Conduction and Convection Heat Transfer

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Lecture- 30
Integral Method for Thermal BL Analysis

Just like for the hydrodynamic boundary layer we have solved the hydrodynamic boundary layer equations by using the integral methods, we will see the corresponding integral method for the thermal boundary layer equation or the energy equation for the thermal boundary layer. So, we will begin with that, Integral method.

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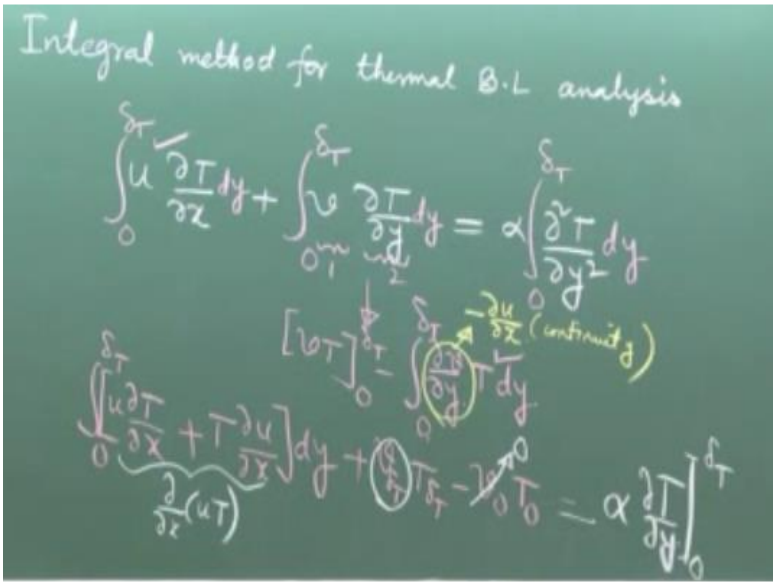

So, we will begin with the thermal boundary layer equation. As a first tip what we should do, we have already done the same thing for the hydrodynamic boundary layer equations. Now you tell what should we do for the thermal boundary layer equation. Integral method, what we are doing what we are doing basically we are integrating it with respect to y across what? Across the thermal boundary layer for the energy equation.

Now, we will integrate this by parts. This is the first function and this is the second function. So, this will be first function into integral of the second minus integral of derivative of the first into integral of the second. And we can make a further simplification in place of del v del y, you can write as minus of del u del x from the continuity equation. Therefore, this equation becomes integral of u del T del x zero to delta T.

We can club this term and this term together, plus T del u del x, plus v at delta T multiplied by T at delta T minus v at zero multiplied by T at zero. That is the stuff. This is equal to the right-hand side. Now these two terms together becomes del del x of T multiplied by u or u multiplied by T. This term is zero because v zero is zero, no penetration boundary condition at y equal to zero.

So, we must estimate what is v at delta T. So how do we estimate what is v at delta T? We will use the continuity equation and integrate the continuity equation across the thermal boundary layer.

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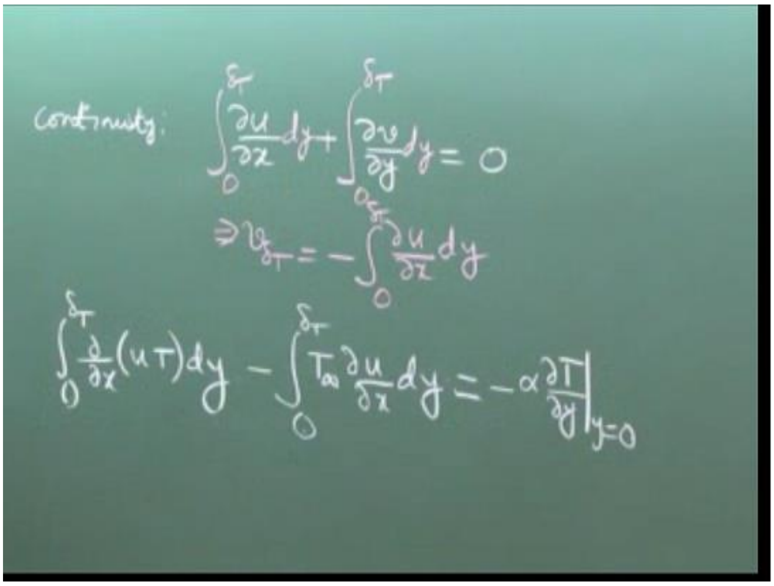

So, continuity- So from here we get v at delta T is equal to minus, what is T at delta T, what is this? This is practically T infinity. So, we can write… Right hand side what is the temperature gradient at delta T, zero, because there is no further variation in temperature. So, this becomes minus. See, just like the momentum integral equation give the wall shear stress directly, this equation gives the wall heat flux because minus k del T del y at y equal to zero is the heat flux at the wall.

It gives it through some integral expression and as we have seen, that if we use an approximate temperature profile, just like approximate velocity profile, then the result may be erroneous for the velocity or temperature but integral of velocity and temperature it is not so erroneous, typically because we are using some complimentary, like some function multiplied by one minus that function something like that. So, with this understanding:

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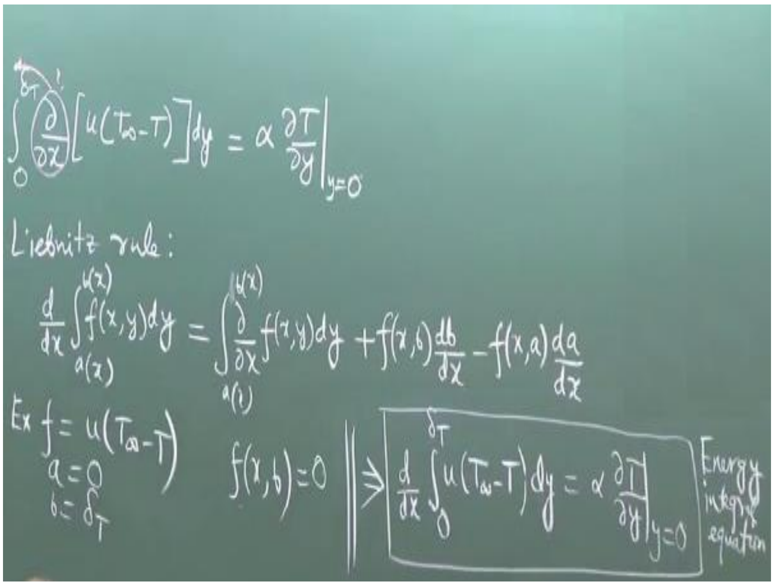

we are combining these two terms because T infinity is a constant, we can take it easily within and outside the derivative without any problem. Next, our issue will be whether we can bring this derivative out of the integral or not. And for that we will use the Leibniz rule to check. So, in this example f is u multiplied by T infinity minus T, a is zero and b is delta T. So, what is f (x, b)? That is f at y equal to delta T, zero because T is T infinity.

At y equal to delta T, T is T infinity. So, this term become zero. What is the other correction term? This is zero because a is zero, so da, dx is zero So that means here also we can just like what we could do for the momentum equation we can take this out of the integral without any problem because the correction terms are zero. So, we can write, this equation is called as energy integral equation.

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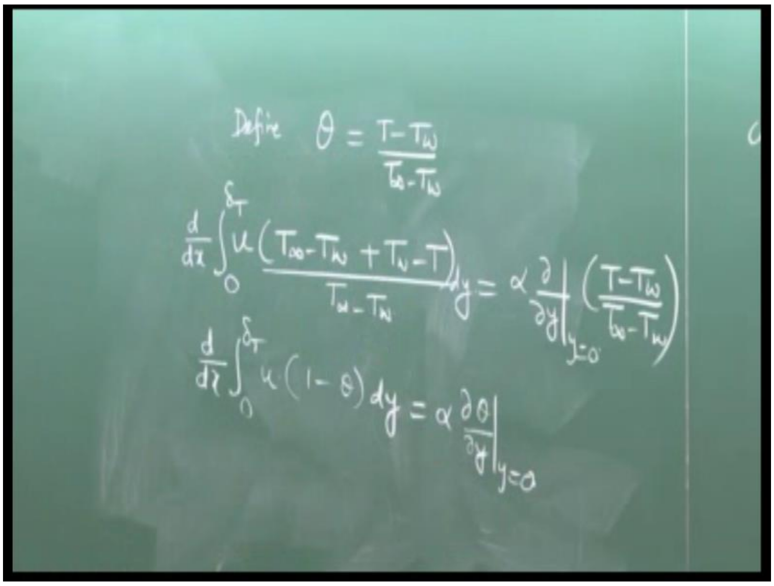

So, we can write this equation in terms of non-dimensional velocity and non-dimensional temperature. So, if you define theta is equal to T minus T wall by T infinity minus T wall. The purpose of defining theta in this way is that this has similar scaling as u y u infinity. See, u y u infinity, zero at the wall and one at the edge of the hydrodynamic boundary layer. This is zero at the wall and one at the edge of the thermal boundary layer.

So, u y u infinity and theta have the same scaling. So, if you use this non-dimensional temperature, then you can write d dx of u…So we have added and subtracted T wall. So, this becomes d dx of zero to delta T, U multiplied by one minus theta, one dy is there. This is the energy integral equation in terms of the normalized temperature. So, we will make an assessment of the situation using the two limiting cases as we had done for the similarity solution, one is Prandtl number much less than one and the other is Prandtl number much greater than one.

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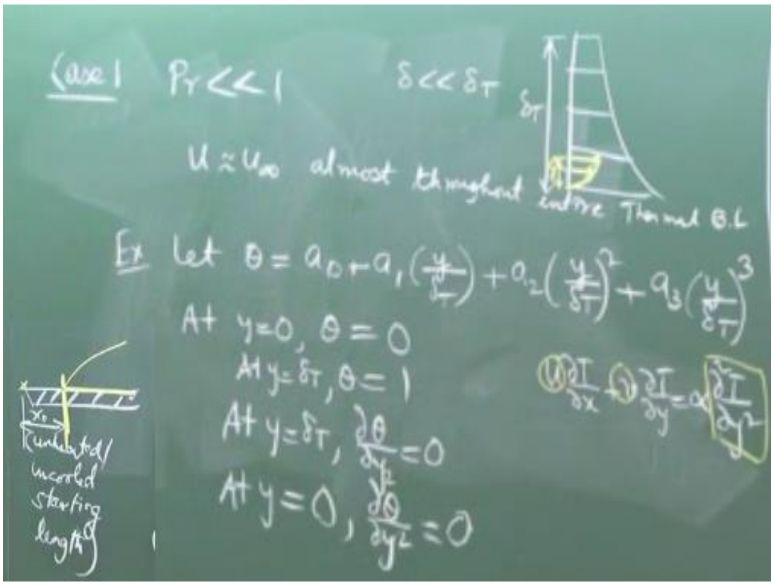

So Prandtl number much less than one means, delta is much less than delta T. So, if this is the delta T then your delta is much less. So almost throughout the thermal boundary layer, u is u infinity. So, u is approximately equal to u infinity, almost throughout entire thermal boundary layer. So that u part is there, what about theta?

So, theta just like we could use approximate velocity profiles for the velocity for the momentum integral equation similarly we can make use of approximate temperature profiles. So as an example, this kind of velocity profile we had taken for the momentum integral equation, similar thing we are taking for the temperature. You could take other profiles but you have to find the constants based on the essential boundary conditions.

So, what are the boundary conditions in terms of priority. What is the most important boundary condition or what are the most important boundary conditions? See, at least two constants if those are there, those should match the values at y equal to zero, and y equal to delta T, that much should be there. So, at y equal to zero, what is theta? Theta is zero. At y equal to delta T theta is one.

Then at y equal to delta T, del theta del y is equal to zero. And the fourth one at y equal to zero, so if you look into the equation, u del T, del x, plus v del T del y is equal to alpha del two T del y two. So, at y equal to zero, both u and v are zero. So therefore, this must be equal to zero. So, in terms of theta. So, you can see, if you cast it in proper non-dimensional form, the boundary conditions look like exactly the same as those where for velocity.

When we approximated the velocity, profile using this formula. Therefore, the constant will also be the same because the similarity is a mathematical similarity. Mathematics doesn't understand that one is heat transfer, one is fluid mechanics. If you bring exact similarity into picture the values will also be similar. So, you will get theta is equal to what was the velocity profile? 3 by 2.

Now instead of y by delta it will be y by delta T for the temperature profile minus half y by delta T whole cube. So, if you work out this by substituting these conditions you will find out exactly these.

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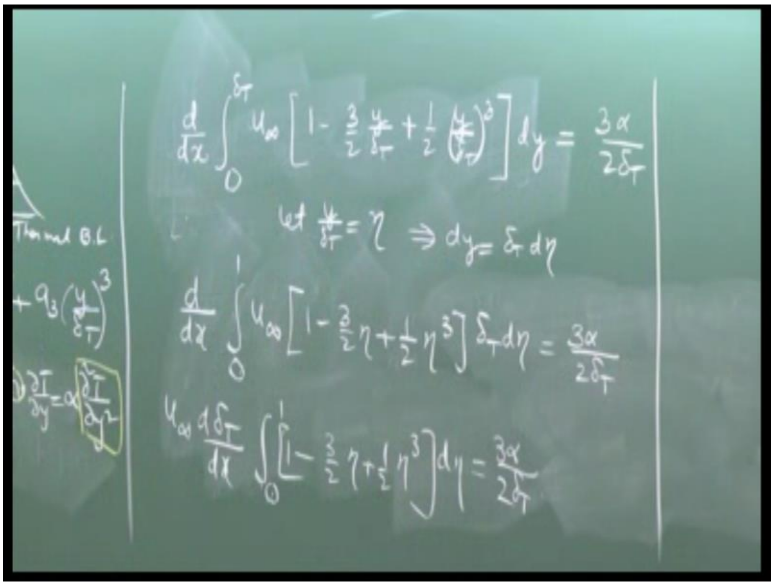

So, we will substitute that here in this equation, d dx of integral zero to delta T, u becomes u infinity, multiplied by one minus theta, okay? So, we have substituted the temperature profile in an energy integral equation. Then let us assume y by delta T is equal to eta. So dy delta T multiplied by d eta. So, d dx of… This is equal to 3 alpha by 2 delta T.

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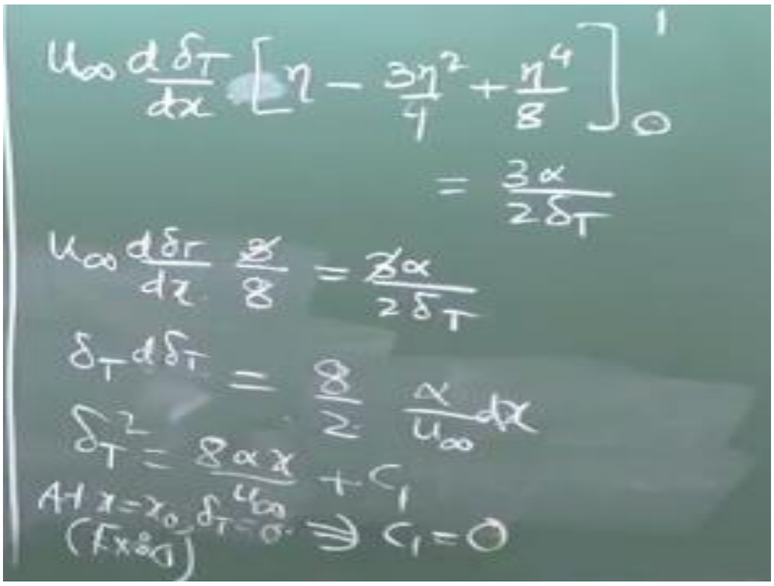

Now we will integrate this. So, if we do it quickly, u infinity, d delta T dx. So, this becomes 8 minus 3, 5; 8 minus 6 two, plus one. So, 3 by 8. So, delta T, d delta T, dx. Now what is the boundary condition to get. This is dx. So, if we integrate this, if we integrate this, then delta T square is equal to 8 alpha x by u infinity plus some constant c1. So, question is what is the boundary condition to get this value of c1?

What is the boundary condition at x equal to zero? See, this is a subtle distinction between the thermal and hydrodynamic boundary layer, at x equal to zero, the hydrodynamic boundary layer thickness must tend to zero. But at x equal to zero, the thermal boundary layer thickness may not be zero. I will give you one example. Let us say, that this is the solid boundary and this plate is heated starting from here.

So, this zone is only heated. So, the thermal boundary layer will develop from here and not from here. Okay? So, the hydrodynamic boundary layer the frictional effect will be failed from, starting from here itself. But thermal boundary layer, the heating or cooling effect of the wall will be faced only from that location from where you start hitting on cooling. So, you may have a length x zero which you may say as unheated or uncooled starting length.

So, we can say that at x equal to x0, delta T equal to zero. As a special example, let us take x zero equal to zero. But this is a special example. Do not think that it is a generalization, x0 could be any value but as a special case we are considering that the entire plate is heated or cooled. So that means, you will get c1 equal to 0.

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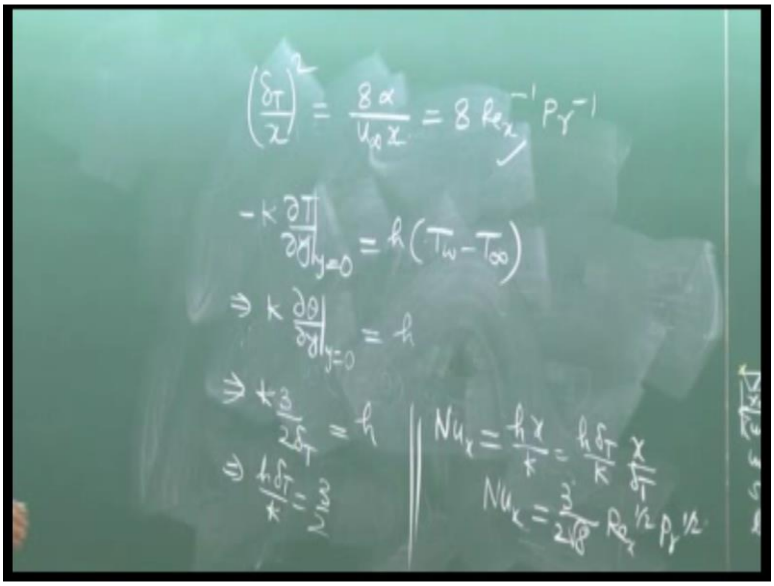

So, delta T by x, is equal to 8 alpha by u infinity x, right? So, this is 8 multiplied by Reynolds number to the core, minus one into Prandtl number to the pr minus one, because this is u infinity x by nu into alpha by nu. Alpha by nu is one by Prandtl number. Okay? Now our objective is to get the Nusselt number. So, we will use the boundary condition at the wall. So, theta is T-T wall by T infinity by T wall.

So, we have brought this within this derivative. Is it true if the wall temperature is a function of this? Yes, or no? Is the transformation from this form to this form valid if T wall is a function of x or not? Doesn’t matter because this is derivative with respect to y. This is not derivative with respect to x. So, if T wall varies utmost it will vary with x, but wall is along x. So, wall temperature cannot vary with y.

So, with respect to y derivative T wall may be a function of x but still it does not have any dependence on y. So, k del theta del y at y is equal to zero is this. This means h delta T by k is equal to 3 by 2. Now what is required is Nusselt number. That is hx by k, Nusselt number at x. So, this is 3 by 2 and x by delta T from this expression, so 3 by 2 root 8 Reynolds number to the power half multiplied by Prandtl number to the power half.

If you recall in the similarity solution it was 1 by square root of pie, right? So, you can compare that how accurate this is as compared to that. Find a numerical values and you will see that the difference is not that much. So actually, the integration has smoothened out many sources of air and makes the result more or less acceptable. Now I will discuss about one very important point before moving on to the other case, Prandtl number much better than one.

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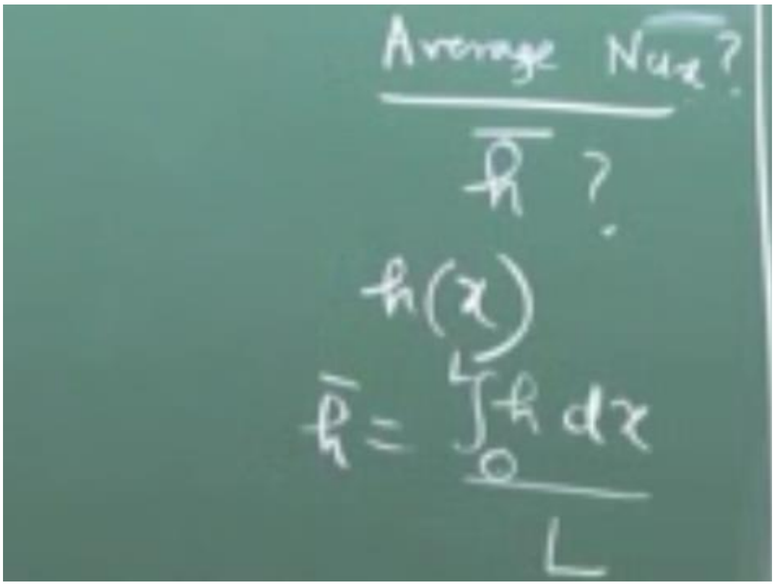

That important point is that how to get the average Nusselt number, why average Nusselt number is important because Nusselt number is a function of x. So, along the length of the plate it will vary. So, if somebody asked that what is the average heat transfer coefficient, how will you calculate it. So, in other words, what is the average heat transfer coefficient. So, you can see that age is a function of x, why h is a function of x?

You look at this expression, h it is a function of delta T and delta T is a function of x. So, you can write h as a function of x by substituting delta T as a function of x here. So, h average is equal to integral of h dx from x equal to zero to L divided by the total length. And Nusselt number average is based on this h average. Okay? So many times, this is a calculation that engineers need to predict what is the average heat transfer from the wall to the fluid or fluid to the wall depending on what is heated and what is cooled.

So, for that h as a function of x gives you the local variation, but you may also be satisfied with the average trend and for that you have to do that integration. So that being a trivial integration because delta being a polynomial symbol, so how delta will vary with x. So, delta square is equal to, delta square is proportional to x or delta T square is proportional to x. So, delta T will vary with square root of x, x to the power half.

So, the boundary layer profile that we draw is actually x to the power half. The profile of the boundary level, this one that we draw, it is actually function of the form x to the power half. Hydrodynamic boundary layer and thermal boundary layer for Prandtl number much less than one. So, delta T as a function of x when you substitute some constant into h to the power half that will give you h, will vary with x to the power minus half.

And then that integration you can do. Okay? Now we will do the next case which is a little bit more important than the previous case. “Professor - student conversation starts” No. That is alright. So, your question is that delta T is equal to zero if the heating or cooling starts from there also it is zero. But here the problem of growth of thermal boundary layer starts from here. So, your domain starts here.

This is not a domain of your interest because in this domain there is no heat transfer. You have to see that what is your domain of heat transfer. So, in this domain all temperatures are same. So, there is no heat transfer. So, heat transfer domain starts from where you have the wall temperature different from T infinity. So, within this part of the domain, the wall temperature and T infinity all are the same.

So, there is no heat transfer. So, you have to keep in mind that this equation is applicable in the domain where you have heat transfer. Okay? So that is why you apply with the domain starting from here. “Professor - student conversation ends” Prandtl number much greater than one means delta is much greater than delta T. So, you cannot say, that u is u infinity within the thermal boundary layer. So, we start with the energy integral equation,

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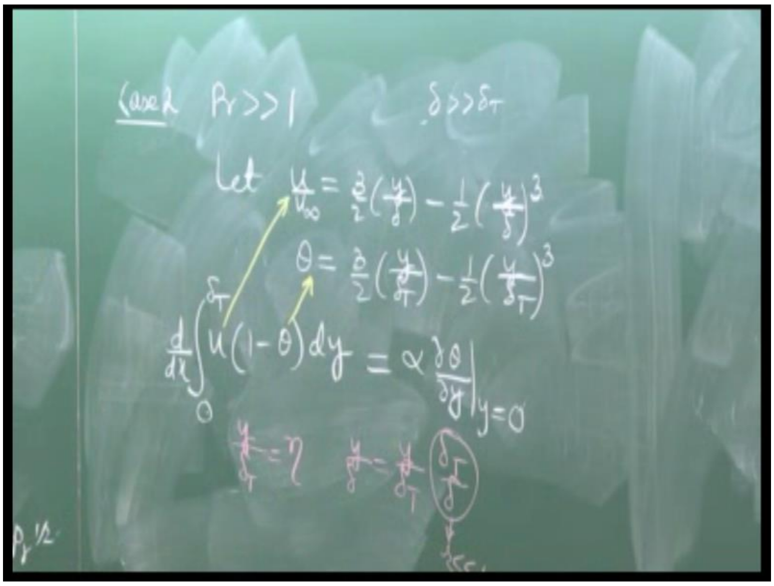

So, in this equation we will substitute this u and we will substitute this theta. So, we will assume that y by delta T equal to eta. So, y by delta is equal to y by delta T multiplied by delta T by delta. Let us say that delta T by delta is equal to a ratio r. What is the value of this r? Small or large? What is r, small or larger? It is small, right? So, this r must be much less than one. So, this we will use for algebraic simplification.

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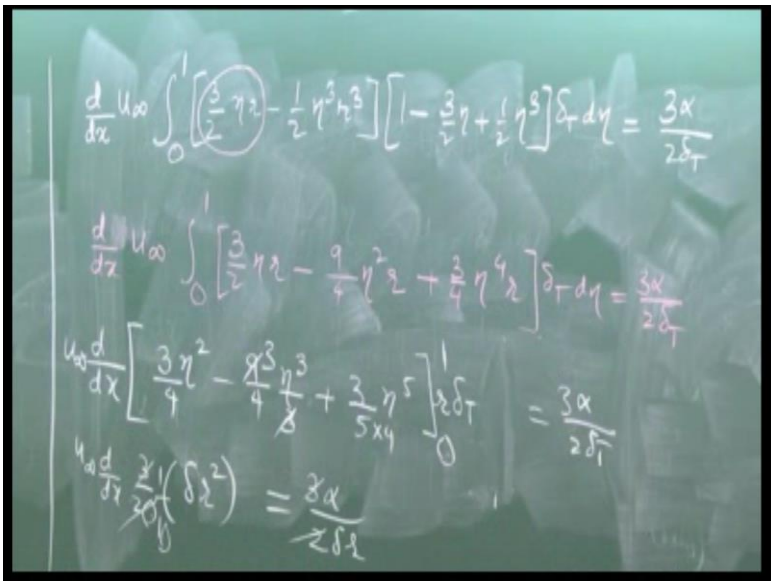

Now, when you multiply this term with this term, out of these two which one will be of dominance? This one will be of dominance because r is expected to be much greater than r cube for small r, right? So, you can write this. So, 3 eta square by four minus 9 by 4 eta cube by 3, plus 3 by 5, eta to the power 5 multiplied by 4. r multiplied by delta T becomes, so r is equal to delta T by delta. Can you quickly do it and tell me what is the value?

So, these two cancels, right? So, 3 by 20. There is one r, right? So, r multiplied by delta T. So, 3 by 20, in place of delta T we can write delta into r, right? So, delta r square. Why we are writing this is because delta we already know. We are interested to solve for r, that will give us what is delta T. So, 3 by 20 multiplied by delta r square is equal to 3… Delta T we have written delta into r, okay? So, these 3 gets cancelled and this becomes 1 by 10.

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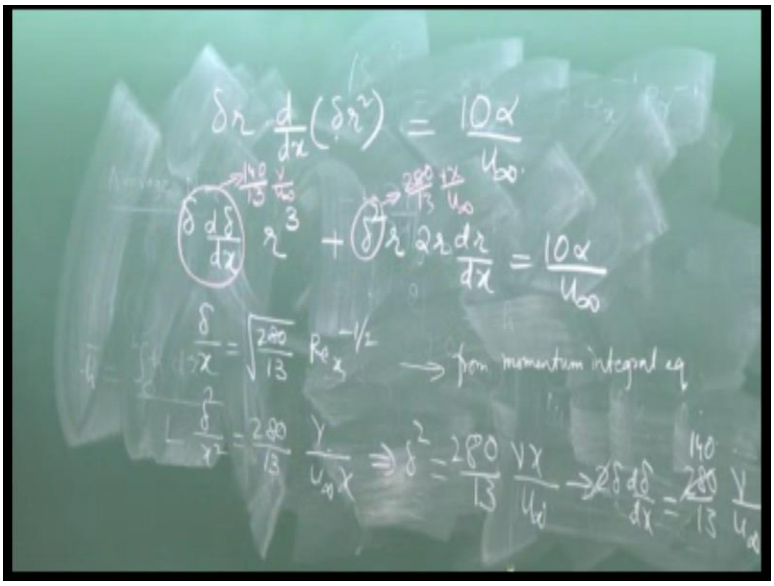

So, u infinity, may be u infinity we will take in the right-hand side. So, d dx of, so we can write delta r into d dx of delta r square is equal to 10 alpha by u infinity. So now we can write delta, d delta dx multiplied by r cube plus delta square r multiplied by 2r dr dx. Now this delta is a function of x, we can get from the hydrodynamic boundary layer equation. So, if you recall, if you had completed the exercise of finding delta as a function of x from momentum integral equation.

In the momentum integral equation, if you substitute u by u infinity is equal to 3 by 2 y by delta minus half y by delta whole cube and integrated you will get this delta as a function of x. So, we will use this, so delta square by x square is equal to 280 by 13 multiplied b Reynolds number to nu power minus one. So, nu by u infinity x, right? So, delta square is equal to 280 by 13 multiplied by nu x by u infinity.

So, you can clearly see that delta varies with x to the power half, the edge of the boundary layer. So, we require delta d delta dx, so if you now differentiate both sides with respect to x, so 2 deltas, d delta dx is equal to 280 by 13 nu by nu infinity, okay? That means delta d delta dx is 140 by 13 multiplied by nu by nu infinity. So, let us write it down there. Then this term delta square in place of delta square what we will write? 280 by 13 nu x by nu infinity.

So here there is one r cube, here you have r square dr dx.

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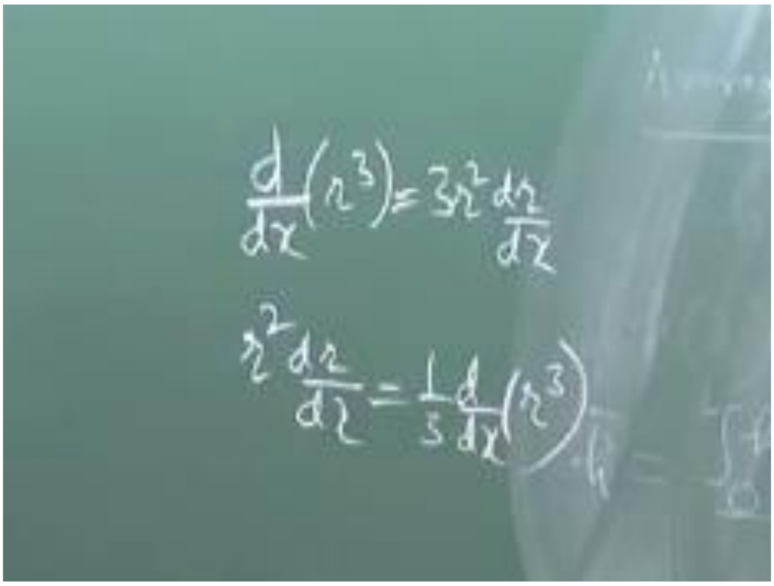

So, what is d dx of r cube? 3 r square dr dx. So, in place of r square dr dx we can write 1 by 3 d dx of r cube, right? So, let us move on to the next step.

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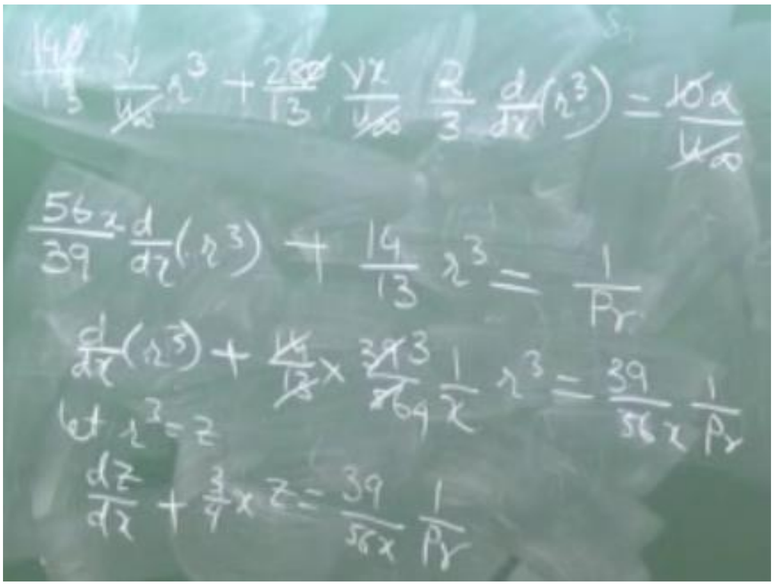

So, 140 by 13 nu by nu infinity r cube plus 280 by 13 nu x by u infinity multiplied by 2 multiplied by r square dr dx is 1 by 3 d dx of r cube, is equal to 10 alpha by u infinity, okay? So, u infinity gets cancelled from all sides, if you take the nu in the right-hand side, alpha by nu becomes one by Prandtl number. So, we can write, okay before that we can possibly simplify a little bit.

So, 56 by 39 d dx of r cube 56 by 39 x, oh sorry, 56 x by 39 d dx of r cube plus 14 by 13 r cube is equal to 1 by Prandtl number. Just check whether this is correct or not? So, we can write d dx of r cube plus 14 by 13 multiplied by 39 by 56, one by x, okay? So, this equation if you take let r cube equal to z, this equation you can write dz dx plus 3 by 4 x multiplied by z is equal to 39 by 56 x into one by Prandtl number.

So, this equation is of the form dy dx plus py is equal to Q. So, the solution of this equation relies on multiplying with the integrating factor. So, let us identify the integrating factor.

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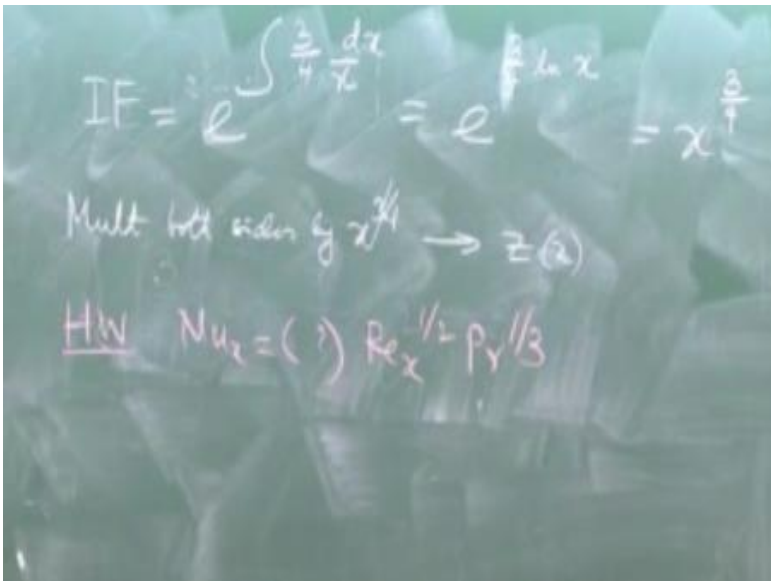

So, integrating factor, e to the power integral, right? So, e to the power 3 by 4 ln x. So x to the power, right? So, you multiply both sides by x to the power 3 by 4 and that should give you what is z as a function of x, remaining is straight forward. See it is important that you work out each and every individual step that gives you a lot of confidence. You will see, I mean many books will write from straight forward calculation it follows.

I mean, as a student I know, whenever I was a student never I found that it is such a straight forward calculation as it is mentioned in the book. So, you would see that the steps that we have gone through there are many steps which if you had tried to do by yourself it would have created a lot of problem and it is important that is why I have done it up to the stage from here.

It is really straight forward because then the remaining is just after putting the integrating factor you can get the solution. So, you please work it out completely and find out. So, this is like the homework that I am leaving on you. Find out the Nusselt number is equal to some constant into Reynolds number to the power half into panel number to the power one third. What is this constant?

It will - If you have done it correctly it will come out to be something in the range of 0.33 something like that. So please do that. Please complete this, but I have done enough number of steps so that I mean the remaining should be easy for you to complete. I mean, essentially for deriving all this as I told you that these derivations are, these are the problems from this part of the course and you must learn how to do these derivations.

No formula based study please. So if you know, or if you learn how to do these derivations maybe exactly these derivations may not come in the exam but if you have followed this you will see that some difference in velocity profile, some different temperature profile, but the method will be very similar. So please gain some confidence and as I want to repeatedly tell you that I do not want to discuss anything in the undergraduate class which I cannot derive in the board.

So, thousands of formula, which I cannot myself derive in the board, I cannot expect that you will remember this and you will derive this and all this. So, whatever are the essential fundamental things which can be derived in the scope of an undergraduate course, that part of convection we are only covering. So, this will give you the foundation for learning the advanced topics in convection. Okay, we will stop here today.

In the next lecture, so what we have done up to today is, we have discussed about force convection over a flat plate. But in engineering maybe another sort of important problem is what happens for force convection inside a pipe or inside a duct or inside a channel, not in an open space but in a confinement. So that is called as internal force convection. So, from our next lecture we will start discussing on internal force convection. Thank you.