Advanced Gas Dynamics
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Lecture – 23
Flow over a Wavy wall: Formulation using Perturbation Theory

Now that we from composter shock tube, right. We started learning the flow property changes across I an expansion fan, we started with learning doing a brief review of how things work in terms of wave propagation, right. And we used linearized equations we introduced the concept of how we could use small perturbations, right. The way property changes in, when there is a wave disturbance or wave is traveling is considered to be a small disturbance over the ambient. So, that is something we sort of dealt with at the time.

So, let us just sort of look a look at a problem and see if we can use that concept here, right. And solve equations and you know get some information about the flow properties. So, say we have a problem like this. So, let me draw this.

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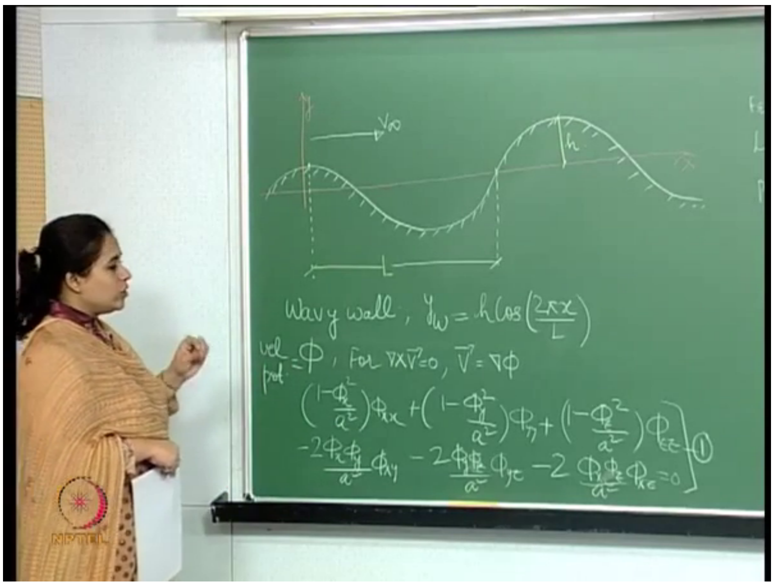

So, let us just say, I need to draw a straight line here. Let us as straight as I can go fine. So, we have x axis like that. So, I am going to draw this. So, what we have here is this essentially this is like this is called this wavy wall problems, this is the wavy wall problem. So, in this here, essentially what this is; let us call this as L which is the wavelength this could this could be land of if you want to choose it that way. This is the amplitude of course, this is the amplitude h. And the Y ordinate is given as Y w or Y wall, right. And this is actually given to us as; so, this is essentially the problem. So, what we need to do here is assume that h is small.

So, essentially considered considering this as a small perspiration problem, find out an expression for the velocity potential, which is phi, right. And the surface pressure coefficient, the pressure coefficient at the wall. So, this is what we need to find out. So, what I will do here is just write out the velocity potential equation just to sort of remind you. So, before I do that. So, essentially what is happening here is that we have a flow, right. We have a flow which comes in like that right. So, incoming free stream is at a mach number of mach number which is m infinity, right. And then we are going to do this you know problem to find out velocity potential phi and coefficient of pressure at the wall.

Now, for Irrotational flow, right; for Irrotational flow we know that the velocity can be written in the in terms of the gradient of the velocity potential right. So, that is essentially for Irrotational flow. And let me just write out the velocity potential equation just to remind you.

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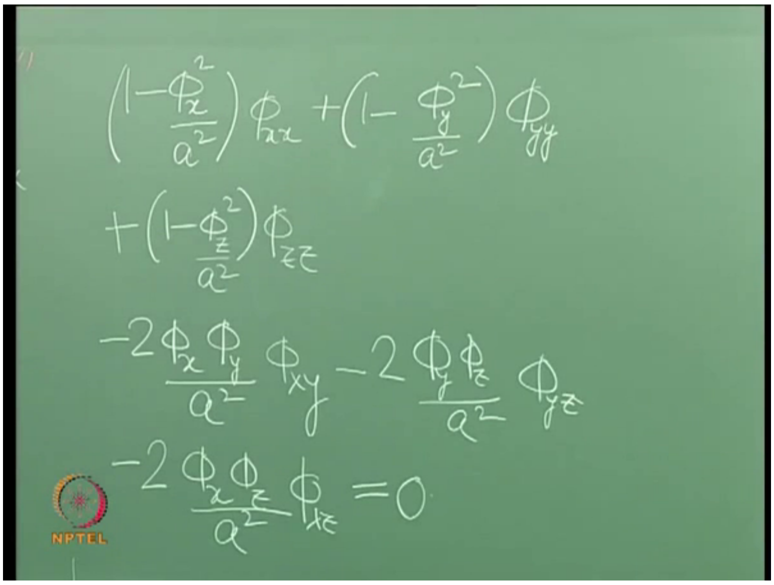

And that is; so, this is essentially the velocity potential equation. So now, let us just so, the basically the point is that when you have a free stream. So, you know we have a free stream and then suddenly it encounters this wall, which is a wavy wall like this. So, how does how do we then how does the then the velocity in that flow field change you know, which was just a free stream and it encounters a wall like this. And in this particular case; so, the hum the nature of the wall or the contour of the wall is given, right. Which is by this expression here right. And so, we need to basically assume that this is a small perturbation. So, what would that mean in terms of linearizing the equation that we will use. So, this is our problem. So, let us go ahead and try to solve this.

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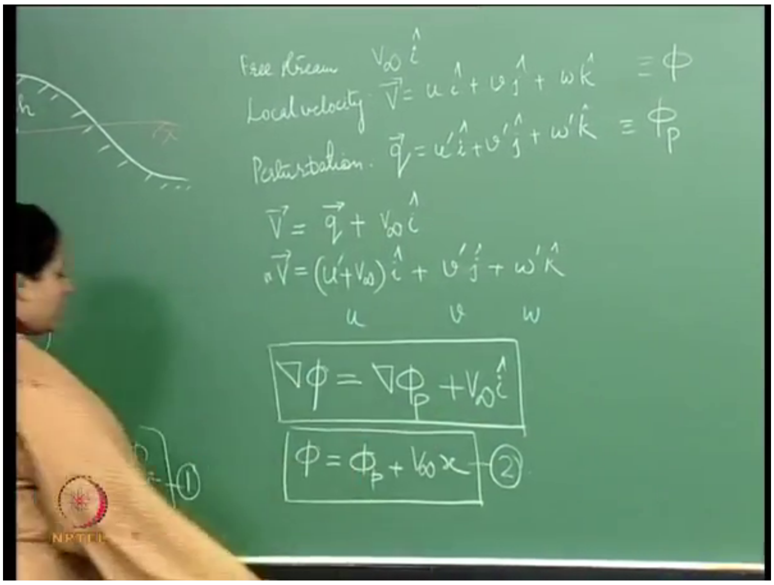

So, let us write our free stream. So, our free stream is essentially in vector form this is how it is, right. This is my free stream, and say the local velocity. So, then the local velocity right. So, local velocity we can call that this is equal to, this is a total local velocity. So, then let us say that the perturbations in the velocities are so. So, the perturbation velocity, you can you can call that that let us say, let us call this is q is say we going to call that u prime ok.

So, essentially, we have a free stream, right. We have a free stream, which has and we have a perturbation velocity which looks like this. And we have a local velocity of this, what is the connection between these? If you if you consider a perturbation theory, as we talked about earlier. So, the free stream is perturbed by this velocity to get the current local velocity right. So, if we do that. So, if we essentially do that; which means that, what we have here in here is that this v is nothing but the, isn't it? So, this is my local velocity which is the free stream plus the perturbation, right. Which means that what we have here is this. What we have is u plus. So, essentially let us say right.

So, this is the so, therefore, this is the local velocity which means were essentially this is my u, this is my v, this is my w, which is the local velocity. So, I have the free stream which is perturb using this perturbation velocity, we get the total local velocity. So now, as you can see that for the if you have a velocity potential like this, you can represent your local velocity in terms of gradients of that potential. So, similarly we can have a perturbation potential corresponding to you know this velocity.

So, let us say that the we have a perturbation potential which is say this is corresponding to a perturbation potential phi p, and this is corresponding to a perturbation potential sorry this is corresponding to potential phi. So, which would mean, right, which would mean if you sort of you know, look at this equation over here. So, essentially, this is the local velocity which is equal to the perturbation velocities, right. Plus, the free stream isn't it? This is what we get. So, this is my local velocity which is corresponding to this perturbation potential corresponding to this potential, that is equal to the perturbed velocities which is given by the gradient of a perturbation potential, and this perturbation is applied to the to the free stream right.

So, therefore, this is what you get. So, hence from here we can write if you integrate this now. So, therefore, say phi is equal to phi p. So, essentially, we need to find out a potential for our wavy wall problem over here. You know, we need to find out the potential so that once you find the potential we shall know the flow properties, the velocities as well right. So now, what we see over here is that the surface potential for the wavy wall can be written as a the component, the component of the free stream plus a perturbation potential. So, as long as we can find the perturbation potential, we should be now I be able to find the surface potential for this kind of problem.

So now we have this. So, let us say call this as. So, let us say, let us call this equation out here as 1. And we should call say this equation as 2. So, we can now write. So, therefore, you see this is the velocity potential, and correspondingly this is a velocity potential equation, right. Now we said that using a perturbation theory we are able to write this potential as a free stream potential plus a perturbation potential, which is what we derived here in this particular equation.

So now what we will do is we will use this equation into 1 and see what we get. So, this is going to be well it is almost like a half page equation, but I will write it out. And just to sort of I am scare you. And then will see what we will do with this. It is important to write this. So, so if I were to write this.

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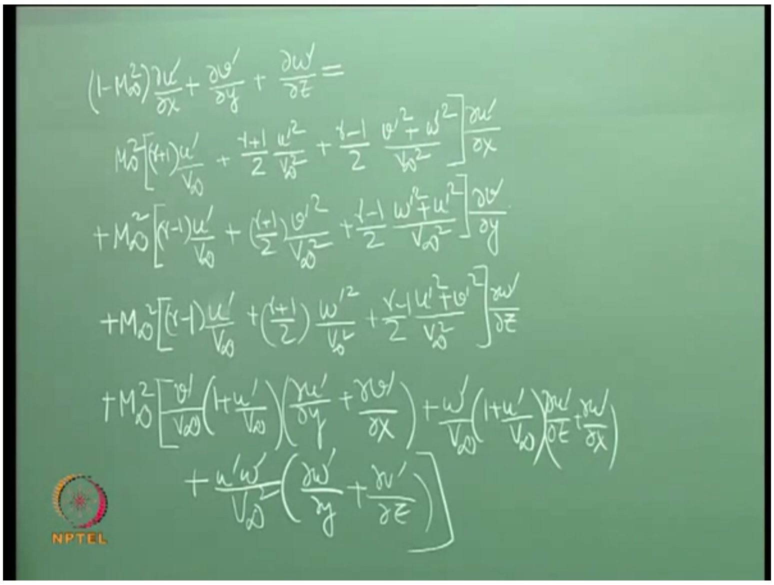

So, essentially, I am, right. I am using the surface potential equation that we got here into the velocity potential equation. So, it looks like this. This is the left-hand side right. So, what we get here is on the right-hand side; this is going to take me a while to even write this. So, so we have got the second term here, which will be well, there is of course, a certain.

Then so, if we hopefully you are all awake by this time. So, this is what it looks like, if I incorporate if I incorporate the perturbation potential, right. Into the surface potential equation, then the velocity potential equation looks like this. So, this is an exact equation as you can see over here it is an exact equation. And although the left-hand side out here which is this is linear, clearly the right-hand side is not, you can see this.

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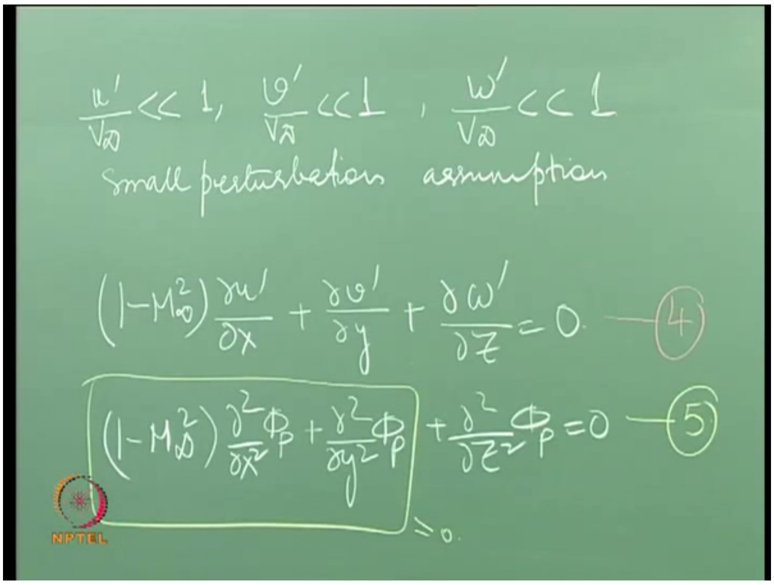

So, therefore, what we are going to do over here is bring in the a small perturbation theory. See if I do that small perturbation theory essentially will mean right. So, this and hence their squares and you know, higher orders will be you know negligible.

So, this is my small perturbation, this is my small perturbation assumption. So, if I use the small perturbation assumption here, then what I should be able to do over here? As you can see if you look at each term over here. So, the all these terms here. So, say let us look at this. So, we have you dash by v infinity square of that term and we have squares etcetera, etcetera, etcetera. So now, all of these terms so, these terms can be neglected, right. Because we are using small perturbation theory and if we do that what we see is that essentially what we are left out with ah this equation. So, if I call this equation as say 3, if I leave this out all I get from there is, right.

So, this is this is what we get and let us call this equation is 4. This is what we get, and of course, we can also. So, we basically have written our velocity potential equation in terms this you can see this now reduces if I use a small perturbation theory, this is our velocity potential equation essentially is in terms of the perturbed velocities right. So, I can also write this; I can also write this as in terms of the perturbation potential, right. Perturbation potential which will be, right and let me call this as 5. So, what you basically see over here is that these 2 equations here 4 and 5, right, which we get by using a small perturbation assumption into the exact non-linear equation surface potential equation in 3. So, unlike 3, these 2 are linear equations using the small perturbation theory, right. And in here so now, we are going to you say. So, basically, we have a and an expression in terms of this.

Now, if. So now basically we will go ahead and try and solve this. Now for this particular case again let us take the 2D version of this, which would be essentially this equal to 0. See, if I take that how do I go ahead and solve this.

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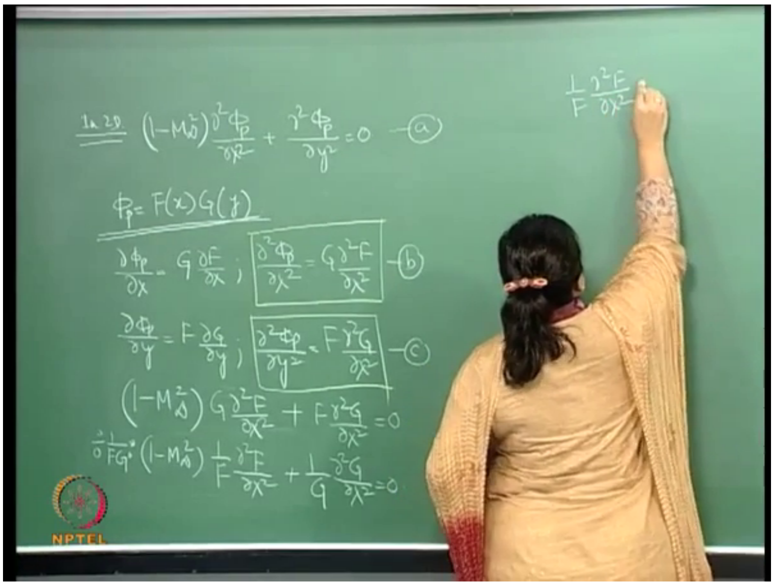

So, let us go ahead and do that, right. So, essentially what we have in 2D, this is what we have. So, this is what? This is the governing equation now.

So, how do we go ahead and solve this now what we will use here is a method of the separation of variables. So, this is the mathematics part of it. So, this is the part of calculus if you not you know, if you know she do not remember it I think you should just go and run yourself through it you know one should be fine. I will walk it through the whole procedure of this, but I will not go into details of the mathematical you know, derivations and all of that. But will work will work step by step through the process of it.

So, using the method of separation of variables, the way we will basically, right. To say right. So, F and G are just functions as so, F is a function of x and G is a function of y. So, this is separation of variables if I do that right. So, if I do this essentially. So, you know if you see here. So, essentially what we get is this, isn't it? Right and therefore, So, similarly if you look at this del G. So, this is; so, this is what we get from here. Now if I do this. So, let us now use. So, let us say. So, this is say our equation a so, we have this expression here and this expression here using a separation of variables. So, if I do this. So, therefore, I will use p and c into a. So, if I do that what do I get? So, if I do that what essentially, I get is right. So, delta phi del x 2. So, that is delta phi del x 2. So, which is plus delta phi del v del Y 2, which is this thing.

So, again now what we can do is divide this whole thing by F G if I divide this whole thing. So, if I divide by F G. So, this is the relationship that we get and finally, what we will do is we divide this whole thing again will take. So, finally, we will also we can also write this as; we will just take this term divided by this term throughout. So, again what we will get is and let us call this as say d.

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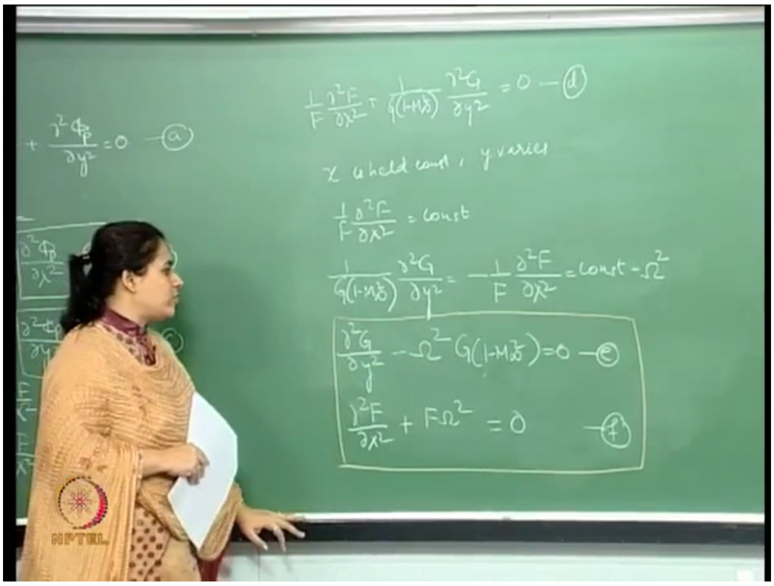

Now this is a now let us say that x is held constant x is x is held constant. So, F is all say x is held constant. And so, Y varies so, we will hold say x is constant, right and Y varies right.

So, which means that; so, this is a constant say and let us call this as so, this is a constant. So, which means that so, again so, if that is true. So, therefore, from d what we get is that 1 by, right. And this is constant right. So, we saying that we would hold x constant and Y will wary. And so, what we see from d is that this term is we just take this on the right-hand side which is this is a constant. And let us call this as we just call this as omega square, you know just to say it is say omega square and if I do that then from here. So, basically, we get to relationships do not we. So, for example, if you look at one of this. So, essentially what we get is that this is equal to a constant, and this is also equal to a constant.

So, we get 2 relations from this. So, which are if you if you just sort of look at this from this what we get is. So, this is one relationship and let us call that a say e. And the other one we get from here which is; so, essentially these are my 2 relationships. So, using the separation of variables F and G being arbitrary functions and what we get from here is to these relationships, where this F and G arbitrary functions and F and omega here is a constant.

So now if you can find appropriate values of F G and omega, then we should be able to sort of solve this equation. Now the reason we write this equations in this stuff in the we wrote this in this form because this has a readymade solution from us this is from our calculus right. So, we have a readymade solution for this which will be; so, the solution to these equations say e and f is given in this form.

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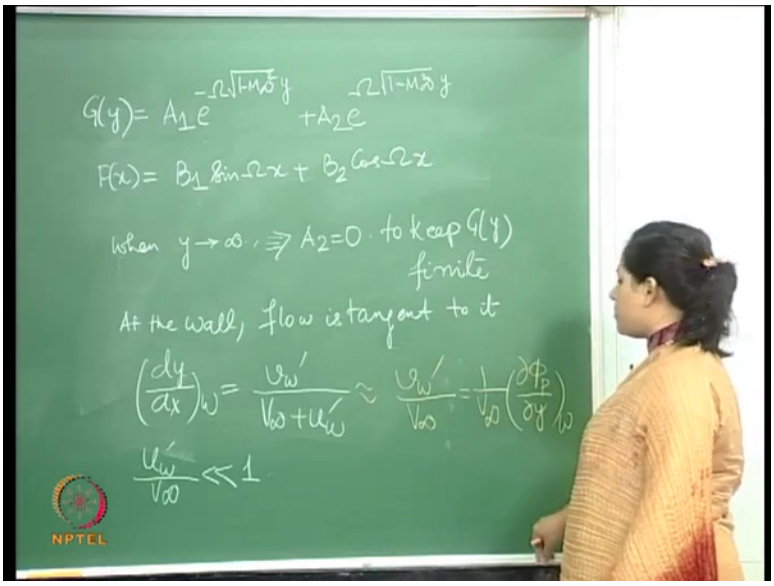

So, then so, G solution is given as again we have constants and Fx has.

So, this is my readymade solution. So, if we can find appropriate values of A 1 A 2 B 1 and B 2, then we should be able to find out F x and G y and hence we will be able to find a solution for our velocity perturbed velocity potential. Because phi p is equal to Fx into G y. So, let us see if we can do that. So, usually how we go about this is that we try to see if we have enough boundary conditions, right. Then we can see if you can find appropriate values for this.

Now, you see of course, phi p. So, even if we have. So, v p is a finite value, right. Which is the velocity perturbed velocity potential, but add velocity potential out here has a finite value, for all cases of x and y. So, we have nothing here to tell us that the velocity potential blows up or becomes indeterminate or discontinuous.

So, therefore, even when Y tends to infinity, right. We have a finite perturbed velocity potential. Now let us look here. Now when Y tends to infinity, look at G y, now this term becomes nearly 1, right. So, because this term becomes to the negative infinity. So, this will this term this exponential over here will tend to nearly 1. Whereas, what happens to this term here, this tends to infinity as well right. So, therefore, if to keep this as really finite right. So, in this case v it this means that we have to set A 2 equal to 0. Because otherwise right. So, therefore, when Y tends to infinity, what we see over here is the Y tends to infinity G y will tend to infinity if we have A 2. This first term out here. So, this exponential will tend to nearly 1, but this exponential out here will tend to infinity.

So, therefore, unless and until we have A 2 0 we will not have a finite value for G. So, therefore, if Y tends to infinity then A 2 out here should be 0 to keep it finite, all right. So, then now the next thing is; so, we have we have a wall here, right. We have a wall given overshare and the equation of the wall is also given to us.

Now, we have this flow moving over the wavy wall. So, definitely so, the when the way say. So, if I take a stream line which is say on the wall itself how will the streamline look like. So, let me draw some streamline vectors, right. So, this will be my local velocity vector then this will look like my local velocity vector so on and so forth.

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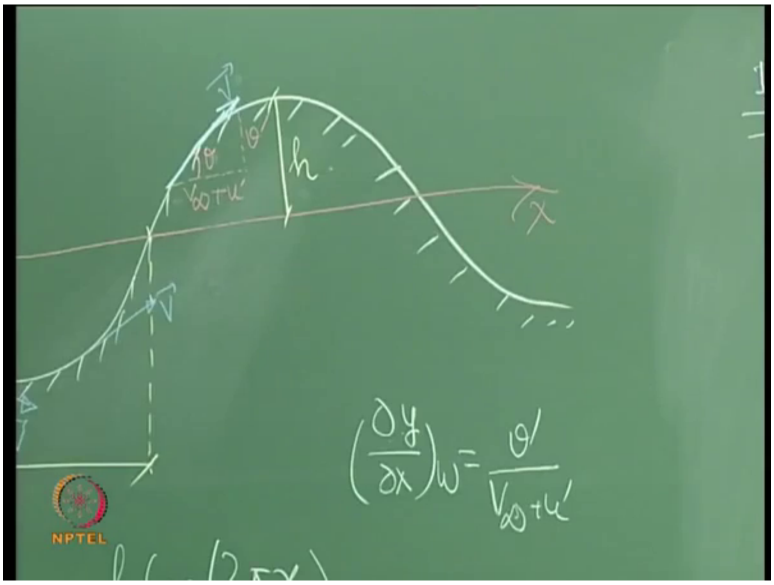

So, essentially what I am doing over here, I am just saying that my velocity vector is tangent at the wall if the velocity vector is tangent to the wall right.

So, therefore, the flow is tangent to it right. So, therefore, another boundary condition is flow is tangent to it right. So, therefore, now in here if we sort of go back over here. So, this is my say let us look at for example, this velocity vector over here. Or say let us look at this velocity vector over here now. So, this was the original this was the original streamline, right. How do we get this local velocity? So, we get it with a stray with this free stream, which has been perturbed by u prime, and this is the v prime in the vertical direction, right.

So, this is essentially my this is essentially how the velocity profile looks like. So, this is the local velocity. So, if I am a sort of highlight this a little more. So, this is my this is my local velocity vector, right. And if I see the components of it the horizontal component is the x component is the free stream plus the perturbed velocity perturbed velocity x velocity, and in the Y direction is the perturbed Y component of the velocity. So, therefore, if I have to take the tangent right. So, what does the tangent mean over here? So, at this point therefore, at the wall can I write it as this right. So, this becomes the tangent.

So, therefore, that is what I am going to do over here. So, tangent to it, so essentially, what I am saying is that dy by dx right at the wall is yeah, well I use a subscript w for wall. So, essentially this is the wall velocities here. So, if I do that now what we will do here is essentially consider small perturbations, right. If we consider small perturbations what we know here is that that u wall, u prime right. So, which would mean that I can actually write this adds, right. I can actually write it like this, right. If I can write it like this, then this I can also write in terms of the velocity potential perturbed velocity potential which will be right.

So, therefore, slope the slope out here is the v w prime this, and then I write v w in terms of the perturbation potential gradient of the predominant potential in the Y direction at the wall. So, this is an expression that we get. Now if I do that let us. So, therefore, call this expression as label it as 1. Now so, this is from the velocity component in the velocity consideration. Now the wall the contour of the wall by geometry in terms of x and Y is also given to us, right. Since that is given to us we can also do this, right. A since that is given to us.

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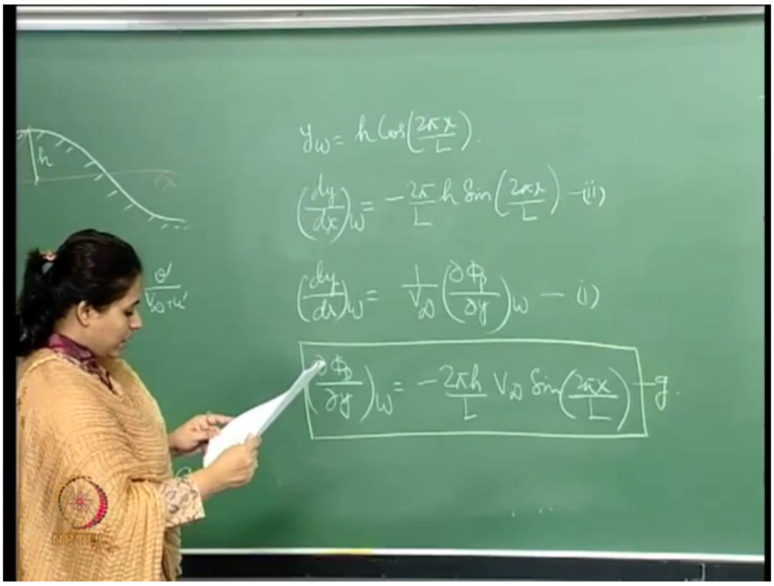

So, which is right. So, therefore, if I consider the slope of the wall, right. What I guess is right. So, this is what we get and let us call at this as 2.

So, this is from the geometry consideration of the physical wall itself, and from the velocity considerations. So, let me write that again. So, from the velocity considerations again what we got is that the slope is this, right. And this was 1. So, clearly this is from the velocity consideration and this is from the geometry consideration. So, let us put these 2 together right. So, basically though therefore, if this is equal to this right. So, if I you if this is equal to this. So, therefore, what we get from here is that del phi p del y, right. At the wall is equal to minus 2 pi h, right. Into v infinity sin ok.

So, we have and let us call this as say g. Now if I do this and of course, we have reduced we our A 2 has basically gone too. So, A 2 has basically now gone to 0. So, there are p so, we have this expression right. So now, the next thing is that essentially having done this; so, now we know that phi p is F x and G y.

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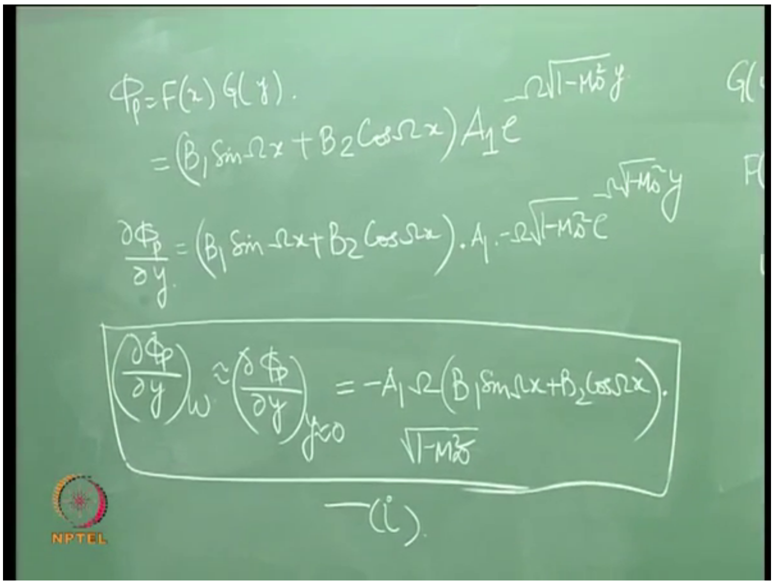

So, how do we write this? So now, Fx is essentially, now G y, G y if you look at this we said A 2 is equal to 0. So, therefore, we are left with this. So, which is A 1 exponential right so, we have this.

So now what we will we will do here, now what we will do here is that we will take you will find out del phi p del Y, is that if I do that from here what I get is this does not change. So, if I do this what I get is A 1 in to into Y. Now what we will do is; now we need basically at the wall at the wall, right. Now at the wall now let us come back here. So, essentially, we need the del phi p del Y at this wall.

Now we have assumed small perturbation, we which means that h is very small, which means that. So, if I use a small perturbation theory. So, then we say that h is very small. Which means that this wall is very close to Y is equal to 0. So, if I take here. So, del phi p del Y at the wall would also essentially mean that we are saying del phi p del Y for the Y for the Y direction to be almost 0.

So, if I do that what do I get here? What I get from here is minus A 1 into this. I think I missed out inter into this. So, let us call this as; that is called this relationship as this is call this relationship as I. So, essentially now what we have done is; now we have a del phi del Y relationship which we derived considering the physical slope which was given to us, and the velocity considerations here and we got an expression for that in terms of x, right. X here and this expression.

Now, from here now then again this is from our map that we did. So, using this relationship that a velocity potential, which we are using over here is essentially written in terms of this F and G we input that in here, and then we wrote an expression for del phi del Y. Now we need the del phi we have an expression a specific expression for del phi del Y at the wall. So, using a small perturbation theory we said that this del phi del Y at the wall for a small perturbation, which means h is nearly 0 which would mean that it is nearly 0 to the Y ordinate be 0 and we get this expression. So, next step is of course, to equate these 2. So, we will continue with this in the next lecture.

Thank you.