Advanced Concepts In Fluid Mechanics
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Lecture - 46
Thin Film Dynamics (Contd.)

In the previous chapter we proceeded somewhat towards deriving the final form of the thin film equations. In the present chapter, to complete the final form, we will first rewrite the equations derived so far so that we can go ahead further with those equations.

Governing equations:

Continuity equation: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

x-momentum equation: $$0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x' \tag{2}$$

y-momentum equation: $$0=-\frac{\partial p}{\partial y}+\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}f_y' \tag{3}$$

where f_x' and f_y' are the dimensional body forces per unit volume.

Boundary conditions:

Tangential force balance (at the interface): $$\frac{\partial u}{\partial y}=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma \tag{4}$$

Normal force balance (at the interface): $$p_s-p_0=-\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}+p_{ex} \tag{5}$$

Kinematic boundary condition: $$v_i=\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\right)_i \tag{6}$$ (‘*i*’ indicates interface)

Along with that we have no-slip boundary condition, i.e. at $y=0$, $u=0$.

These equations eventually can be manipulated to obtain a velocity profile. We are just recapitulating the things discussed in the earlier chapters before writing the final expressions. We first integrated the *y*-momentum equation (3) to get pressure as a function of *y* as $p=\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}f_y'\left(y-h\right)+p_s$. Then we substituted it in the normal force balance equation (5). At $y=h(x,t)$, the pressure p is equal to p_s the expression of which is given by $p_s = p_0 - \frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2} + p_{ex}$. Substituting this expression of p_s we got the final form of the pressure distribution. So the pressure distribution was obtained by combining the *y*-momentum equation and the normal force balance boundary condition. Once the pressure profile was obtained, the expression of the pressure (*p*) was differentiated with respect to *x* and then it (the expression of $\frac{\partial p}{\partial x}$) was substituted in the *x*-momentum equation. That gave a second order partial differential equation $0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon^2 l_c^{\ 2}}{\mu u_c} f_x{}'$ for the velocity *u*. It was then integrated twice with respect to *y* with the no-slip boundary condition and the tangential force balance boundary condition to get the velocity profile. All these equations were derived in the previous chapter. Just for the sake of continuity we are reproducing this in the present chapter. The final form of the velocity distribution is given by $u=\left(A+\frac{\varepsilon^2 l_c^{\ 2}}{\mu u_c} f_x{}'\right)\frac{y^2}{2}+C_2\, y$ where we have two parameters *A* and C_2. The expression of *A* is given by $A=\frac{\partial p}{\partial x}=-\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y{}'\frac{\partial h}{\partial x}+\frac{\partial p_s}{\partial x}$ and the expression of C_2 is given by $C_2=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma - Ah - \frac{\varepsilon^2 l_c^{\ 2}}{\mu u_c} f_x{}' h$. Since $h(x,t)$ is a function of both *x* and time, therefore, both *A* and C_2 are functions of *x* and time.

We have written all of the governing equations, boundary conditions and the velocity profile so that we can see that which equations or conditions are utilized and which are not utilized yet. In the process of deriving the velocity profile, we have integrated the *x*-momentum equation (2) twice with respect to *y*. But we have not yet used the continuity equation (1) as well as the kinematic boundary condition (6). The kinematic boundary condition at the wall boils down to the no-penetration boundary condition. So, now, we use the continuity equation; we integrate the continuity equation over the thickness of the film $\int_0^{h(x,t)}\frac{\partial u}{\partial x}dy+\int_0^{h(x,t)}\frac{\partial v}{\partial y}=0$. If we try to take the derivative $\frac{\partial u}{\partial x}$ outside the integral sign we will not be able to do that. The reason is we have a limit of the integration $h(x,t)$ which itself is a function of the independent variable *x*. Therefore, we cannot just willingly bring the partial derivative $\frac{\partial u}{\partial x}$ in or out of the integral. We have to use the rule of differentiation under integral sign or the Leibnitz's rule. The Leibnitz's rule is given below

Leibnitz's rule: $$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x} dy + f(x,b)\frac{db}{dx} - f(x,a)\frac{da}{dx} \qquad (7)$$

In the general form of the Leibnitz's rule the independent variables are *x* and *y* while in our case the independent variables are *x* and time *t*. Another difference is that instead of ordinary derivative, here it will appear to be partial derivatives. So, for our present problem, we can write $\frac{d}{dx}\int_{a(x)}^{b(x)} u\,dy = \int_{a(x)}^{b(x)} \frac{\partial u}{\partial x} dy + u(x,b)\frac{db}{dx} - u(x,a)\frac{da}{dx}$ where the function $f(x,y)$ is replaced by the velocity u. The two limits of the integration are h and 0 respective which means $b = h$ and $a = 0$. The velocity u at $y = h$ is equal to u_i which is the velocity at the interface, so, $u(x,b)\frac{db}{dx} = u_i \frac{\partial h}{\partial x}$. However, the velocity u at $y = 0$ is equal to zero because of the no-slip condition, so, $u(x,a) = 0$. Also the lower limit of the integration (*a*) here is a constant (i.e. 0) and its derivative with respect to *x* $\frac{da}{dx}$ is equal to zero. So we get $\frac{d}{dx}\int_0^{h(x,t)} u\,dy = \int_0^{h(x,t)} \frac{\partial u}{\partial x} dy + u_i \frac{\partial h}{\partial x} - 0$. Now we need to think about whether we can write the operator $\frac{d}{dx}$ on the left hand side of this expression or we need to replace it by the partial derivative $\frac{\partial}{\partial x}$. If we had only one independent variable x, then we could keep it as the ordinary derivative. But we have two independent variables *x* and *t* and therefore, the ordinary derivative $\frac{d}{dx}$ should be replaced by the partial derivative $\frac{\partial}{\partial x}$, so, $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\,dy = \int_0^{h(x,t)} \frac{\partial u}{\partial x} dy + u_i \frac{\partial h}{\partial x} - 0$ (i.e. meaning same in spirit but conceptually different). When we wrote $\int_{a(x)}^{b(x)} f(x,y)\,dy$ in the general form, $b(x)$ was function of *x* only. But in the present problem $b(x)$ is equal to $h(x,t)$ which is a function of *x* and *t*. So the derivative cannot fundamentally be an ordinary derivative because of its dependence with the independent variables *x* and *t*.

With this the evaluation of the first part of the continuity equation $\int_0^{h(x,t)} \frac{\partial u}{\partial x} dy$ is completed, i.e. $\int_0^{h(x,t)} \frac{\partial u}{\partial x} dy = \frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy - u_i \frac{\partial h}{\partial x} - 0$. So question arises about the second part which is $\int_0^{h(x,t)} \frac{\partial v}{\partial y} = [v]_{y=0}^{y=h(x,t)}$. The velocity v at $y = h$ and at $y = 0$ are equal to v_i and zero respectively because of the no-penetration boundary condition, so, $\int_0^{h(x,t)} \frac{\partial v}{\partial y} = [v]_{y=0}^{y=h(x,t)} = v_i - 0$. These two conditions are kinematic boundary conditions. So the kinematic boundary condition is used to simplify the integral $\int_0^{h(x,t)} \frac{\partial v}{\partial y}$ of the continuity equation. So the simplified form of the integral of the continuity equation is given by $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy - u_i \frac{\partial h}{\partial x} + v_i = 0$ or, $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy = u_i \frac{\partial h}{\partial x} - v_i$. The only remaining boundary condition is the kinematic boundary condition at the interface (6) which reads as $v_i = \left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right)_i$ with '*i*' indicating the interface. From this condition, $v_i = \left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right)_i$ we can write $u\frac{\partial h}{\partial x} - v_i = -\frac{\partial h}{\partial t}$. Now we substitute the condition $u\frac{\partial h}{\partial x} - v_i = -\frac{\partial h}{\partial t}$ in the equation $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy = u_i \frac{\partial h}{\partial x} - v_i$ and we get $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy = -\frac{\partial h}{\partial t}$ or, $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy + \frac{\partial h}{\partial t} = 0$. So we have now only one step remaining to complete the governing equation for the film thickness which is to evaluate the integral $\int_0^{h(x,t)} u\, dy$. Integrating the expression of the velocity profile $u = \left(A + \frac{\varepsilon^2 l_c^2}{\mu u_c} f_x'\right)\frac{y^2}{2} + C_2 y$ with respect to y we get $\int_0^h u\, dy = \left(A + \frac{\varepsilon^2 l_c^2}{\mu u_c} f_x'\right)\frac{h^3}{6} + C_2 \frac{h^2}{2}$. Substituting this expression of the integral in the equation $\frac{\partial}{\partial x}\int_0^{h(x,t)} u\, dy + \frac{\partial h}{\partial t} = 0$ we get

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[\left(A + \frac{\varepsilon^2 l_c^2}{\mu u_c} f_x'\right)\frac{h^3}{6} + C_2 \frac{h^2}{2}\right] = 0 \qquad (8)$$

where A and C_2 are again functions of *x* and *t*. If we integrate this equation (8), we will get h as a function of *x* and *t* which is the solution of the thin film equation. This appears to be a bit involved because this is the most general equation that we have written after considering all boundary conditions, all body forces and all possible physical effects combined together. But in some cases certain boundary conditions may become redundant; for example, the curvature effect may not be important sometimes. In that case the normal force balance boundary condition becomes redundant. So the general form of the thin film equation (8) can be simplified to a large extent in case we are solving specific problems where certain effects are negligible.

In the next chapter, we will try to address one such problem where we will consider the gravitational spreading of a drop. We will consider a drop on a surface and then allow the gravity to play its role so that the drop is spread. We will find out the dependence of the height of the droplet as a function of position and time using the thin film model and that will be discussed in the next chapter.