Advanced Concepts In Fluid Mechanics
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Lecture – 29
Potential Flow (Contd.)

In the previous chapter, we discussed about the definition of a complex potential function which is inclusive of the velocity potential and the stream function. Now we need to discuss very carefully about the commonality between the velocity potential and the stream function. One interesting confluence is we have already discussed that except for the stagnation point, they are orthogonal to each other. We considered a two-dimensional, incompressible and irrotational flow. For two-dimensional and incompressible flow we can write $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$. For irrotational flow we can write that the curl of the velocity vector is a null vector, i.e. $\nabla\times\vec{V}=\vec{0}$ which is equivalent to the form of $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$ for a two-dimensional situation. Now if we substitute the expressions of the velocities u and v in the equation $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$, we get $-\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0$ which is clearly the Laplace equation in two-dimension. So we have $\nabla^2\psi=0$ for two-dimensional and incompressible flow. For two-dimensional and irrotational flow we can write $u=\frac{\partial\phi}{\partial x}$, $v=\frac{\partial\phi}{\partial y}$. For incompressible, two-dimensional and irrotational flow (i.e. if the additional consideration of incompressibility is taken into account) we have $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ which after substitution of $u=\frac{\partial\phi}{\partial x}$, $v=\frac{\partial\phi}{\partial y}$ becomes $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ or, $\nabla^2\phi=0$. Interestingly, both ψ and ϕ satisfy the Laplace equation in two-dimensional plane. So, if we write $\nabla^2\phi=0$ and $\nabla^2\psi=0$ we can also write $\nabla^2\left(\phi+i\psi\right)=0$. That means we can use the complex potential function $F=\phi+i\psi$ as a basis to capture the field variables ϕ and ψ. This is the first aspect. Besides, if $F=F_1$ is a solution or $F=F_2$ is a solution then we have $\nabla^2 F=0$, or, $\nabla^2 F_1=0$ and $\nabla^2 F_2=0$.

Also, $F_1 + F_2$ will be a solution of this equation, i.e. $\nabla^2 (F_1 + F_2) = 0$. Not only that, if there are other solutions like $F = F_3$ or $F = F_4$ then those will also become the solution just by simply adding. This linear superposition works because the Laplace equation is a

linear partial differential equation. Laplace equation is a beautiful equation because form a mathematician's or an applied mathematician's perspective it might have different applications but generically it has the template form of $\nabla^2 (\ \ ) = 0$. Whereas, from the point of view of Physicists and Engineers it can applied on one side to solve the problems in irrotational flow in Fluid Mechanics; on another side in Electrical Engineering to solve the electrical potential distribution where there are no volumetric charges. So we can clearly see that the Laplace equation address a wide range of Physical problems and therefore, the solution of the Laplace equation is very important in applied mathematics community. Here, we are simply using the superposition and this superposition gives a very simple but an elegant way of approaching the Laplace equation. Now we will try to find out the functions F_1, F_2 etc. for simple types of flows. Let us consider the example of uniform flow; uniform flow is the simplest inviscid and irrotational flow. So, we will try to find out the function like F_1. There are other types of inviscid and irrotational flows which will give the complex potential functions like F_2, F_3 etc. Eventually we will build up on the system by considering a superposition of those things and that will be our objective.

Our objective will be to solve for ϕ and ψ. Now question arises about the importance these two variables ϕ and ψ. From our earlier discussions we can remember that the contour of a body itself is a streamline. The reason why the contour of a body itself is a streamline is that, by definition of a streamline there cannot be any flow which penetrates through it. A physical body cannot be penetrated by a fluid until and unless there are holes in the body. So we may recognize out of here is that, by obtaining stream functions we can also solve an inverse problem. We can tell that what could be the contour of a body passed which the flow is being described by the stream function. The reason is that a special case of the stream function (stream function being set to be equal to zero) will represent the contour of the body. For example, we can have a question about what could be the nature of the potential flow passed a circular cylinder. To understand that we can make a superposition of certain simple types of this potential *F* (i.e. elementary types of complex potential *F*); we can superimpose them together and try to obtain a contour of a body which looks like a circular cylinder. So our objective will be to find the superposition that gives the flow passed a circular cylinder or the flow passed an elliptic cylinder.

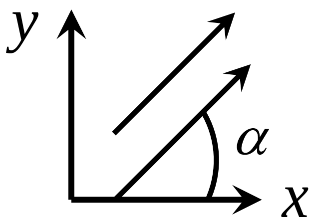

Figure 1. Example of uniform flow making an angle α with the *x* axis.

Now we will start with the first example which is the example of uniform flow. So, to model a flow, we are actually interested about the stream function and the velocity potential. In this example we know the velocity because the flow is uniform. Let us consider this uniform flow makes an angle α with the *x* axis. The *x* axis and the *y* axis are shown in figure 1 and there is the uniform flow velocity u_∞. So, $u = u_\infty \cos\alpha$ and $v = u_\infty \sin\alpha$. Now the objective is to find the complex potential corresponding to this. From the discussion in the earlier chapter we have $\frac{dF}{dz} = u - iv$ (if $F = \phi + i\psi$, then $\frac{dF}{dz} = u - iv$ which was derived earlier). Substituting $u = u_\infty \cos\alpha$ and $v = u_\infty \sin\alpha$ in the expression of $\frac{dF}{dz}$ we get $\frac{dF}{dz} = u_\infty \cos\alpha - i u_\infty \sin\alpha = u_\infty e^{-i\alpha}$. Let us take the simple example of the special case $\alpha = 0$ which is the uniform flow along the *x* axis. In that case we get $\frac{dF}{dz} = u_\infty$. Integrating both sides we get $F = u_\infty z$; there will be an additional constant which can always be set equal to zero as a basis. So, $F = u_\infty z = u_\infty (x + iy) = \phi + i\psi$. So, isolating the real part and the imaginary part we get $\phi = u_\infty x$ and $\psi = u_\infty y$.

Now let us try to draw $\phi = \text{constant lines}$; $\phi = \text{constant lines}$ means $x = \text{constant lines}$. Similarly, $\psi = \text{constant lines}$ means $y = \text{constant lines}$. So, $\phi = \text{constant}$ lines are parallel to the *x* axis while $\psi = \text{constant lines}$ are parallel to the *y* axis. $\phi = \text{constant lines}$ are shown in figure 2 as $\phi = c_1$, $\phi = c_2$ and $\phi = c_3$ respectively while $\psi = \text{constant}$ lines are shown in figure 2 as $\psi = k_1$, $\psi = k_2$ and $\psi = k_3$ respectively. Overall this figure is called as the flow net with orthogonal ϕ 's and ψ 's are together laid out to form a network. So this is the simplest type of flow but at least we can get a complex potential $F = u_\infty z$ for uniform flow along the *x* axis. Before getting into the next example, which may demand the use of the polar coordinates, we will try to write the function $\frac{dF}{dz}$ in terms of the polar coordinates.

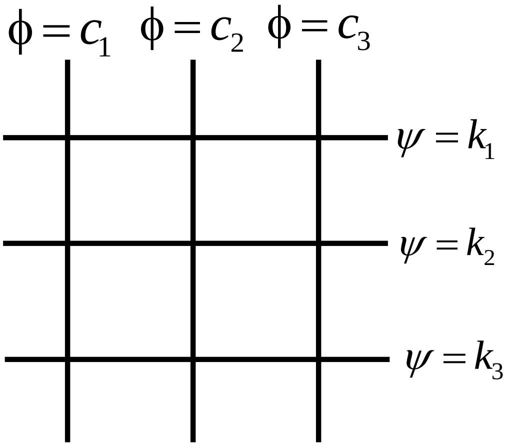

Figure 2. Flow network which is a combination of $\phi = \text{constant}$ and $\psi = \text{constant}$ lines.

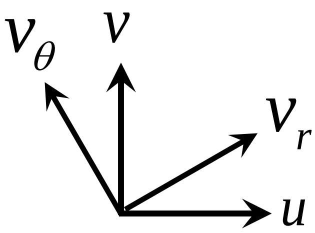

Figure 3. Representation of the velocity components in the polar coordinate system.

In case of polar coordinates we have v_r and v_θ as the velocity components instead of *u* and *v* velocity components. This is the same thing but expressed in *r*-*θ* coordinate form where the angle between v_r and *u* is *θ* as shown in figure 3. Now we resolve the v_r and v_θ velocity components along the *x* and *y* directions and we get $u = v_r \cos\theta - v_\theta \sin\theta$ and $v = v_r \sin\theta + v_\theta \cos\theta$. Now we recall the expression of $\frac{dF}{dz}$ which is given by $\frac{dF}{dz} = u - iv$. Substituting $u = v_r \cos\theta - v_\theta \sin\theta$ and $v = v_r \sin\theta + v_\theta \cos\theta$ in the expression of $\frac{dF}{dz}$ we get $\frac{dF}{dz}$ in terms of the velocity components v_r and v_θ as

$$\frac{dF}{dz} = v_r \cos\theta - v_\theta \sin\theta - i\left(v_r \sin\theta + v_\theta \cos\theta\right) = \left(v_r - i\,v_\theta\right)\left(\cos\theta - i\sin\theta\right) = \left(v_r - i\,v_\theta\right)e^{-i\theta}.$$

With this in purview we will consider our next example which is called as the point source. A point source is a point from which we have a radial spreading of flow as shown in figure 4. Let us consider the source is located at the origin. Then the velocity distribution reads as $v_r = \frac{c}{r}$ and $v_\theta = 0$.

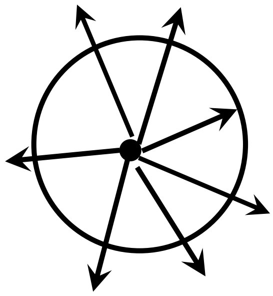

Figure 4. Example of a point source with the source being located at the origin.

From the expression $v_r = \frac{c}{r}$ we can see that at $r = 0$, there is a singularity but close to $r = 0$ we have a very high radial velocity v_r. Also this radial velocity v_r decreases with $\frac{1}{r}$ as the radial location *r* increases. This may seem to be abstract but what is not abstract is the flow rate (q_r) which can be obtained from this velocity distribution. At a radial location *r*, the flow rate q_r is given by $q_r = v_r \times 2\pi r$ where the length perpendicular to the plane of the figure is chosen as 1. There is no need of breaking it up into θ-elements because v_r is not a function of θ, v_r is a function of *r* only. Substituting $v_r = \frac{c}{r}$ in the expression $q_r = v_r \times 2\pi r$ we get $q_r = \frac{c}{r} \times 2\pi r = 2\pi c$. This gives rise to an interesting observation that the flow rate (q_r) is not a function of *r*. Although the velocity v_r is a function of *r*, the flow rate (q_r) is not a function of *r*. We can write this flow rate q_r as the universal flow rate *q*, i.e. $q = 2\pi c$, or, $c = \frac{q}{2\pi}$. So, the constant *c* in the expression of v_r can be replaced by a more meaningful parameter $\frac{q}{2\pi}$.

In this scenario we can use the polar coordinate more effectively. We recall the expression of $\frac{dF}{dz}$ in the polar coordinate which is given by $\frac{dF}{dz} = (v_r - i v_\theta) e^{-i\theta}$. Here we substitute $v_r = \frac{c}{r} = \frac{q}{2\pi r}$ and $v_\theta = 0$ to get $\frac{dF}{dz} = \left(\frac{q}{2\pi r} - 0\right) e^{-i\theta} = \frac{q}{2\pi r e^{i\theta}}$. Now $r e^{i\theta}$ is equal to the complex number *z*; the equivalent representation of the complex number *z* in the *r*-θ plane is given by $z = r e^{i\theta}$. The term $e^{i\theta}$ is equal to $\cos\theta + i \sin\theta$ and thus, $z = r e^{i\theta} = r(\cos\theta + i \sin\theta) = r\cos\theta + i r \sin\theta = x + i y$ where $x = r\cos\theta$ and $y = r \sin\theta$. Using this expression we can rewrite $\frac{dF}{dz}$ as $\frac{dF}{dz} = \frac{q}{2\pi z}$. Integrating both sides we get $F = \frac{q}{2\pi} \ln z$. So we can sum it up and say that the complex potential function *F* for a source will be related to the flow rate *q*. Sometimes *q* is called as the strength of the source. Now question arises about the corresponding scenario when it is a sink instead of a source. If it is a sink instead of a source, then everything will remain the same with just *q* will be replaced by – *q*. Now if there is a source of strength *q* located at $x = a$, then we will just require a translation of the coordinate. Let us define $X = x - a$; when *x* is equal to *a*, *X* is equal to zero. In terms of *X*-coordinate it will become $F = \frac{q}{2\pi} \ln Z = \frac{q}{2\pi} \ln(X + iY)$. In the present example, Y is equal to zero, so we get $F = \frac{q}{2\pi} \ln X = \frac{q}{2\pi} \ln(x - a)$. Similarly, wherever the source is located, we can make a shifting of the coordinate system. If it is located on the *y* axis at $y = b$, then we can introduce $Y = y - b$. In the generic representation of the function F, we should express *F* as $F = \frac{q}{2\pi} \ln(X + iY) = \frac{q}{2\pi} \ln(x - a + iY)$. Since *F* is a complex potential function, it should include the *y*-component also. But in the present example it really does not matter. It is therefore better to express in a generic way using the combination of *x* and *y* components. Then the special case will be when there is a source lying on the *x* axis or lying on the y axis. These special cases are treatable by reducing it to either x or $i\,y$. If it is located along the x axis, then z will be simply equal to x and if it is located along the y axis, then z will be simply equal to $i\,y$. In the present example, it will be $F=\frac{q}{2\pi}\ln Z=\frac{q}{2\pi}\ln(z-a)$. Let us take an example of the y axis to make the understanding more clear. Define $Y=y-b$; new Z will be equal to $Z=x+iY=x+i\,y-i\,b=z-i\,b$. So the part $z=x+i\,y$ is not disturbed but the disturbed part will be a along the x axis or becomes $i\,b$ along the y axis. So in our present example the complex potential function is $F=\frac{q}{2\pi}\ln(x+i\,y-a)=\frac{q}{2\pi}\ln(z-a)$ where X is shifted to become equal to $x-a$. The term $z=x+i\,y$ is present there and the parameter a is isolated which reflects the shift of position in the x axis. So, a indicates the shift in the location along the x axis. Overall, we have the form of the complex potential $F=\frac{q}{2\pi}\ln\left(z+(\ \ )\right)$. The term z will always be there irrespective of whether there is a shift along the x axis or along the y axis; the definition of $z=x+i\,y$ will always be present. Besides, there will be an additional term (denoted by the circled part in the expression of $F=\frac{q}{2\pi}\ln\left(z+(\ \ )\right)$) indicating the shift in either in the x axis or in the y axis. At this stage we stop the discussion of the present chapter which will be continued with more examples on superposition of these basic flows.