good as 2*integral of 0 to H u dy/2H so half is good enough and you may substitute u that is c/2 mu*integral of 0 to H y square-H square dy/2H. So let us just complete this expression. You have u average=c/2mu * this sorry 2 is not there in the denominator this 2 is not there because it is already canceled.

So c/2 mu * the integral of that so H cube/3-H cube/H right. So this becomes - of two third so that becomes –cH square/3 mu. So you can write c as 3 mu u average-3 mu u average/H square.

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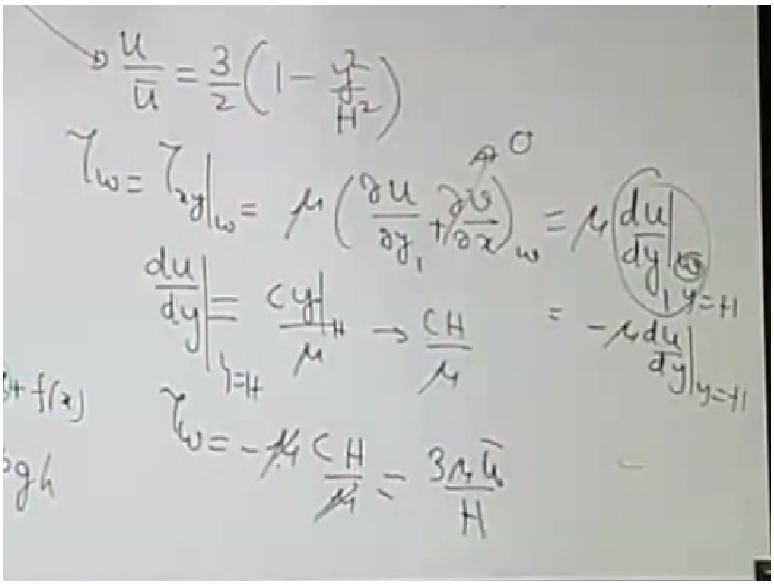

And that if you substitute in the expression for u, you will get from this expression u/u average is 3/2*1-y square/H square and clearly the form shows that is a parabolic velocity profile. So that is the first objective that we have satisfied that we have found out what is the velocity profile. For engineers, the next important objective which is quite straightforward and one step from this is to find out what is the wall shear stress?

So what is the wall shear stress? So the wall shear stress is basically tau xy at the wall that is for a Newtonian fluid okay. Remember that is mu * partial derivative of ui with respect to xj+uj with respect to xi so just in terms of u and v and x and y, these are substituted. At the wall v is 0 and in fact we do not care whether v is 0 at the wall because for fully developed flow v is 0 everywhere.

So we have straightaway whatever effect of v is there 0, so it is as good as mu du/dy because u is the function of y only mu du/dy at the wall, wall means you have y=H. So what is mu du/dy at y=H? So you find out what is du/dy. So du/dy is cy/mu. So du/dy at y=H is cH/mu and c we know is that -3 mu u/H square. So we can write this c in terms of tau wall or tau wall in terms of u average from this expression.

One important thing is we have not substituted this with a proper sign. Remember that when you write tau as mu du/dy then that y is what a coordinate which is perpendicular to the wall from the wall to the fluid. So we are considering the top wall here, ideally that y coordinate should have been this one but our y coordinate is opposite to that. So it is better to consider a separate coordinate y1 which we are talking about here, which is just like the y1 coordinate which is perpendicular to wall and towards the fluid.

So this is basically at y=H or y1=0 but y1 and y they are just oppositely oriented. So you can write y1=nothing but H-y so when you write this, this du/dy so this may be written in terms of –mu du/dy at y=H. Though it is mu du/dy y1 at y1=0 that is as good as so this + or – sign is with the convention of the normal of the wall. So that means du/dy at y=0 that is cy/mu that is c at y=H that is cH/mu. So what is tau wall? –mu*cH/mu right.

So –mu du/dy and c you can write as 3 mu u bar /H square. So 3 mu u bar/H so that is the wall shear stress. Sometimes we write the wall shear stress in a non-dimensional form. So this is the dimensional form. So the non-dimensional form of the wall shear stress is something what engineers use as friction coefficient.

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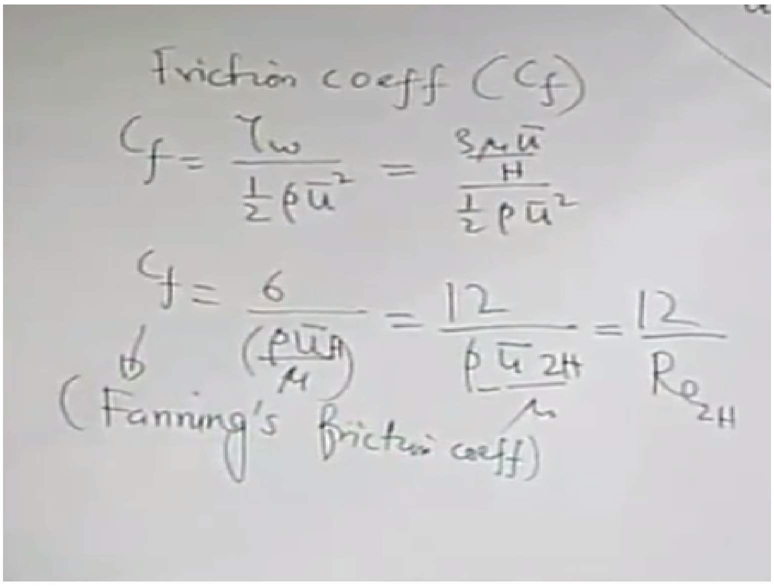

So friction coefficient say Cf that is defined as a non-dimensional form, so wall shear stress/1/2 rho u average square. So see rho*velocity square is a unit of pressure, stress is also unit of pressure so this is unit less, 1/2 as a sanctity that is like sort of represents a normalized effect with respect to the kinetic energy but like it is for just for non-dimensionalize even one may omit the 1/2.

It is just to keep the sanctity of the sort of sense of kinetic energy. So this you have 3 mu u average/H/1/2 rho u average square. So Cf is 6/rho*u average*H/mu okay and if you want to write it in terms of the width of the channel, this is 2H so this you can write as 12/rho u average*2H/mu and importantly this rho u average*2H/mu it represents what? It is like if you see rho*velocity into the characteristic width of the channel/mu. It is a Reynolds number.

So this is 12/Reynolds number based on the length scale 2H, which is the distance between the 2 plates. So the non-dimensional friction coefficient sometimes this is known as Fanning's friction coefficient because fanning was the person who first introduced this coefficient Fanning's friction coefficient that is 12/Reynolds number based on the length scale which is the distance between the 2 plates.

See for a flow taking place internally, the velocity that you are taking the Reynolds number is average velocity because otherwise velocity varies over the section. So you have to take some reference. The reference is the average velocity, rho and mu are properties and the length scale is the characteristic length of the system, which is 2H or even you may write it in terms of H.

But then you have to clearly specify tha what is the length scale that you are taking for the Reynolds number usually 2H is taken as the length scale. So let us stop here today and we will continue with the exact solutions in the subsequent lectures. Thank you.