Conduction and Convection Heat Transfer
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Module - 07 Lecture - 20
Review of Fluid Mechanics - I

In the previous lecture we were discussing about the continuity equation and we will take it up from there. So, the continuity equation was derived from the Reynolds transport theorem by applying the principle of conservation of mass.

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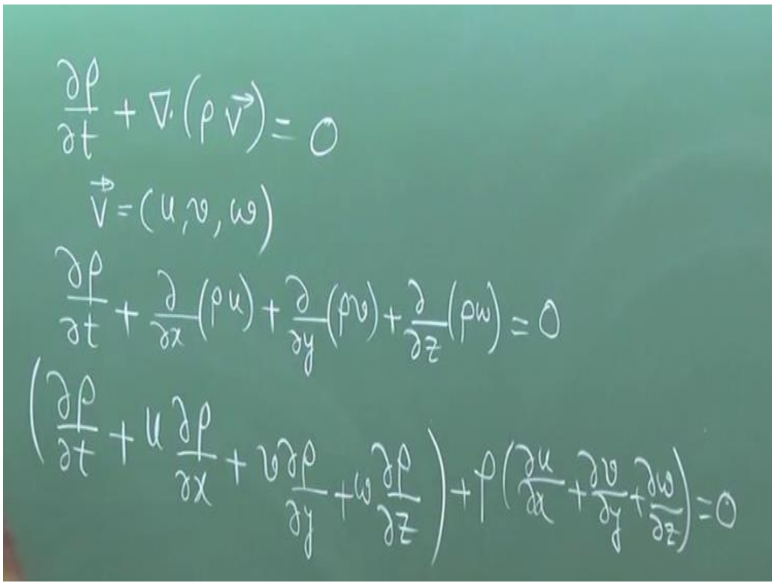

Now, we will try to develop a bit of insight into this equation. Let us say that you are considering a Cartesian system, where the velocity vector will have three components, u, v, and w, u along x, v along y, and w along z. So, it is possible to write, so you can write this as, what we have done is we have expanded each term in terms of the product rule of the derivative. So, we have clubbed three terms here and the remaining three terms with rho.

Now, we would be interested to consider the special significance of these terms, okay. To understand that, let us take an element with the length of delta x along x, delta y along y, and delta z along z. Overtime, it is possible that this element, this fluid element in this special case of consideration of a fluid will deform.

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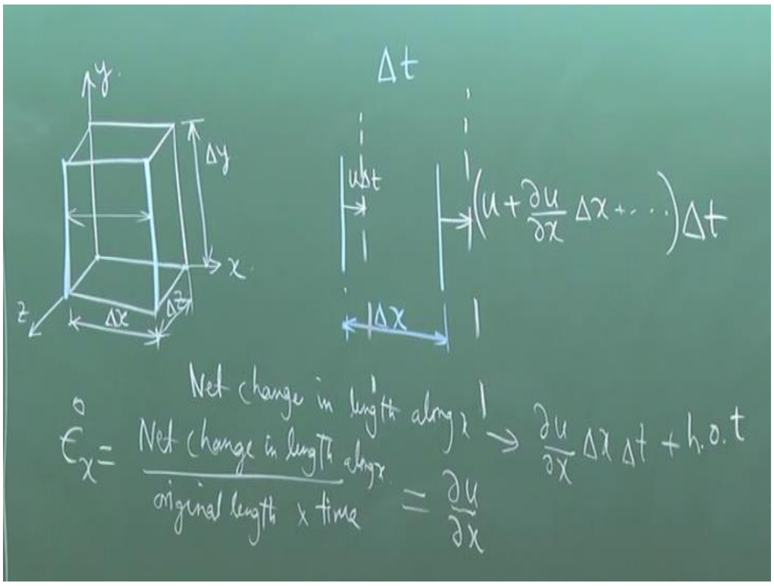

Now there are virtually possible mechanisms of deformation. But we can talk about two types of deformation broadly, one is linear deformation, another is angular deformation, these two types of deformation, we can talk about. So linear deformation will give rise to a change in volume and that is what is the matter of our interest to calculate.

So just for clarity, let us draw these two lines separately, these two vertical lines, which I have marked with blue colour, let us draw them separately. So, they were at a distance of delta x apart. Now let us say that a time delta T has elapsed, over a time delta T this line element will go along move along x by a distance of what? What is the displacement of this line element, u into delta T, where u is the velocity along x, right.

This line element will also be displaced, so it will be displaced by an amount, what is the displacement of that? The displacement of this is u into delta T, what is the displacement of this, what is the velocity of this line. If the velocity of this line is u, what is the velocity of this line, u plus del u del x into delta x plus higher order terms in the teller series, this into delta T.

So, what is the net change in the length, this was the length along x, what is the net change in length. That is the difference between these two displacements, that is the net change in length. So net change in length along x that is, plus higher order terms. So net change in length per unit original length per unit time, what is this? This is equal to, because if you now divide both sides by delta x and delta T and take the limit as delta x delta T, all tends to zero, the higher order terms will be tending to zero.

So, this is nothing but what, this is the so-called strain rate of the fluid element along x, epsilon dot x. For fluid, it is the rate of strain that is important and not the strain, because fluid deforming. So, if you allow more and more time, it will be more and more strain. So total strain depends on time, but the rate at which it is straining is more important, because that gives how fast or how slow it is changing in its dimensions.

Now this is about the linear strain, what about the volumetric strain. There is a volume element here, what is the change in volume.

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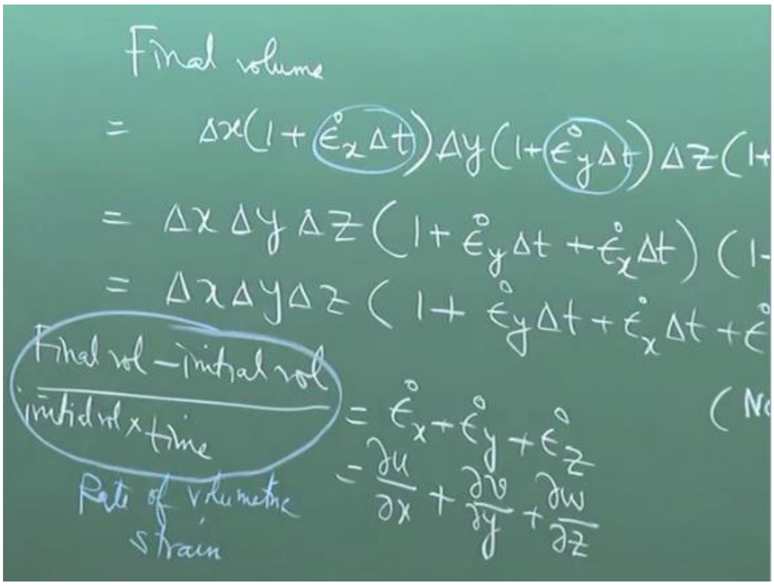

So, let us write the final volume, this is the new delta x, right. This is the original delta x plus the change in delta x, right. So, this is the new delta x. Similarly, you have a new delta y, and you have a new delta z. So, you can simplify this a little bit, so while simplification we have neglected the product of these two terms. That is of the order of delta T square as compared to the other terms, which are of the order of at the most delta T.

Again, we can do one more step, so all these are neglecting of the order of delta T square, okay. So final volume minus initial volume by time, divided by initial volume, what is this? What is this physically? Physically this is the rate of volumetric strain, final volume minus initial volume divided by initial volume divided by time. So, if you subtract the initial volume and divide by this and time you will get epsilon dot x plus epsilon dot y plus dot z, okay.

So, we have shown that epsilon dot x is del u del x, similarly epsilon dot y is del v del y and epsilon dot z is del w del z, okay. So, this what does it indicate, this is rate of volumetric strain so it is clear from here that this term, which is put in the square bracket represents the rate of volumetric strain. For what kind of situation, you do not have any change in volume of fluid element, when the flow is incompressible.

Remember there is a very important distinction between incompressible fluid and incompressible flow. Incompressible fluid is a fluid where or rather a compressible fluid is a fluid where density is a significant function of pressure, whereas when we were talking about incompressible flow, incompressible flow means there is no change in volume of a fluid element.

So, it is purely a kinematic parameter whereas when you are talking about incompressible fluid you are trying to see whether density is a strong function of pressure or not. If density of the fluid is a strong function of pressure it is compressible, otherwise it is incompressible fluid. On the other hand, incompressible flow means there is no change in volume of a fluid element. So incompressible flow is a kinematic parameter, it is a kinematic constant, right.

We have to keep this in mind. So, this if it is incompressible flow, this term must be zero for incompressible flow. So, this does not follow from conservation of mass, right. We have shown that this follows from pure kinematic constraints. So many times, there are many misunderstandings, one misunderstanding is that divergence of velocity, which is the term in the square bracket is zero, which follows from the conservation of mass from incompressible flow.

It is not true, it is following purely from kinematic constraints. We can relate that with conservation of mass in some way or the other by coupling that with the continuity equation that is fine. But fundamentally it has nothing to do with the conservation of mass. It has something to do with change in volume of a fluid element, which is a purely kinematic parameter.

Now if this is zero for incompressible flow, then these also must be zero for incompressible flow, because sum total has to be zero, zero for incompressible flow, right, because sum total has to be zero. So, if the term in the square box is zero this curly term should also be zero.

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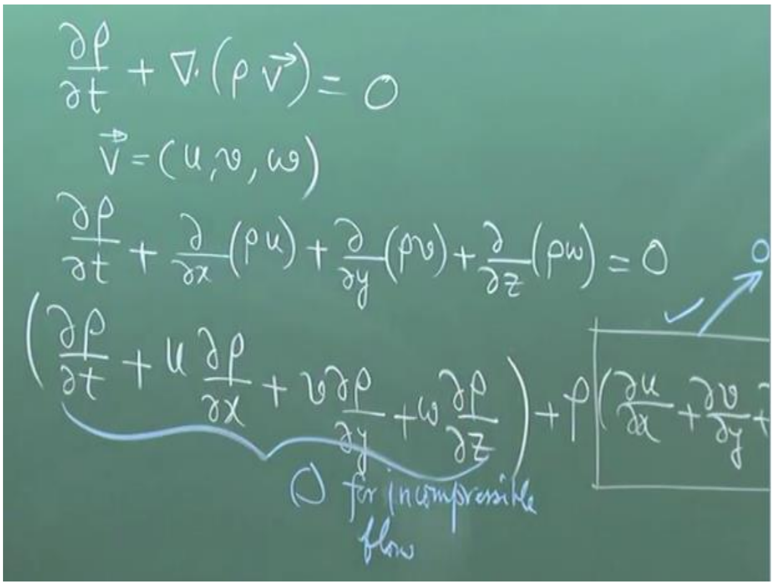

So, what can we tell from this, see there are some many misunderstandings propagated by standard books, even very good books. And one of the misunderstandings is that, in books it is written assuming incompressible flow we take density as constant. This is completely wrong, you can see from here that for incompressible flow this term has to be zero, that does not means that density has to be constant.

I will you an example, when density is not a constant but still it is incompressible flow, I will give you an example, but first you can appreciate mathematically. That this equal to zero has to be true for incompressible flow that does not mean that rho has to be constant. But it is true that rho is equal to constant is a special case of this, because if rho is equal to constant this is trivially satisfied, right.

So, rho is equal to constant is a special type of incompressible flow, so that is a constant density incompressible flow. But you can also have variable density incompressible flow. So, let us take an example.

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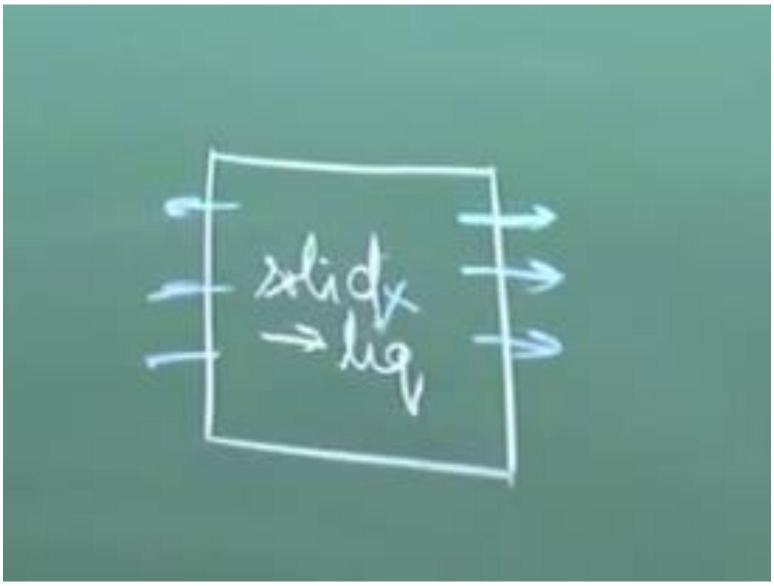

Let us say that you have a domain in which some solid phase is converted into liquid phase, right. Let us say that it is some metal where the solid metal when is converted into liquid it is expanding. So, when it is converted into liquid and it is expanding and this is the control volume what will happen, first this volume could accommodate the solid. When it is expanding to become liquid this volume cannot accommodate that much of liquid.

So, liquid will flow from different boundaries, okay. Because it cannot be adjusted with in this volume, so what it should do, it should escape from the volume, okay. So, what does it means, it means that see look at this term, del rho/del T, what is this, this is rate of change of density at a given location due to change in time. You have what del rho/del T, because at a given instant of time with in the control volume.

The solid has changed to liquid, which are of different density. So, you have this term, then you have this flow terms u del rho/del x, v del rho del y, and w del rho del z. These are advective components of the change in density. So, it says that for incompressible flow the sum total has to be zero, but this is possibly balanced by these two, these three terms. If this is plus this is minus so that, if this is plus A this is minus A so that sum total is zero.

So here neither of solid nor of liquid can be thought of as compressible phases, right. These are the normally incompressible phases. But here you have a density change. So, it is possible that you have incompressible flow, but with variable density. So, we should keep in mind that we should not have the prejudice that incompressible flow means constant density flow. Incompressible flow may also mean variable density flow.

But a special case of incompressible flow is constant density flow. Finally, we will give a mathematical perspective to this constant.

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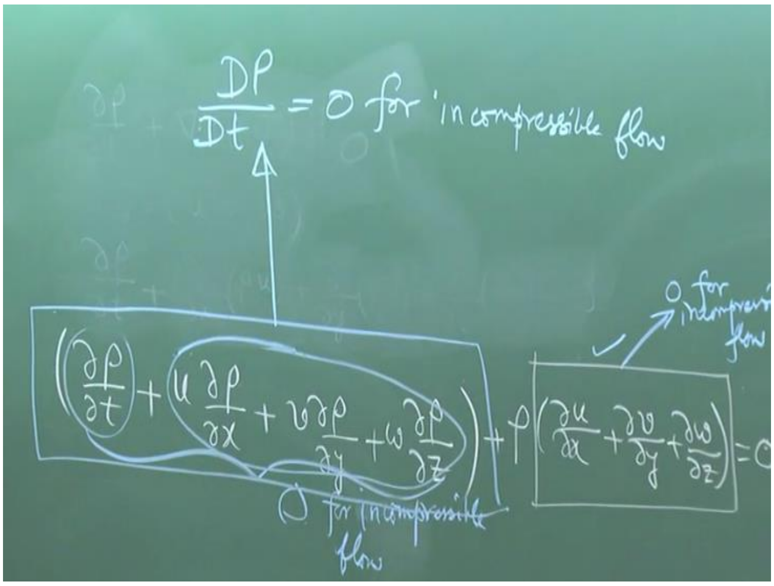

So, if you write this particular term in short hand notation you can write that the total derivative this is the total derivative, right, you in your earlier course on fluid mechanics you have learnt that total derivative of velocity is acceleration, right. In place of rho if you write u then that is acceleration along x, right. So, this is just D/Dt operator operating along rho, this is the unsteady component.

And this is the advective component and the component due to flow, okay. So, sum total of these two is zero for incompressible flow. So, we have discussed about conservation of mass, next we will discuss about conservation of momentum.

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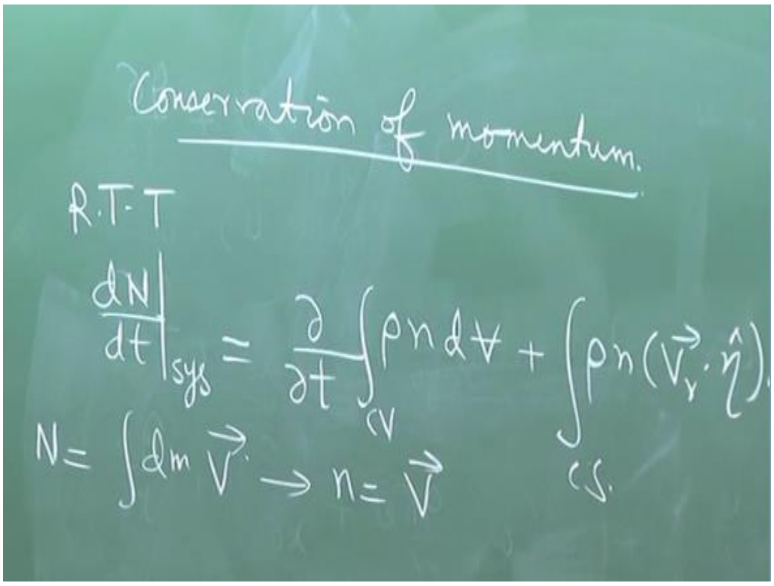

So, we will start with the Reynolds transport theorem, so if you recall this N is the extensive property of the system, n is N per unit mass, and Vr is the velocity of the fluid relative to the control volume, and eta is the unit vector outward normal to the control surface. So, for linear momentum conservation N is what MV, right, or you can write integrate of dmV, because every mass of the fluid element may have a different velocity.

But loosely MV is fine, I mean, because I mean there are many things which we write rigorously, there are many things which we do not write very rigorously, still it is okay if we mean there are same kind of thing and as I said go ahead with the conceptual understanding.
So, this means that what is small n, V, so you can see that this becomes a vector equation, because this V will have its own components.

So, let us write the different components of the vector equation. So, let us write the x component, now we will make the two important assumptions that we made for working out the details of the continuity equation, that stationary control volume and non-deformable control volume.

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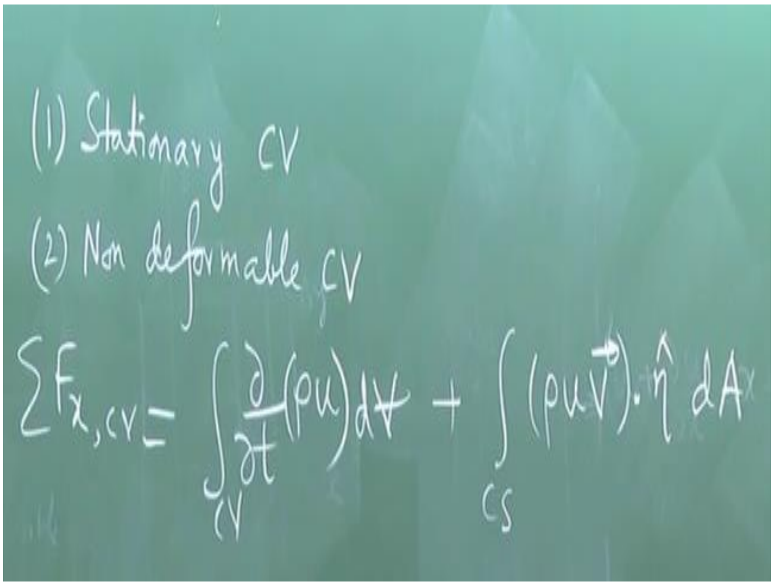

So non-deformable control volume means we can take this derivative within the integral, because volume is not a function of time, and stationary control volume means V relative and V at the same. Not only that, what is the left-hand side, this is the, see this is why we apply the Reynolds transport theorem. For this, you can directly apply the Newton's second law, but not for this term, because this refers to a control mass system, this is rate change of linear momentum of the system.

So, this is the resultant force along x acting on the system. And because while deriving the Reynolds transport theorem we have taken a limit as delta T tends to zero. The control volume almost coincides with the system. Therefore, the force on the system and force on the control volume are the same. So, this is resultant force acting on the control volume.

So, we can write resultant force along x on the control volume is equal to, so this term you can use the Divergence theorem, to convert the area integral to the volume integral. So, this will be integral of Divergence theorem, by Divergence theorem. So right hand side we have been able to express all the terms in terms of volume integral, left hand side also we will be attempting to do so.

So, the first important conceptual thing to understand is how to represent forces in terms of different integrals. So, forces in continual mechanics when we say continual mechanics, it can be mechanics of solid or mechanics of fluid whatever. You will have two types of forces.

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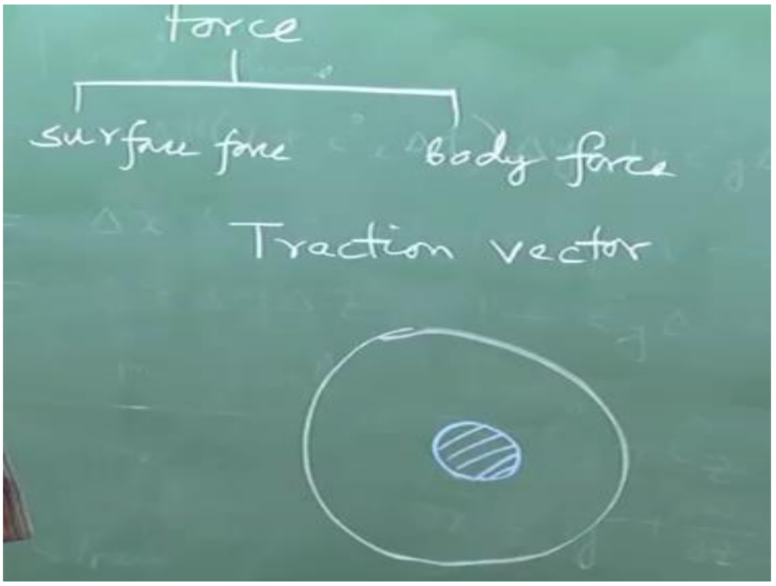

One is surface force another is body force. By the name it is clear that surface force will act on the surface element, and volume body force will act on the volume elements. So, we will define something called as traction vector. So, what is a traction vector, let us say that you have an element like this may be a solid or it may be a fluid whatever. Now you take a small volume out from it.

So, if you take a small volume you have to represent the force exerted by the remaining part of the volume on this one. This is just like drawing a free body diagram, if you isolate one part of the system, then you have to show all the forces exerted by the other parts on that part. That is the concept of a free body diagram. So, this is just like a free body diagram, let us say you have isolated this part and you are interested to draw the free body diagram of that.

So, if you do that, let us say that T is a force per unit area, at the given point.

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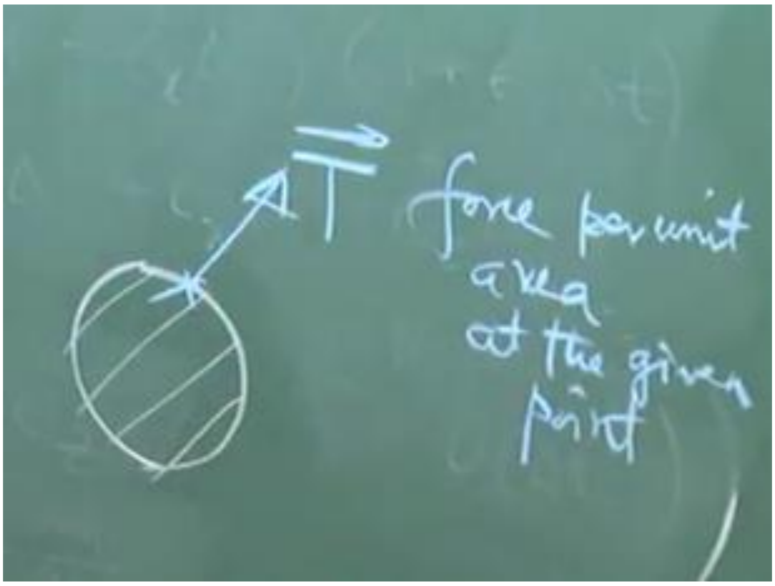

This is a force. This is not actually any external applied force. This force is because of the interaction between the shaded volume with the remaining part of the volume. This is like action reaction type of force, okay. Now, this is not the total force. This force per unit area, okay. So, if this is force per unit area that means the total force will be dependent on the area that is chosen here. Let us say, you choose an area.

If you choose an area dA, it may be a differential force dT whatever. Now, this differential force dT will depend also on the orientation of this area, right. Because at a given point, if you choose the same area, but with different orientation, you will get different force. So, this force at a given point does not only depend on the location of the point, but also on the orientation of the area chosen to calculate the force.

To specify that, you use a superscript eta with T. This eta is a unit vector normal to the surface dA, okay. So basically, this is a vector because it has its components, but it is a special type of vector where it is actually something more general than a vector in a sense that it also depends on orientation of the area to calculate the force. So, you can write the traction vector using this notation, but in reality, it is very difficult to deal with surfaces, which are arbitrarily oriented.

So, to systematize the entire behaviour we intend to take surfaces, which are either having normal along x or normal along y, or normal along z. So, we consider those surfaces, then we can represent the behaviour of any arbitrary surface as a function of the force on those surfaces. That we will show later on. But we will first take an example. Let us say this is x axis. From now onwards, we will try the patient use a notation, which is called as index notation.

So, in index notation we will write x axis as x1, y axis as x2 and z axis as x3. So, we will use one index i, xi, i = 1 means x axis, i = 2 means y axis, i = 3 means z axis, okay. So now this has how many surfaces. This volume, this has six surfaces. The speciality of these surfaces is that their normals are either along x or along y, or along z. So, for this surface, this traction vector is alternatively replaced by a notation of tau-ij.

It is represented by a notation tau-ij, what does this i indicate, I will write it separately, tau ij.

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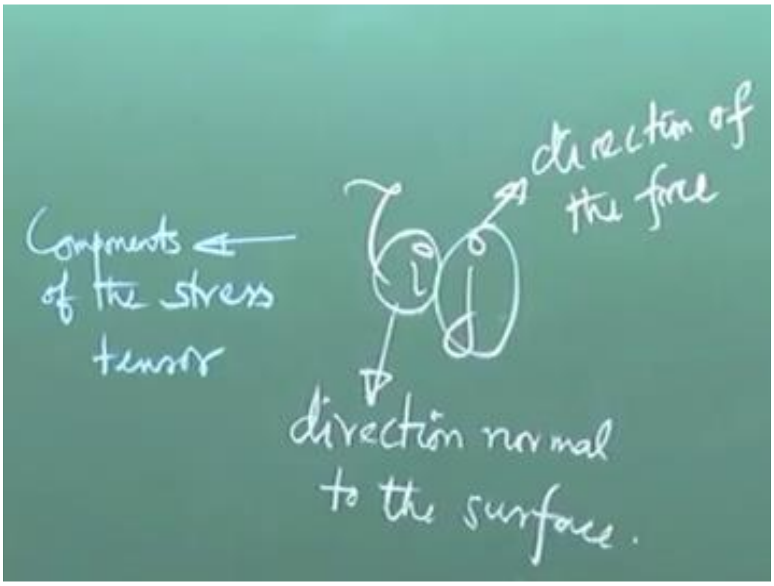

Very important in terms of the notation tau-ij. It is again force per unit area, what. So just like traction vector or traction vector on special surfaces, which are normal either along x or along y or along z, so this i represents the direction normal to the surface and j represents direction of the force. So, you can see that this tau-ij, which are called as components of the stress tensor, is not a vector.

This is the first thing that we understand. So, this is something a little bit more general than a vector. Why this is more general than a vector? A vector for its specification requires how many indices? It requires one index for its specification. Let us say a force f. if you write a force f, if you write f with subscript i, then i = 1 means x component, i = 2 means y component, i = 3 means z component.

But here you require two indices for its specification, why? Because it acts like a vector in a sense that it has its own components, but this individual components also depend on the orientation of the area chosen to calculate the force. So, direction normal to the surface that appears as another index. So, it requires two indices for its specification that makes it a second order tensor.

So, it is very difficult to define at this level what is a tensor and I do not want to get into that abstract mathematical discussions here. It is not very important, but we have to understand that at least the tensor is something which is somewhat more general than a vector. So, but vector is a special type of tensor. So, this example tau-ij, this is called as second order tensor because it requires two indices for its specification.

A vector requires one index for its specification so vector is a first order tensor. A scalar requires no index for its specification. So, scalar is a zeroic order tensor. So, in this way you can also have higher and higher order tensors. In the equations that we are deriving, we will come across another tensor, which is a fourth order tensor, which require four indices for its specification, but we will discuss it at an appropriate time.

But I mean just you will be having two indices does not mean that it is a second order tensor, so there are several other properties, a second order tensor maps a vector on to a vector and we will show that how this maps a vector on to a vector.

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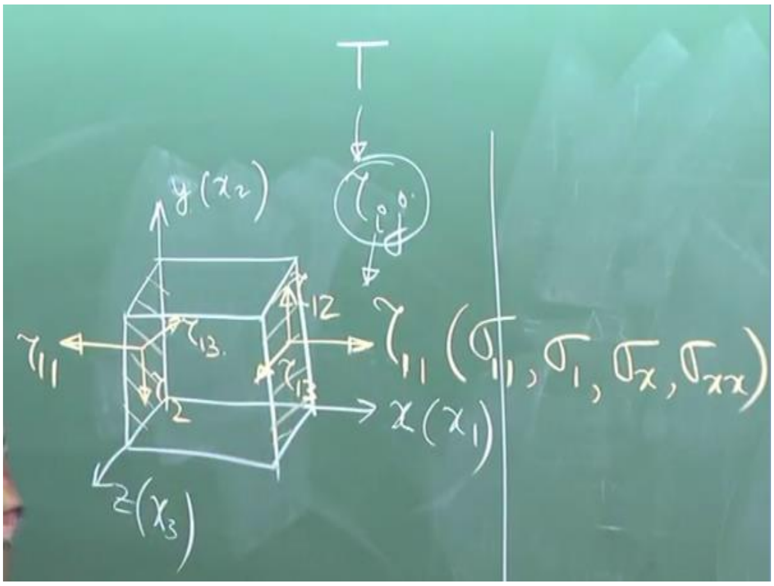

Now, what we will do is we will try to draw the various components of the stress tensor on this element, so let us take this surface. I will just write tau, you have to tell what will be the indices. What will be the first index, first index is normal to the surface, which is under consideration. This surface has normal in what direction? 1, x1 right, so the first index is 1, okay. The second index is what?

The second index is the direction in which this force is acting, so what is that? That is also 1, so this is tau-11. In books, correspondingly it is also sometimes. You can see this is the normal component of stress, so in books, sometimes to write it as normal component of stress, it is sigma-11, or because 11 is repeating sigma-1 or sigma-x or sigma-xx. These are different notations used in different books, okay. Then think about this.

This is tau12, direction normal to the surface is 1 and direction of the force is 2. Similarly, this is tau13, right. So, let us now consider this surface. There is a sine convention that if the outward normal of the surface is along a positive direction, then the force will also be shown in the positive direction. Otherwise if the outward normal of the surface is in the negative direction, we will show the force in a negative direction.

So here the outward normal to the surface is negative direction, so we will show the force in the negative direction, so this is tau11, this is tau12 and this is tau13. Similarly, for other four surfaces, you can write these things, I do not want to make this figure clumsy by doing the same thing. I mean these two examples should be good enough and you should practice to draw these or represent these forces in other four surfaces also.

So that you get a good grasp on this representation. Now, this is so far so good, so this tau-i is a notation we can use for those surfaces, which have normal either along x, or along y, or along z, but in reality, any surface, any arbitrary surface will not have normal along x, or along y, or along z, so how will we represent that.

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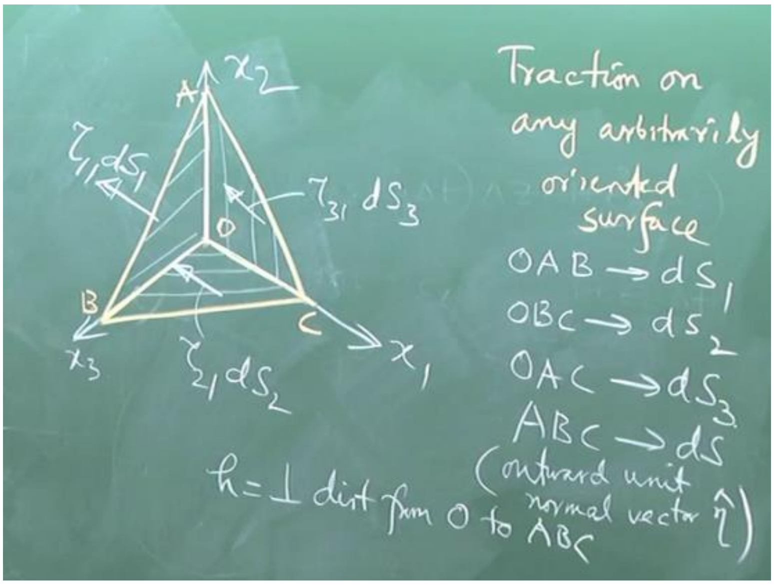

So, to understand that this is the traction on any arbitrary surface, arbitrary oriented surface rather. So, let us that we have a volume taken, which is like this. How many surfaces does this volume have? Four surfaces. Can you tell that why have we chosen this type of surface? So, in this volume, there are three surfaces, which have normals along x1, x2, and x3, but the fourth one is not.

The surface ABC is having arbitrary orientation and the other surfaces have orientation with normal along x1, x2, x3, so for other surfaces, we can represent it by the tau-ij notation, but not the surface ABC and by writing equation of the equilibrium for the entire volume, we will be able to represent the force on an arbitrarily oriented surface in terms of the force on the other surfaces.

That is the motivation of taking these type of element. So, let us try to represent the force along x. Let us try to represent the force along x. So, for this surface, OAB let us say that the surface OAB is ds1. What is the force on this? First of all, how do we represent it? What is the direction normal to this surface? Negative of x1, right. So, the force along x1 will be shown along negative of x1 by sine convention, so this is tau11, right.

First one is direction normal and second one is it is the action of the force into this is per unit area, so tau11 into ds1. Let us say that we consider OBC as ds2. This is the surface OBC, so what is its outward normal, negative of x2. Negative of x2 is its outward normal. So, the force along x1, this is along negative x1. This is tau21 ds2, but along negative x1. The third say OAC, let us say the area is ds3.

So similarly, for this also the force is tau31 ds3. Let us say the ABC is ds and ABC let us say as outward unit normal vector of eta.

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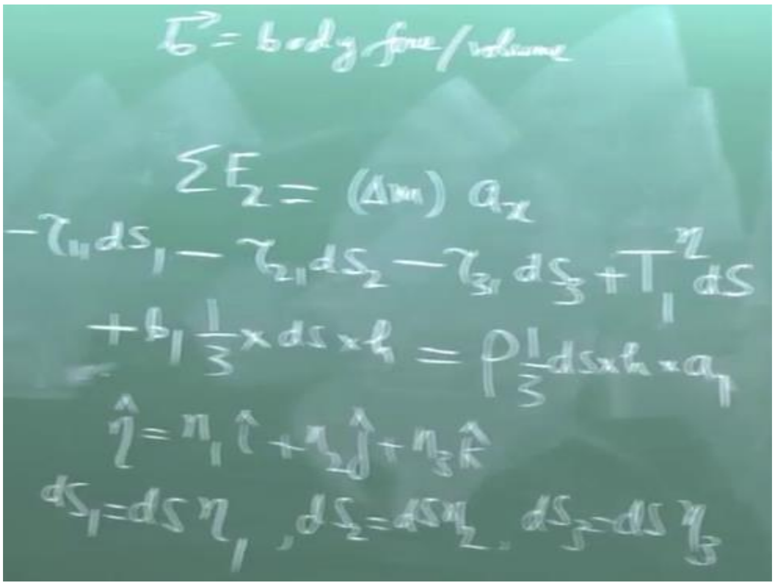

Let us write the equation of motion for this element, so we can write resultant force along x is equal to the mass, let us say delta m is the mass times acceleration along x. So, what is the resultant force along x. Minus tau11ds1, minus because it is along negative x, minus tau21ds2, minus tau31ds3, so these three surfaces we have represented. The fourth surface ABC, how will we represent that.

How will we represent the force on ABC, we will use the traction vector notation because it is an arbitrarily oriented surface. So plus, T with subscript 1 because this is the component of the force and eta superscript, which is the direction normal into ds, ds is the area of ABC. This is the surface force, then there is also a force, which is a volumetric force on a body force.

We have discussed little bit about that, but usually body force is represented by typically either force per unit mass or force per unit volume. So, let us say that B1 or B is a vector which is body force per unit volume. So, B with subscript 1 is the component of the body force per unit volume along x. What is the volume of this element? Let us say that h is equal to perpendicular distance from o to ABC.

So, one-third into ds into h that is the volume, right, h is a perpendicular distance from o to ABC. This is the resultant force. This is equal to what? The mass, rho into acceleration along x, a1. Now let us write this eta as eta1i plus eta2j plus eta3k. eta1, eta2, eta3 are the direction cosines of the unit vector, which is normal to ABC. Then, you can easily see that what is the surface ds1? Ds1 is nothing but the projection of the surface ABC on the x2, x3 plane.

ds1 is the projection of the surface ABC on the x2, x3 plane, right. So, their normal components are such that you can say that ds1 is equal to ds into eta1, right. If you take a dot product of this with i, you will get eta1. Dot product of this with i means basically component of eta in the direction of i. Similarly, ds2 is equal to ds into eta2 and ds3 is equal to ds into eta3.

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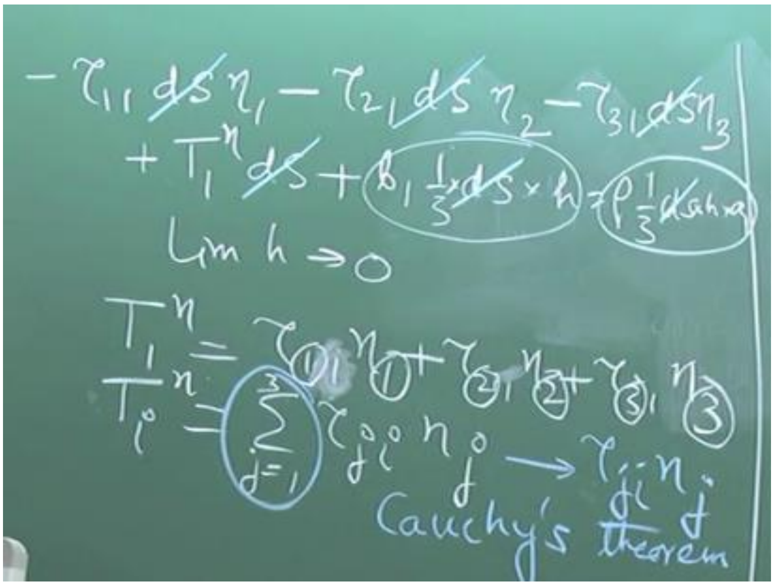

So, you can write minus tau11 ds eta1 minus tau21 ds eta2 minus tau31 ds eta3 plus T1ds plus B1(1/3) ds into h is equal to rho into 1/3 ds h into a1, right. Now you can cancel ds from all terms, so we cancel ds. So, you can then, what is our interest? Our interest is at this point o, we are trying to represent the force on the arbitrarily oriented surface in terms of the force on the other surfaces, at the point o.

So, at the point o, if we consider, which would take the limit as h tends to 0. So, take the limit as h tends to 0. So, if you take the limit as h tends to 0 what will happen, the entire volume will shrink to a point. The entire volume will shrink to the point o. So, if you take the limit as h tends to 0, this term will be 0 and this term will be 0. So that means even if there is a body force, even if the fluid element is accelerating.

It will not matter at the end for what we are deriving. It is true even if the fluid is accelerating, it is true even if it is not accelerating. It is true even if there is a body force, it is true even if there is no body force, because eventually these terms are getting cancelled. So, you can write T1 eta is equal to tau11 eta1 plus tau21 eta2 plus tau31 eta3. See this can be written as summation of tau-ji eta j, tau-ji eta j where if you use this index i instead of 1.

So, 1 is force balance along x. Instead of that you can make force balance in any arbitrary direction i, just replace 1 with i, so instead of force balance along x, you just write force balance along i. So, replace 1 with i and the remaining term summation over j, for j is equal to 1, 2, 3. This is j = 1, sorry this is i = 1. This is j = 1, this is j = 2, this is j = 3, okay. So, this is j – 1, 2, 3.

Einstein introduced a notation, when he said that when there is a repeated index like this, we will not use this summation, we will just write it as tau-ji eta j, with an invisible summation. Summation is not written, because there is a repeated index, the notation is that we will always consider as summation over the repeated index. This is Einstein's index notation and that is what is commonly used in the literature for describing these terms.

So, whenever there is any repeated term and there is no sigma, you have to keep in mind that there is actually a sigma. It is not represented. That is what Einstein introduced. So, this equation is a remarkable equation, which relates the traction vector on any arbitrarily oriented plane with the traction vector or stress tensor components on surfaces, which are oriented either along x or along y or along z.

So, you can represent force on any arbitrary oriented surface, surface force on any arbitrary oriented surface in terms of the stress tensor components, which are specific to surfaces having normal either along x1 or along x2 or along x3. This in continuous mechanics is known as Cauchy's theorem, one of the very fundamental theorems and in the next lecture, we will take it from here.

And see how we can use the Cauchy's theorem to represent the forces in the equation of motion or the conservation of linear momentum that we have written using the Reynolds transport theorem. Thank you very much.