Advanced Gas Dynamics
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Lecture – 12
An introduction to Expansion waves

So we have kind of called it quits with shock waves right. So, let us sort of you know begin with the expansion waves right let us see what; that means, and so let us you know it is usually called Prandtl Meyer Func expansion waves.

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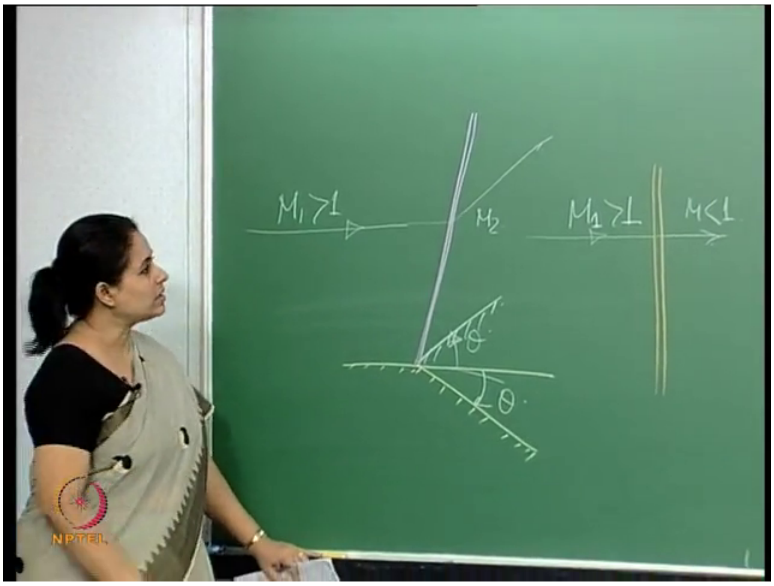

So, let us begin now what we have done. So, if you have say a supersonic you know flow say if you have say a supersonic flow right and then you have a you know an obstacle like that. So, you have a compression corner like that there is a shock waves right shock wave like that it turns the flow etcetera or if you have a an incident flow like that you have a normal shock right. So, this is supersonic and then this here becomes sub sonic and this here for a you know for say an op an you know blunt body could be a combination of sub sonic and supersonic and depending on that this Mach number M 2, you know depending on whether this is an oblique shock is a strong shock or a weak shock you know based on that this can be supersonic or sub sonic.

So, basically what we have done here is that we have a flow which comes here and we kind of block it, we block it and hence we are changing these properties here. Let us try to do exactly opposite to blocking it instead of blocking it this way why do not we try this instead of blocking it right. So, let us try this let us expand it right instead of blocking it this way let us expand it right. So, we were say blocking it by a theta like this instead let us allow it the corner to expand it and let us see what that what; that means,. So, basically say if I have a flow like that right if I have here flow like that and say it encounters a corner like this a corner like this right.

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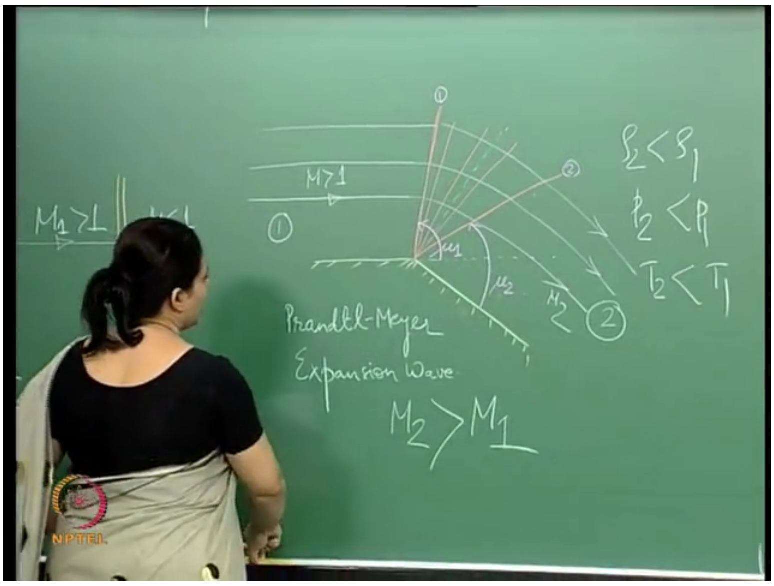

So, what we have seen so far what we have seen so far is this, we have seen this so that there is a there is shockwave which comes instead now we are saying instead of blocking it blocking the flow like that let us expand it. So, we have something like this.

So, in this case what will happen is that these will that this will that the flow will actually expand over this here and then continue to flow past it. So, hence the namebasically you know expansion fan. So, let us sort of you know you know drawn it in in let us begin. So, let us say right and say you have this. So, this is a region on which it sort of expands this is the region in which it will expand. So, what happens is that this comes like that it moves like this and then goes past it. So, then this is my M 2. So, similarly we can have other streamlines which comes like that and goes past it and it expands over this region and moves past it. So, what we are interested now is in this region. So, we will talk about this you know we will detail this a little more. So, usually this is you know, but kind of diagrams that you will see in you know general books etcetera. So, this region is essentially the shock wave sorry the expansion wave right.

So, this is actually called the Prandtl-Meyer; Prandtl-Meyer Expansion Prandtl Meyer expansion wave. So, now, essentially as we spoke in the talked about in the last lecture that this a these are a collection of Mach waves. So, essentially the Mach wave is very very weak for the limiting condition of an oblique shock right. So, therefore, if you increase the deflection angle a little bit. So, then you will start getting a shock and you get the shock becomes stronger. So, the weakest shock or the limiting condition of a shock is the Mach wave. So, these are the fine disturbances right. So, if you have a supersonic flow like that and which you are making it pass through a corner like this you are making it expand through a corner like this. So, if you will do. So, over a region of disturbances which will look like this?

So, like we you know should like we discussed yesterday right. So, these are lines which are basically for different Mach waves. So, in this case so therefore, if I were to let me sort of define this. So, say these are 2 you know the first and the last ma Mach numbers. So, therefore, this angle right this angle the angle of the first 1 is let us call say this is the Mach wave angle this is mu 1 and angle that this is making is mu 2.

So, basically what you are saying is that this right. So, what basically is that? So, let us say this is the you know this is say 1 and 2 right. So, the starting. So, starting Mach wave is making this angle with this surface right and the end Mach wave is making mu 2 with the with this particular surface, because what you see here is that this Mach wave is therefore, here you know in the surface and then it expands the surface is actually changed which is this.

So, therefore, we take the angle with this surface here this is essentially my Mach wave. Now so what exactly is the so if I consider this as the first and first region and second region over here. So, what we will see is that what we will see over here is that. So, then now if this is the case the continuity equation still holds.

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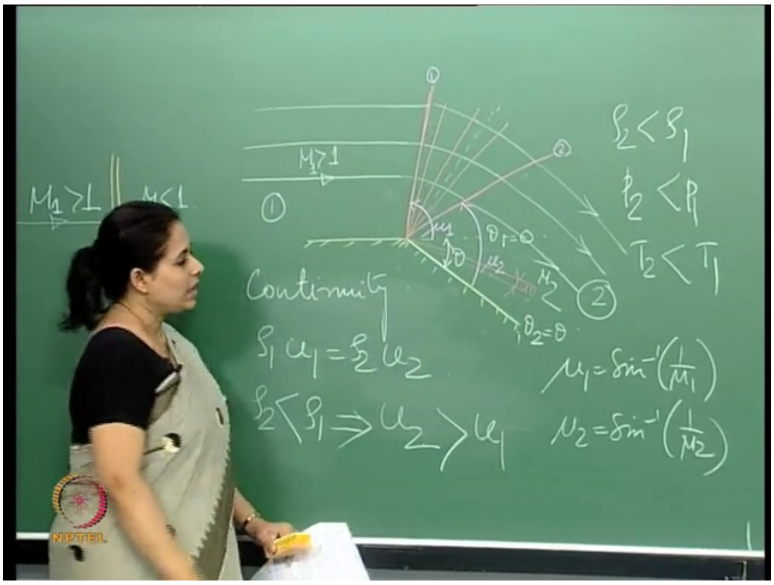

So, we have the continuity equation right which is so rho 1, u 1 is equal to rho 2 u 2 right, but what happens here is that rho 2 is less than rho 1, what this implies is that u 2 is greater than u 1. So, what this is meaning is that the flow what is coming here actually starts moving faster beyond a moving faster due to the expansion fan. So, this is what we get from here. So, the upstream side here so say this is says M 1. So, this is M 1 and here it is M 2.

So, let us call this is basically M 1 and this is M 2 right. So, just to get all the parameters in place so mu 1. So, mu 1 is equal to sin inverse 1 by m 1 right and mu 2 right essentially these are Mach waves. So, the Mach waves angle are given by this this we discussed the last lecture. So, this is we this is how the structure looks like.

So, now, therefore, what we are seeing is that due to this turning angle. So, let us say let us call this as a theta let us call this as theta. So, what we are seeing is that again if we turn this flow right turn this flow by an angle theta there is again a change in property the change is properties in this particular case is due to this expansion fan. So, clearly what we need to do now is find out you know the relationship between the change in properties and this theta right. So, how by how much if we change how by for say an infinite decimal change in theta what is the corresponding change in the properties right that is what we been doing so far.

So, we will do it for this case as well. So, now let us say now let us look at just say 1 Mach wave and a certain few things here.

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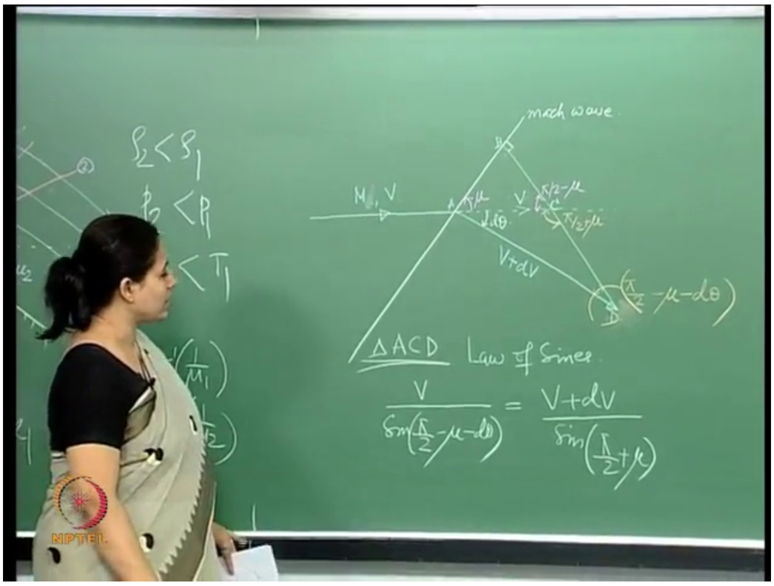

So, say this is a Mach wave right. So, we have this is a Mach wave let us just draw at here right this is a Mach wave and this is an. So, say this is your this is just for a general Mach wave right. So, then this is the Mach number and this is the incident velocity and it encounters the Mach wave and then it you know. So, therefore, the velocity here increases is not it what we showed here was that as the flow goes past the Mach wave the velocity is increasing it moves faster beyond the Mach wave. So, is the velocity is it the incoming velocity is V. So, it encounters this Mach wave it starts travelling faster right. So, it starts travelling faster.

So, let that be V plus you know an incremental change in the velocity and this Mach wave is making. So, let us look at these angles here. So, say this angle is mu this angle is mu, now let us sort of complete this rectangle here. So, let us look at this. So, now, if we are looking at this geometry over here. So, let us look at say. So, if we look at this triangle over here. So, this angle is 90 minus mu is not it. So, this angle minus mu which makes this angle what. So, if that is so and let say that this angle let us say that this has been turned this has been turned by say d theta.

So, this Mach wave is able to turn the flow by an angle an incremental angle d theta, then what does it make what is this. So, now, if this is it, let us consider this point as A C. So, sorry let us consider that as D. So, if we consider the triangle A B D then this angle turns out to be this angle turns out to be phi by 2 minus mu phi by 2 minus mu minus d theta right this is just simple from this geometry.

Now, if I do that if I do that. So, this is basically. So, this was the velocity this was the incoming velocity. So, now, a this is say the incoming velocity V, now let us consider the triangle A C D look at the triangle A C D, if I do that and using the law of science this is from basically the math right. So, if I use the law of science what we get is that V by pi by 2 minus mu minus d theta is equal to V plus d V by sin pi by 2 plus mu right this is what we get..

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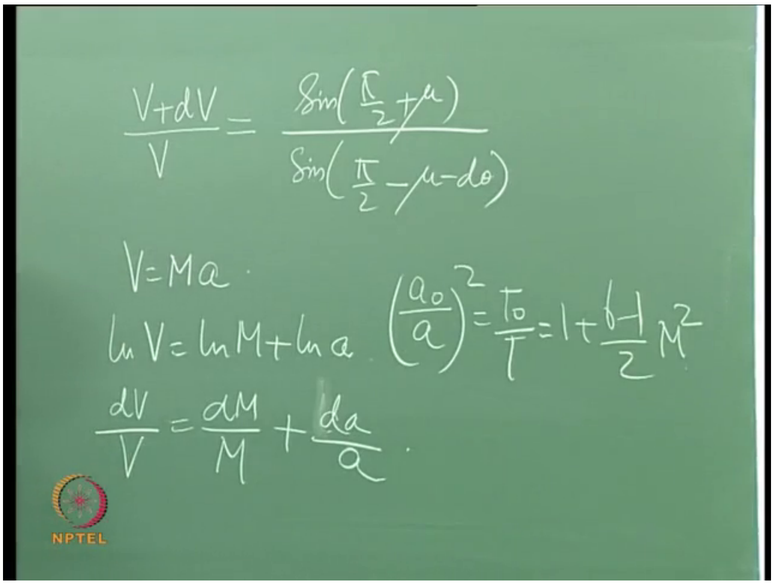

So, therefore, therefore, we can write that V plus d V by V say V plus d V by V is sin pi by 2 plus mu by sin pi by 2 minus mu minus d theta. So, the basically this is what we get and if I you know do this. So, here now if I write this out, I am use this now we also know that V is equal to. So, now, this we also know this right velocity is Mach number into speed of sound now from the so if I differentiate this so what I get is sorry. So, if I take log here and then and then if I differentiate right. So, what I get is d V by phi is d M by M plus by a right, then for a calorically perfect gas and for adiabatic energy equation what we get. So, what we get is this these are relations which we sort of you know being doing so far. So, hopefully you can recall. So, gamma minus 1 by 2 M square ok.

So, essentially what I am saying is that. So, we get this relationship now when we have this here. So, what we will basically do is write if I were to expand this etcetera this equation then I will be able to write it like. So, let us erase this now. So, we have these 2 relationships we have this and this out here.

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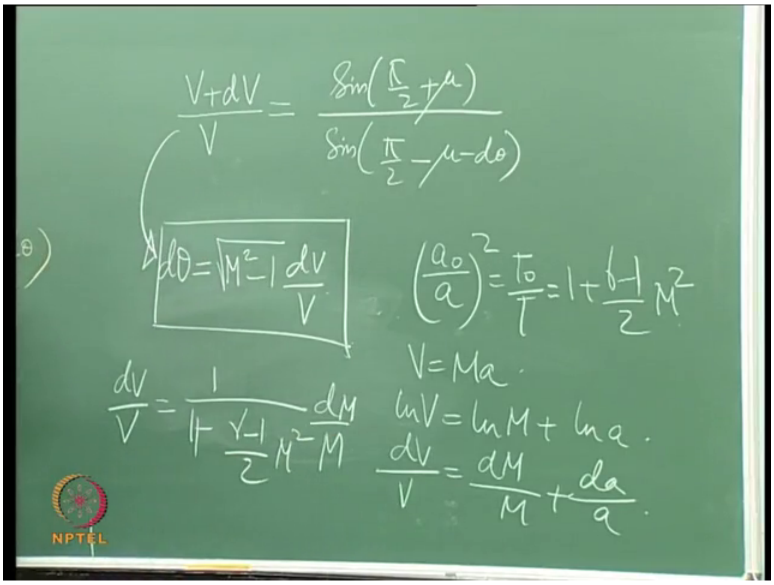

Now this relationship I can write in this form. So, d theta is equal to M square minus 1 d V by V. So, this relationship if I expand etcetera so I am I can write it in this form, but again if I were to write this o over here right now d V by V right.

So, therefore, this is a relationship now we just saw we just saw actually that V is equal to M a we said l n V is equal to l n M plus l n a. So, therefore, if I differentiate I get d V by V which is equal to d M by M plus d a by a. Now using therefore, d V by V is equal to this relationship, now then in this I am going to use this relationship over here with the combination of this and this I can write d V by V d V by V in terms of the Mach number and I can write that as . So, basically using this so these relationships that I going to change the velocity terms in terms of the Mach number, this is what we get.

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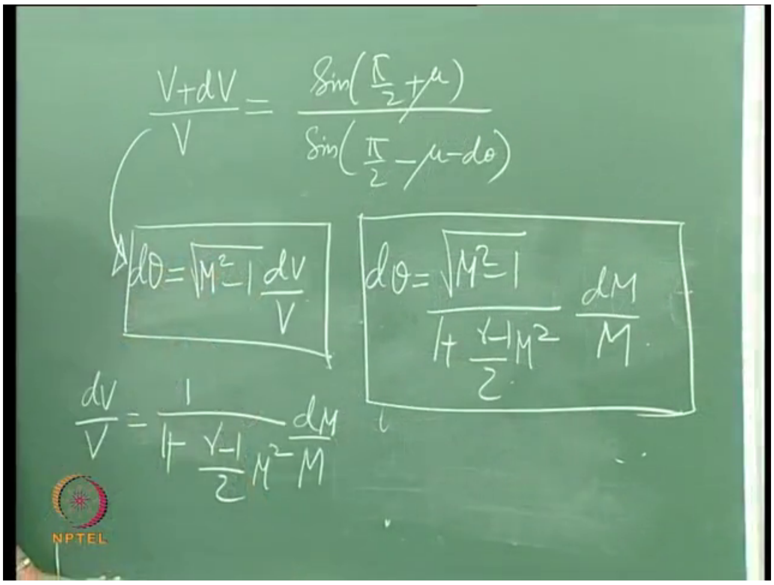

So, therefore, So, what we can write is that d theta is equal to what we get here is that d theta I equal to under root M square minus 1 by 1 plus gamma minus 1 by 2 M square.

So, what we are seeing over here is that for a given Mach number right for a given Mach number if you turn the if you or for example, if you turn the flow for a for an infinite decimal change in the turning angle d theta, this is a relationship between the change in the Mach number and the change in the deflection angle right. So, this is basically the differential equation which governs the Prandtl Meyer the Prandtl Meyer expansion fan right and this is the Prandtl Meyer relationship Prandtl Meyer equation. Now we will work a little bit a along with this. So, therefore, what we see over here essentially if you come back to this diagram here.

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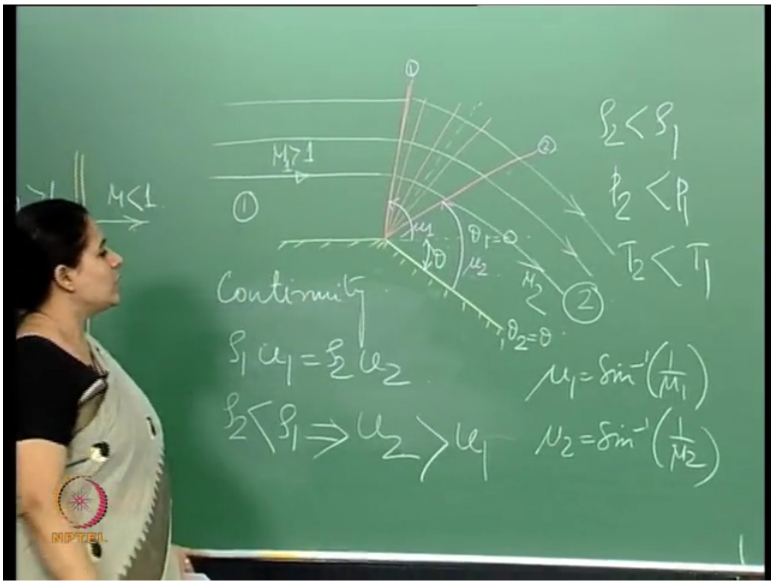

So, say therefore, say this 1 let us just say that this is a theta 1 and this is theta 2 of course, theta 1 is 0 and theta 2 is theta. So, what is happening is that we have a Mach number, we have an incident the incident Mach number of M 1, it encounters this expansion corner and it goes from theta 1 to theta 2 and what have found out is that for if I have this incident wave and I right. So, I have an incident wave and I deflected it by or I expand it by an infinite decimal angle d theta, if I do that then the relationship of this ma this this change in the deflection angle with the Mach number on the Mach number beyond the expansion wave.

So, therefore, what we can say here is that. So, let us sort of write that again. So, having done this this is like basically for a single right.

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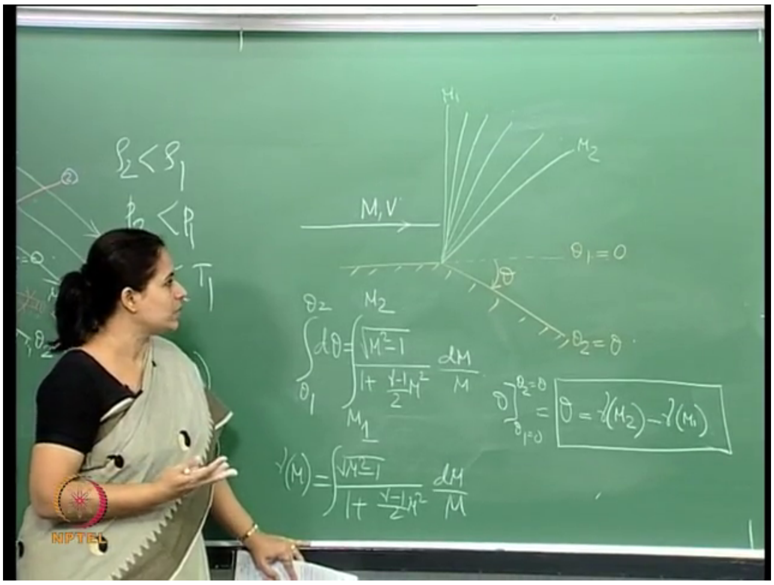

So, what we have is that we have this Mach number, we have this Mach number right and we have these are for an expansion fan. So, it goes through that and so this 1 is theta 1 this 1 is theta 2.

So, this is equal to 0 this is equal to theta that is this angle is equal to theta right. So, therefore, the relationship that I have is that d theta is equal to under root M square minus 1 by 1 plus gamma minus 1 by 2, M square d M by M. So, this is a relationship. So, therefore, an incremental deflection the corresponding changes in the Mach number. So, in this case, what we will do is I think for a total change in angle from theta 1 to theta 2 all we will do is integrate this ok.

So, if I integrate this between theta 1 to theta 2. So, if I integrate this from theta 1 from theta 2 it basically goes from M 1 to M 2. Now this integral this integral is called as the Prandtl Meyer function. So, this integral, which is a function of the Mach number, is not it. So, this integral is called the Prandtl Meyer function and if we do this integration what we get is this let us write this out.

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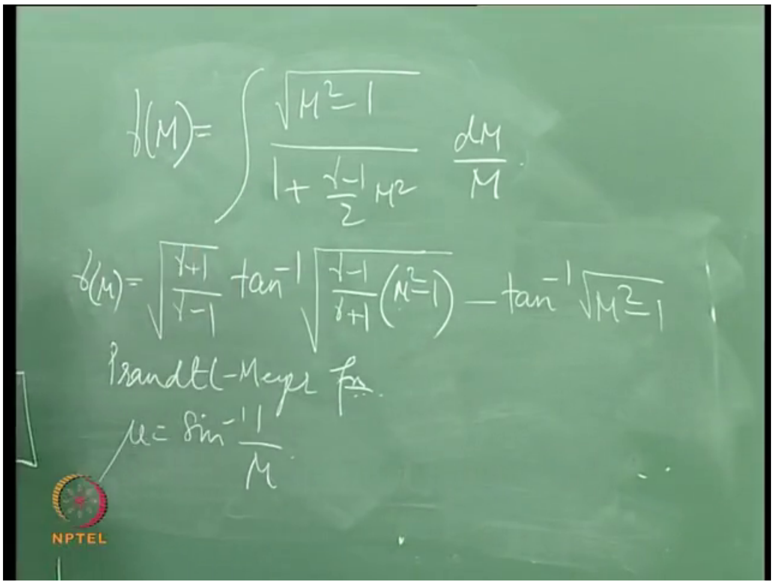

Basically this is the integral right this is the integral and if I do this. So, I am not sort of doing this steps or whatever you can you know try it on it is own you please. So, this is tan inverse which is gamma minus 1 gamma plus 1 M square minus 1 minus tan inverse M square minus 1 right.

So, therefore, this is the, this is the Prandtl Meyer function. So, Prandtl Meyer functions. So, as you can see that if for a given Mach number you have a different Prandtl Meyer function. So, Prandtl Meyer function and there is. So, this basically what we are talking about here is that this is the expansion fan. So, expansion fan basically consists of several Mach waves' right and this the limited Mach numbers are M 1 and M 2. So, all these have different a Mach numbers right. So, these will basically go through a series of this will suffer a deflection this will you know will suffer you know will expand through this fan. So, step by step when it encounters each and every Mach wave right and this Prandtl Meyer function is basically defined for you know every small incremental change in the theta and the Mach number.

So, therefore, if I write this integral as you know in in this form which is a Prandtl Meyer function. So, then coming back here so my total change in theta that is in this case theta goes from theta 1 to theta 2 which is 0 to theta right, which becomes theta. So, then that I can write as Prandtl Meyer function M 2 minus M 1, so M 2 minus M 1. So, this is essentially the relationship, now in here if I have to just sort of yeah. So, now, what we will do is let us sort of do a problem and see that how and how we gonna use or what are in the table and so on and so forth. Now what you will see in the table again now when you see this Prandtl Meyer function.

So, basically what you see is if a Mach number is given then this can be plotted I mean you immediately calculate this for any given Mach number you can calculate this for say gamma is equal to 1.4. So, and also Mach wave angle is sin inverse 1 by M. So, the Mach wave angle also can be calculated. So, therefore, what you have in the standard charts is this what you will have in the standard charts is basically a listing of the Mach number, the Mach wave angle and the Prandtl Meyer function.

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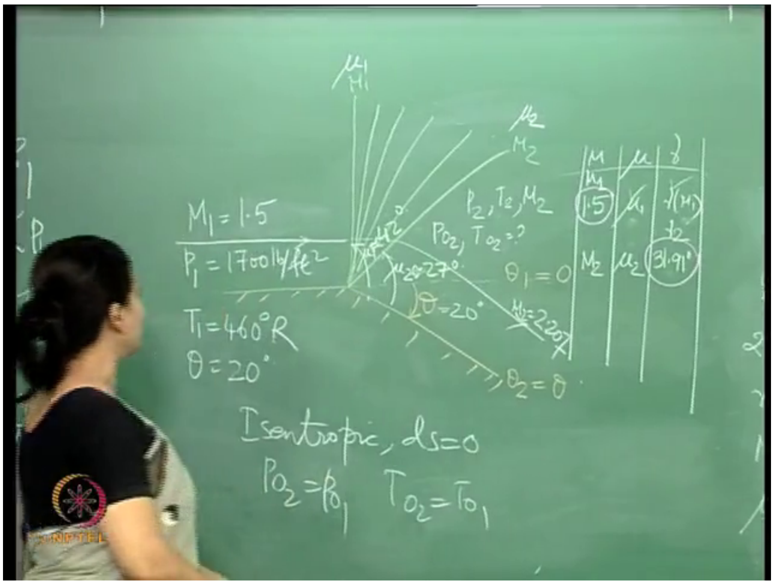

So, this is what you will have for the Prandtl Meyer expansion waves the standard chart will have the Mach number corresponding Mach wave angle and the Prandtl Meyer angle. So, let us therefore, look at a problem. So, let us look at a problem and see how we can use this chart or what else you can use. So, what is given is that this M 1 here what is given is M 1 right M 1 is 1.5 P 1 is. So, you can convert this to S I units I usually do that till I get chance to do this today. So, you can just do that and T 1 this is given in Rankine. So, you can convert that just Google it and you know you should able to convert these to S I unit to kelvin and say newton per meter square.

So, M 1 is this P 1 and T 1 and theta is equal to 20 degrees. So, essentially what we have is that we have this flow this flow which comes in with Mach number this pressure and temperature and it goes through an expansion corner the angle of which is 20 degrees. So, this is 20 degrees what we are asked to do is to find out basically this is the region 2. So, what we need to find out is P 2, T 2, M 2 stagnation conditions and also Mach waves angle mu 1 mu 2, because clearly you need to find out what is the angle what is the angle here mu 1 and mu 2. So, these are things that I need we need to find out.

So, this is basically the problem. So, how do we go about this? So, what do we do and how do we go about this. So, what you can straight away see is that for the Mach number 1.5. So, go to your charts for Mach number 1.5 we can get mu 1 and gamma M 1. So, you can get this from the charts right away just look for a 1.5 Mach number and so let us go step by step here right.

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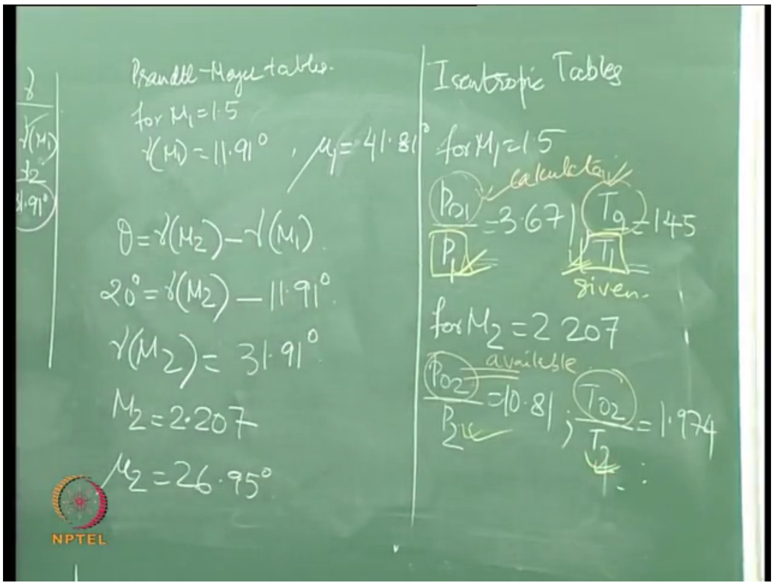

So, let me write this. So, first step is that Prandtl Meyer. So, go to the tables for M 1 is 1.5 you get the corresponding Prandtl Meyer function to be equal to 11.91 degrees which is around 12 degrees. So, this is what you get then and we get also the. So, this is mu 1 actually which we get that as 41.81 degrees which is around 42 degrees. So, therefore, now what we what do we do now. So, that is all the information we will get, that is all we will get at this point of time using Prandtl Meyer tables I can only find out the Prandtl Meyer function and the Mach wave angle.

What do we do now well we had a relationship which is said? So, theta is equal to M 2 minus M 1 right. So, this is what we had derived for our expansion fan. So, we said if it goes through an angle which is deflection angle of theta then that is equal to gamma M 2. So, Prandtl Meyer functions for this minus Prandtl Meyer function for this. So, this angle is known to us theta which is 20 degrees this is something that we do not know this is something that we found out which is right.

So, therefore, what we can do now is find out this which is equal to right this is what we get. So, once we get this. So, now, what we what we do is we just go back to the tables again we go back to this table here, now instead of the Mach number or mu 2 what we have is this this is available to us is not it in the first case this was available to us. So, we found out these in this case now this is available to us. So, this is something that we were able to find out.

So, here corresponding to the Prandtl Meyer function we will go we will go the reverse way and find M 2 and mu. So, what we had here this 1 was M 1. So, we were able to find mu 1 and M 1, now this is mu yeah well we can say this mu 2 right this is what we know. So, correspondingly we will find out mu 2 and M 2 right. So, we do that. So, what find is that the corresponding Mach number is 2.207 around basically 2.21? So, that and the corresponding mu 2 is 26.95. So, the Mach wave angle. So, what we get here is that the incident this this mu is 11.91 right or no 41.81.

So, this was around say 42 degrees and this mu 2 is. So, this this angle is around 27 degrees around 27 degrees this is what we get. Now once we have done this now we are done with the more or less definition of geometry etcetera now we need to find out the properties. So, P 2 T 2 and this stagnation conditions how do we how do we go by doing that. Now as I said that in the outset that all of this is as entropic relationship. So, the flow across this the expansion across this fan is Isentropic. So, it is Isentropic and which means that the entropy changes here if that happens then then what do we do in this case what do we know immediately right because of the Isentropic condition this is what we get right away.

Now, because it is isentropic we will go back to the isentropic tables. So, we go back say to the isentropic table’s right we go back to the isentropic tables and so for the first case now right, M 1 is 1.5. So, for this here, we go from M 1 equal to 1.5 to M 2 here right. So, this goes like that. So, this here this is M 2 is so around 2.2, basically so you the Mach number has increased right. So, let us go at M 1. So, for M 1 equal to 1.5 right M 1 equal to 1.5 what we get is. So, around 3.7 and T 0 1 by T 1 is 1.45 right and for M 2 equal to 2.207 right what we get is P 0 2 by P 2 right which is 10.8 1 and T 0 2 by T 1 T 2 is around 2 . So, how do we go about? So, basically we need to find out P 0 2 and T 0 2 and P 2 and T 2. So, how do you go about doing that? So, what we see over here is that P 1 and T 1 is given right P 1 and T 1 over here, but how do we incorporate that.

Now, what you see here is that from these 2 relationships right P 1 and T 1 is given. So, these can be calculated. So, this 2 you can calculate because these are given is not it this and this is given right. So, you can calculate this P 0 1 and P 0 2 main aim is you have to find out P 2 and T 2 what do we do well because it is isentropic. So, P 0 2 is equal to P 0 1. So, the value that you calculate here you put in here, then T 0 2 is equal to T 0 1 again because it is isentropic. So, again you use this value of T 0 1 which you calculate and put it here. So, once you do that. So, using that what we will get is P 2 and T 2 once you. So, this is basically it has calculated. So, this is available it has been calculated. So, then you evaluate these 2 values when you evaluate P 2 and T 2.

So, that pretty much gives us. So, that is how we go about the calculating the properties across an expansion shock. So, pretty much just use the Prandtl Meyer tables and then go about calculating the property change there is. So, mu 1. So, this is making mu 1 and this is mu 2 right, now let us just calculate the so M 1 essentially this the before this this starting Mach wave that is the angle with the horizontal what is the horizontal? The angle that you know this wave the after wave that meets with the horizontal which is just 27 minus 20 right is not it that is around 7 degrees. So, this angle that it makes here you can calculate this this is just for a slightly this is around 7 degrees.

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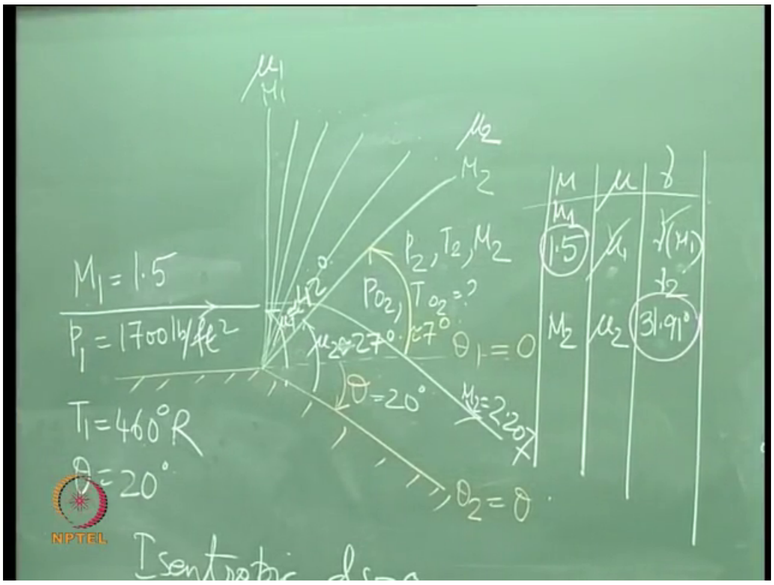

So, we have basically completely defined or the geometry as well as found out the property across this expansion shock the expansion wave sorry. So, essentially this is about you know this is 1 class that we will do on expansion fan what we will move on to now is some unsteady motion right the shock tube which is I think very interesting. So, this is that and I think we will stop here today.

Thank you.