Advanced Gas Dynamics
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Lecture - 26
Supersonic Flow past a 3D Cone: Axisymmetric/Quasi 2D Flow

So I start with something which I think eventually is going to be the way to go conical flow 3 D flow. As you seen so far that the equations involved in in solving you know various sorts of problems are complicated or large you know numerically exhaustive. So, we will instead of go step by step. So, one of the reasons of doing like 2 D flow because it cuts down the; you know really the number of equations or the largeness of the equation.

So, if I do this. So, the simplification of the numerical procedure is something which is sort of paramount when we do 2 D flow otherwise you would have. So, just gone into 3 D flow and done that. So, when I say 3 D flow let us sort of go slow with this and try and understand what I am trying to say. So, let us say today is lecture is about conical flow; now what you I mean by that? let us say is drawn axis system let us call that as ok.

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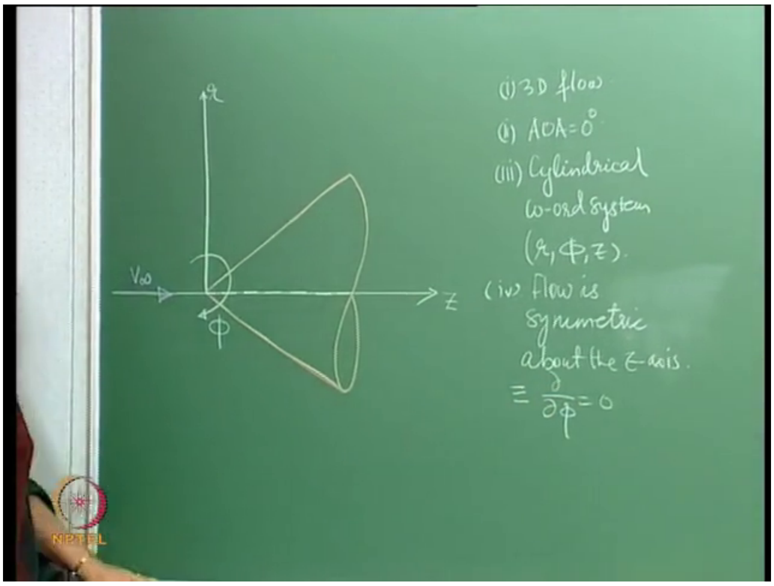

So, now basically what you can see is this is cylindrical coordinate system right. So, we have this this is a radial axis z axis and phi. So, now, therefore, here. So, if I have a body like this; now that is my cone. So, that is my conical body and we have a free stream coming in. So, this is something that we are talking about. So, of flow which would you know go past a body like this. Now let us set of list out some of the properties of the sort of a flow that we are going to look at and then go ahead and look at the equation that we need to solve for it. So, like you say this is a 3 D flow we will we looking at 3 D flow and the body if you look at this is essentially it is the planar curve right and which is rotated about a fixed axis which is z here.

So, in this case; as you can see that z is basically the sort of goes through the middle of the body or which is which sort of is symmetric; the body is symmetric about the z axis or you can say z axis is the line of symmetry of the body and the free stream velocity direction is in the direction of the is basically is in a same as the line of symmetry. So, therefore, in here what I say the angle of attack is 0 degree.

So, essentially what I am trying to say here is that if I take this as the line of symmetry, if I take this as the line of symmetry, then this is making 0 angle of attack. Now 0 angle with this free stream and hence I write as angle of attack is 0 right, then of course, this is cylindrical coordinate system that we are looking at. We are looking at a cylindrical coordinate system yes like I said z axis is the axis of symmetry right and V infinity is a line to z. So, essentially what happens here is that the flow is symmetric in the phi direction right flow is symmetric in the phi direction. So, therefore, rather flow yes say it is in line. So, the flow is basically symmetric about the z axis, right.

So, essentially the flow is not varying in the direction of phi. So, which essentially because this is essentially means that right. So, properties which are you know properties are not varying in the direction of phi. So, now, having said that. So, since we said that the angle of attack is 0 flow is symmetric about the z axis. So, the flow here is 3 D flow the flow here is 3 D flow, but then we really have 2 independent coordinates.

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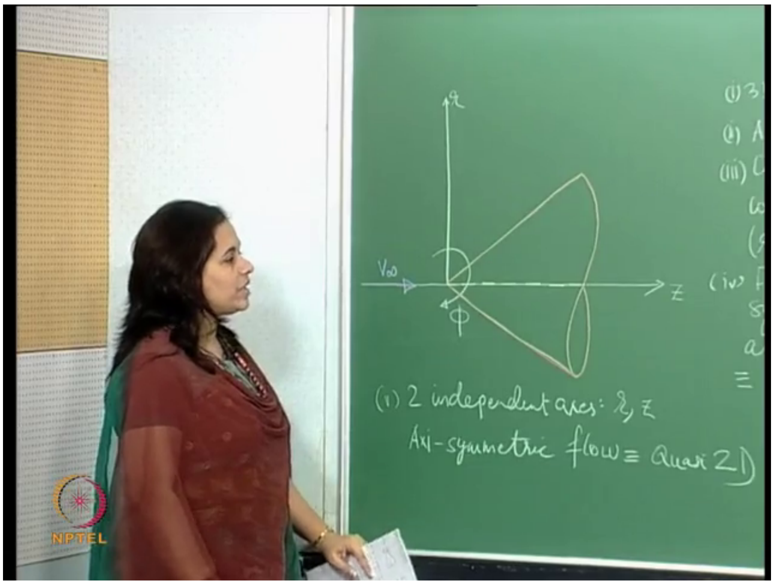

So, essentially what we have here is, it is the 3 D flow we have 2 independent axis right which is; so, it which is basically the r the radial and the z directions. Now in a such a flow; it is the 3 D flow in space with 2 independent axis because a we have a free stream which is in line with the axis of the symmetry, which in this case is the z axis which means that the angle of attack is 0 and what this makes it is therefore, Axi-symmetric. What; that means, is that this is an Axi-symmetric flow right it is an Axi-symmetric flow which also essentially means that it is a quasi-2 d flow right it is a quasi-2 d flow.

So, this is a kind of you know this is a kind of problem that we are going to look at now and hopefully within a another couple of lecture we should really do a full solution we should look at the solution of a full 3 D problem which is not Axi-symmetric, where these 2 are not in line and their angle of attack is on 0 will come to that. So, let us start with something like this now this also is; so, basically what we looking at is you know the problem here is supersonic axisymmetric flow you know which means this right.

Now, let us also look at, some scenes here. So, why is this problem important now; why is this problem important well number one. So, we is you know compare it ease with which we can solve thing exact equations, we are not going to use any sort of linearization for this sort of thing as a lot of practical applications and historical significance now the first solution to a problem like this was given by Busemann in 1929 actually.

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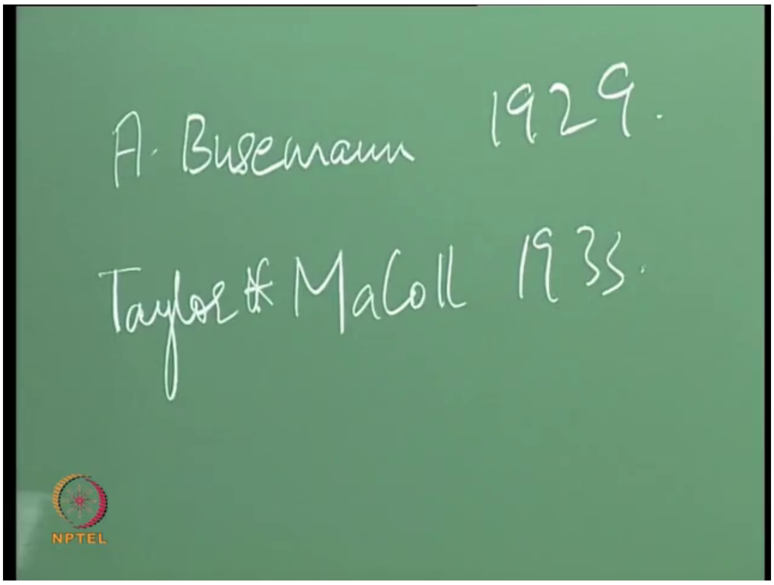

There was this guy called he suggested a solution for this and then finally, in 1933 you know Taylor and McColl basically. So, G I Taylor and McColl; they gave the very significant solution for you know supersonic 3 D axisymmetric flow and that is what we are you know going to look at right. Now when you look at solution and how we go about this such a problem. So, now, every time you see a supersonic sharp nosed V cone; you know you can always think of a solution you can always an you have and you are thinking oh god how do they make this one how do they decide upon the shape well let us see if we can find some answers to that because this is a kind of analysis which basically has applications for you know supersonic flow nozzles and sharp nosed V cos so on and so forth; alright.

So, let us I got some diagrams in place, now first thing first let us look at this.

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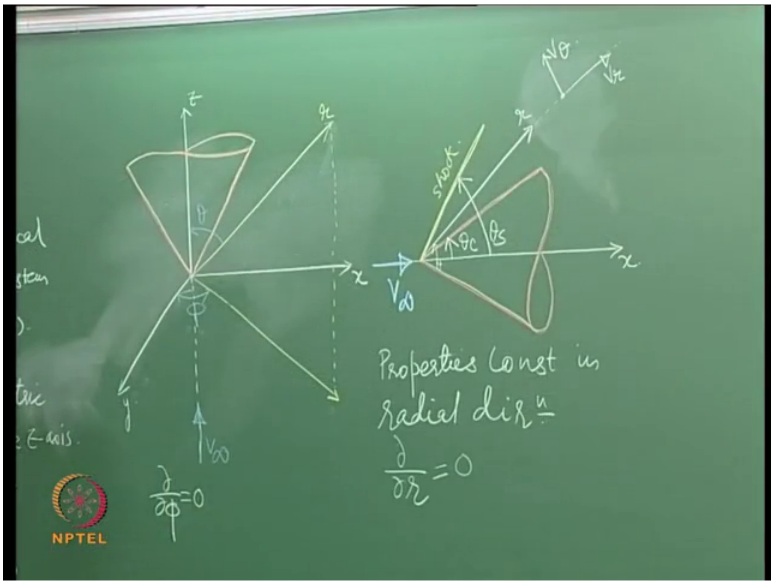

So, essentially what we saying is this right if I have a Cartesian coordinate system. So, I mean basically what I am trying to do here is look at this cone we have this you know 3 D cone, how does it look in you know the space that we are most comfortable with. So, if I have this here and say I have; so, I have. So, basically this is my cone. So, this is my Cartesian coordinate system and this is the picture that I am looking at ok.

So, now what we will do is we will look at the coordinate system that I am talking about got a nice yellow chalk here. So, now, let us use this. So, this is essentially a radial axis and if you drop a perpendicular on to this plane right onto this plane then you essentially get that now let us look at some angles. So, if I have this now this is essentially making an angle phi and this r is making an angle theta. So, this angle is making an angle theta.

So, therefore, now if you look at this. So, as you as you look at this. So, in my x y and z coordinate system this is the cone right then this radial line. So, now, if you basically look at this radial line, if you take the x and z you now coordinate system right you take a line and rotate about say this point and you know fix it at this particular point, take this radial vector and fix it at here and then rotate it about the z axis what you will get is this curve which is this you know 3 D volume here which is the cone right. So, if you take that and you move it this way, then you are able to generate this particular curve. So, therefore, this is a radial vector you drop a perpendicular of that into the x y plane. So, that makes an angle y this makes an angle y with the; this makes an angle phi with direction y. So, and this is essentially the direction of the free stream. So, this is these basically. So, this is the geometrical system that we are looking at, this is the picture.

So, let us go back now see the similarity of this picture. So, same thing now if I do this this is again my x axis right. Now I am going just look at. So, what I am looking at is r phi and z. So, this is the only thing that I am looking at. So, x here, therefore, in this you know axis system. So, therefore, I have. So, if we have that. So, then. So, let us draw this here, this is say you know this is my radial this is my radial vector. So, let us a let us draw that little smaller. So, we have this r and of course, if you have a. So, you have the velocity free stream coming in from here right. So, therefore, that would make what we had been doing so for right. So, this is my shock wave. So, this is my shock ok.

So, now let us look at this now what angles are we talking about here. So, what we a. So, let us call this as theta c this I am looking at the physical picture view this is theta for the cones a and this of course, is theta of the shock angle this is the theta of the shock; shock angle and yes. So, r is basically. So, this r. So, this essentially makes an angle theta everywhere. So, now, if I want to define some well if I want to define say a unit velocity vector. So, let us say I derived. So, if have a velocity vector. So, the radial direction velocity vector is V r and in this direction is in the fero direction is V theta perpendicular to the r direction.

So, this is a kind of a geometry we are we are looking at. So, like we said before yeah del del phi is equal to 0. So, any derivatives in there; there are none actually because it is axisymmetric. So, properties do not change you now in phi. So, these are this is 0. So, also now in this particular if you look at this diagram, now if you have properties in here properties are also constant along a radial direction right it is also constant on radial direction.

So, therefore, so, let us write this you know properties are constant right which means that right. Now armed with that let us begin to look at the equations now that you are going to sort of look at some of the you know pictures here, try to understand the geometry a little bit let us go on and look at some of the equations. So, let us look at this the continuity equation for a steady flow. So, the continuity equation for a steady flow.

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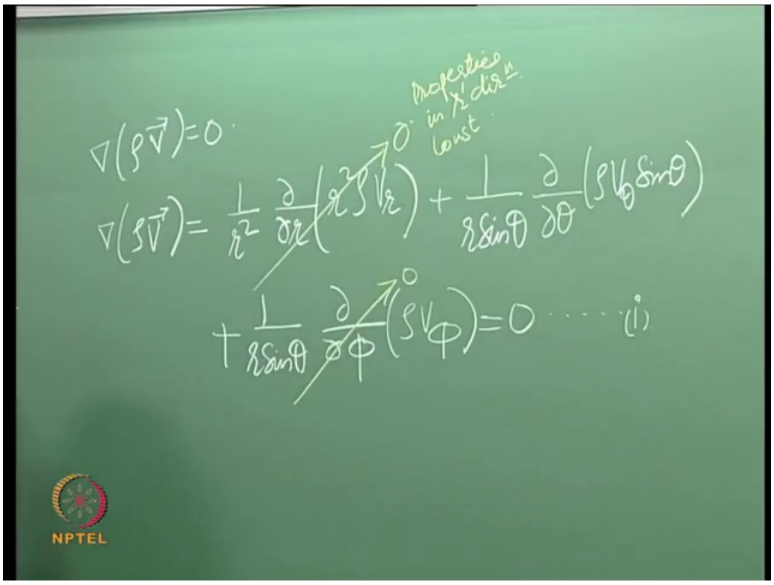

So, now, let us to do that and write this in spherical coordinate system. So, if I write this now do not get scared by this. So, what we get equations; the, if I break this up and write this down in the spherical coordinate system how will this look like.

So, let us look at this. So, you can find this any book I am going to write this. So, that you know we can look at it here and see what we will do with that. So, you can see the terms appearing here. So, now if I you know write this in this direction now let us look at this particular equation over here. If I look at this particular equation here as we have just as we have just said that you know the properties remain if I look at this particular equation the properties in the radial direction remains constant. So, therefore, derivatives there are no derivatives. So, therefore, del; del r goes to 0. So, this here actually goes to 0 and also. So, this is essentially because properties along r direction constant ok.

So, now again r this is an axisymmetric problem. So, it is axisymmetric about the z axis. So, therefore, does not vary in the phi direction. So, again this goes to 0 right and this is become. So, this sort of goes to should be write this yeah. So, if I have this now before I do that before I do that that is do see; let us go and expand this and then we will apply this then let us go and expand this because you might miss out a term a if I do that if I do this.

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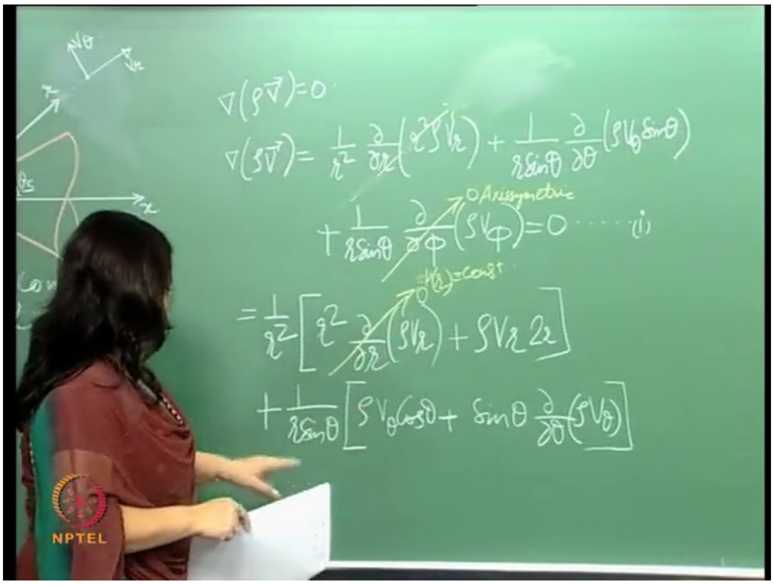

If I am go if I am going to expand this what I get is this ok.

So, let us sort of expand this here. So, what we get is r square that plus the second term is one by r sin theta which is rho which is rho phi theta cos theta sin theta del V theta and will leave that if it is. So, this one is of course, axisymmetric. So, this is axisymmetric and this term you see if I expanded. So, I get this right if I get this you see I would have lost out on this term if I put that in here.

So, now will go ahead and say that this is 0 because properties along r are constant. So, so let say the property is in r direction r constant. So, therefore, we go had we get rid of this. So, alternately. So, the equation that we come up with therefore, if I look at this. So, I have 1, 2 and 3 terms here. So, if I look at this what I get is this.

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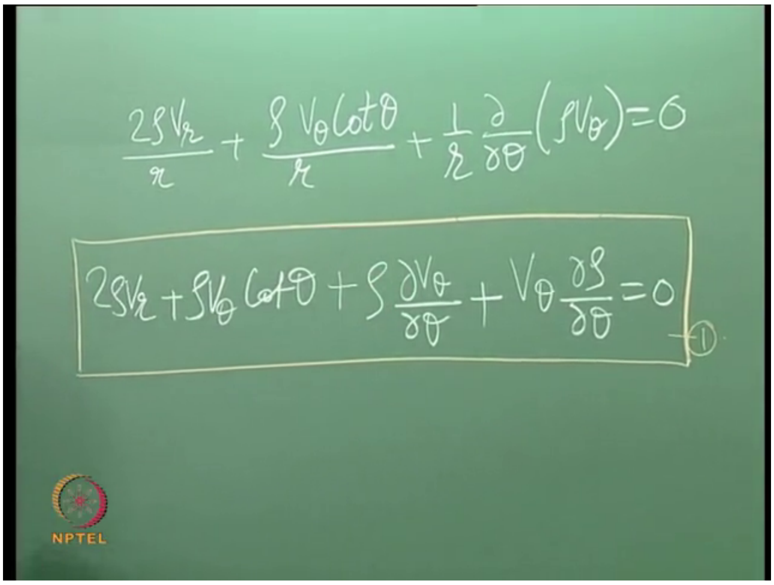

So, this and then I can just multiply it by you know or through a what I get this. So, if I differentiate this as well. So, what I get is right. So, this is the equation this is from the continuity equation that we are going to try and. So, let us call this here. So, this is what we get from the continuity equation for axisymmetric supersonic flow. So, we get this particular equation.

Now, if you remember if you remember the Crocco's theorem which was giving us a connection between the thermodynamic properties and the kinetics of the flow. So, if you remember right or let me instead of write it out again and remind you. So, let us go here. So, if you remember this.

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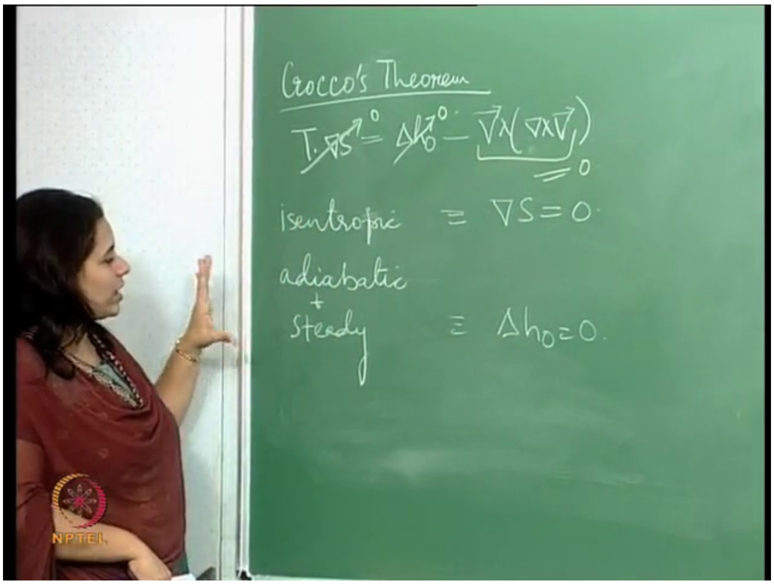

This is our Crocco's theorem . So, which was. So, essentially the momentum and energy equations combined. So, we ca this is the Crocco's theorem, right. So, it connects the thermo dynamic properties with the kinetics of the flow right now we know for this here. So, the flow are here is isentropic the flow out here is isentropic which implies which implies what and it is adiabatic and steady we consider adiabatic and steady with implies that implies it is constant right. So, this is what in implies if I look at this equation from here. So, essentially what I have is both this terms go to 0 which means which means that this 2 is equal to 0 right.

So, therefore, if I have for a supersonic axisymmetric flow which is isentropic adiabatic and steady then this wholes which is you can see this is an expression for the irrationality of the flow. So, an supersonic axisymmetric flow which is isentropic adiabatic and steady is also irrotational; now let us see what this means in terms of our velocity components V r V theta so and so forth.

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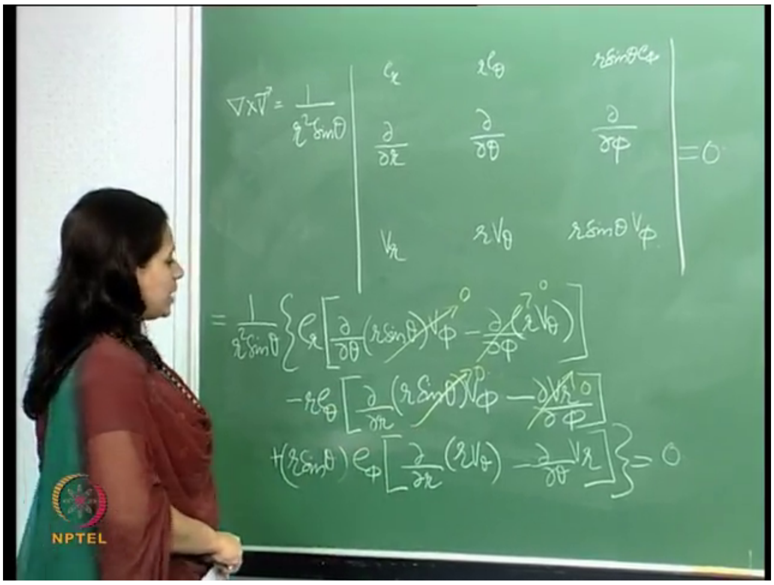

So, del cross V is this and so, in this ration basically have del del r you have del del theta del del phi you have V r r V theta r sin theta V phi and what we just found out that this is also equal to 0 this is also equal to 0. So, what we get out of here. So, if I am to going to go ahead and expand this. So, what I get from there is this what I get from here is this right.

So, if I expand this if I expand this whole thing what I get is this you can cross check this you can just cross check that let me just do this for you. So, e r. So, e r basically I do this here. So, del del theta of r sin theta V phi. So, which is this minus del del phi of r V theta that is what we get minus e r that which is these 2 on the here this is the. So, del del r of r sin theta V phi minus del phi of V r and you know. So, on and. So, forth r is this out here. So, r sin theta and e phi is del del r of r V theta minus del del theta of V r. So, that is it this is what we get from here right and V theta, but there is really no V phi because there it is axisymmetric. So, the V theta is not going to be here. So, we do not have any V phi. So, let us sort of this goes to 0 here goes to 0 del del phi del del phi is axisymmetric. So, that goes to 0 del del phi axisymmetric that goes to 0.

So, you can see what we are left out with we just left out with r 0 here. So, what we get is this what we get is this.

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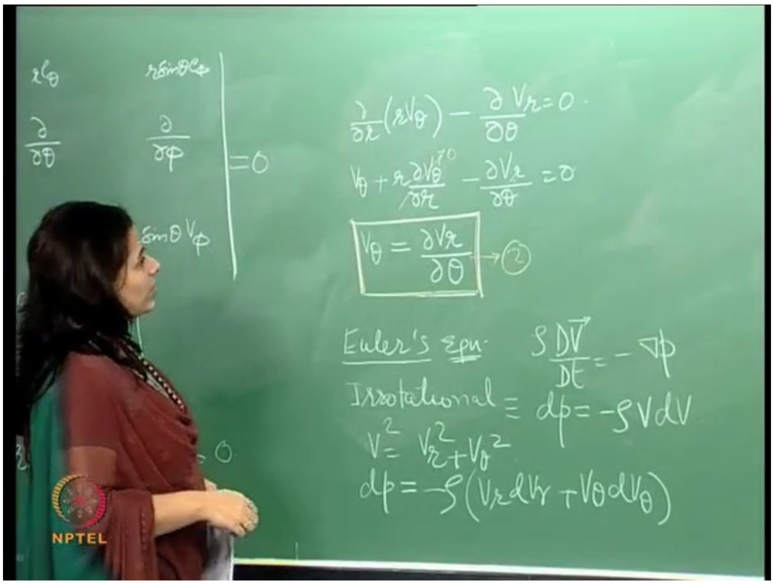

Del del r of r V theta. So, del del r of r V theta minus del del theta of V r is equal to 0 that is what we get let us expand this first one. So, what we get is V theta plus r del V theta del r now again in the radial direction properties are constant. So, therefore, this also goes to 0. So, what we essentially get out of here is that V theta is equal to; now this is interesting. So, in the theta direction which is normal to the r direction. So, or this we can this as 2. So, all we basically do is take a derivative of the radial velocity in the theta direction.

So, essentially what we are saying is that if we have radial vector like that a radial vector like velocity like that we are looking at the change of this radial velocity in this theta direction when I change the theta like that what is the change in the a radial velocity vector. So, that is what this is essentially telling us. So, Euler's equations. So, on the; I am not writing the body forces then what we get is. So, this is irrotational. So, we have irrotational flow which has been right. So, what; that means, is that. So, if it is irrotational I can write the change in pressure in terms of the velocity change on the density right. So, if I do that ok.

So, now let us see how we will sort of a write out the Euler's equation in terms of V r V theta and so on so forth. So, so let us just say here. So, I am you know V square out here you know let say. So, V square in here is V r square that makes sense. So, the velocity here is radial velocity and the; you know curve you know radial velocity and the theta velocity direction. So, if I do that if I write that. So, therefore, d p; d p is equal to if you look at this equation if I look at this I can write in terms on in the r and the theta direction right. So, if I write this now let us also recall that we know let us recall that let me use now.

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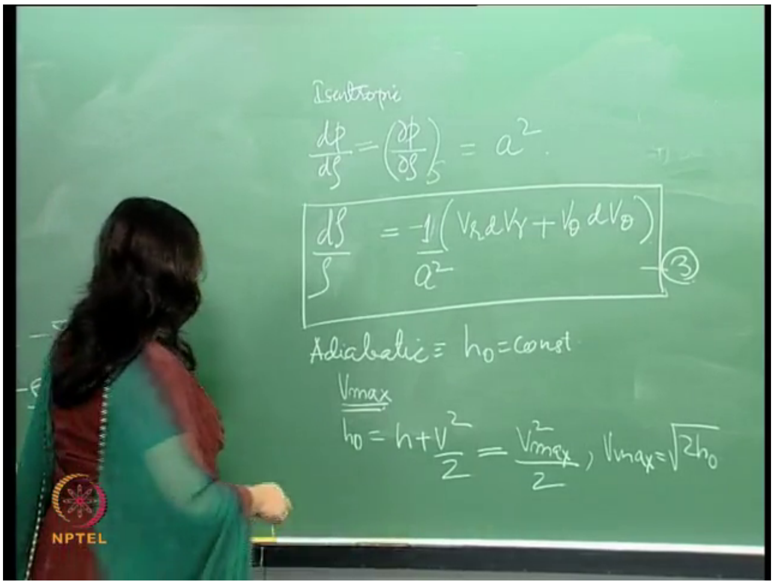

Now it is isentropic; right, it is isentropic. So, therefore, we also know that right. So, we now this from before now having said that. So, therefore, let us write this as. So, d rho into a square is equal to d p and for d p let us put in this value in here. So, to do that. So, basically what we get is the which is minus rho into this or I can write this as or I can write this as this equation I can write as this is d rho into a square is equal to this right or I can bring this here right making that as one making that as one I will take this a square down here. So, then this is what we get. So, d rho; let us call this as 3. So, let us we. So, we get this is this equation out here now we saw it is adiabatic. So, we said it is adiabatic. So, therefore, the right. So, adiabatic which means that enthalpy constant right. So, if we if we if we do that let us define another reference system in terms of the velocity right.

So, let us you know new reference velocity. So, let us call that as a Vmax. Now Vmax is essentially the velocity is the maximum velocity right which is which is reached from the reservoir conditions right which is reached from the reservoir conditions. So, that the temperature out here is 0. So, that the enthalpy is 0 and this is the maximum velocity. So, which means you know that. So, h naught right. So, which will be right. So, the maximum. So, h naught is nothing, but this.

So, I look at and this is c p into t in temperature c p into t right. So, therefore, I reach the maximum velocity the entire enthalpy is coming from the velocity and the temperature is 0. So, what we get here is. So, this is my new you know reference this is another new reference velocity. So, this is velocity which essentially I am reaching I am reaching by expanding the flow from reservoir conditions to a place where the temperature is 0.

So, what you can see from here therefore, what you can see is that Vmax is equal to 2 times the reservoir enthalpy right. So, another 2 times the reservoir enthalpy now again if I use for a calorically perfect gas. So, let us just ref rewrite this here. So, I am going to use this place.

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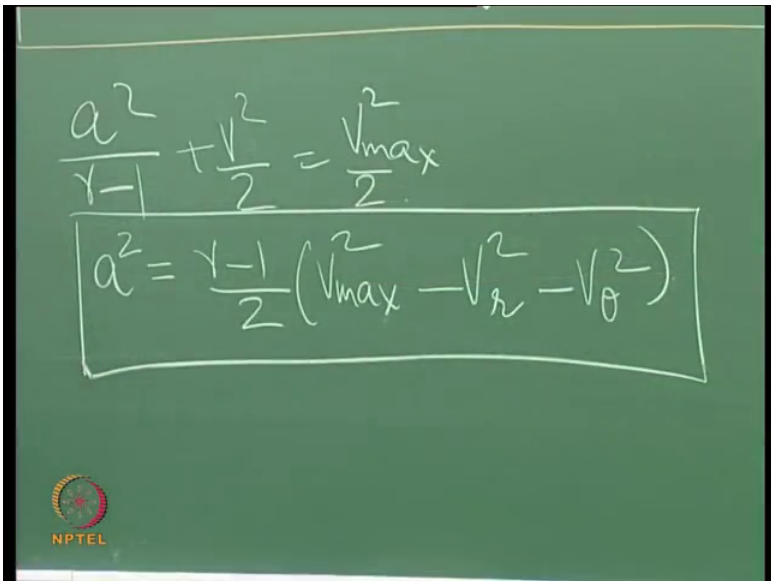

So, for a calorically perfect gas again; right. So, I can write this. So, in here from here what I will write is what I will write here is this a square right which I can write as and this I can write as V r square plus V theta square right. So, if I write this then what I get. So, I get a value for this a square. So, a square I take this a square and put it in this equation 3 out here if I do that let see what we get I take that value of a square which we just got in terms of Vmax, V r and V theta and we will put in in here.

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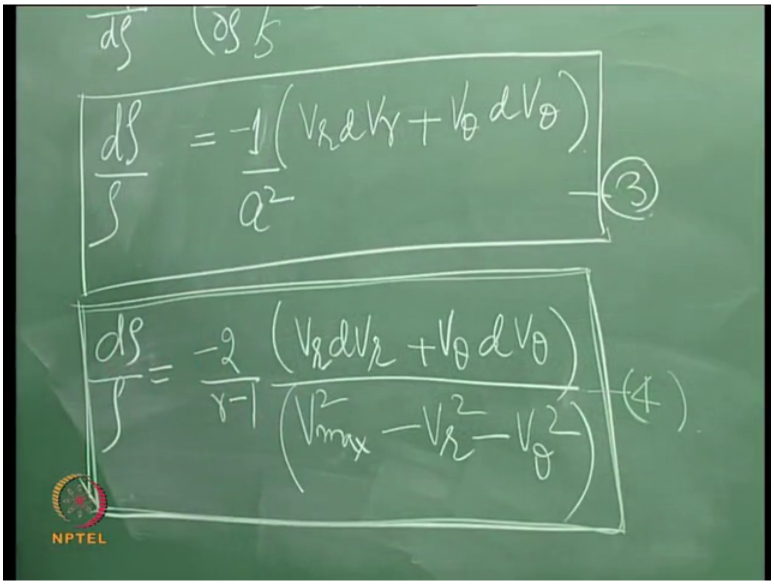

So, what we get is therefore a square is and we get and this Vmax square is nothing, but 2 times the 2 times the reservoir enthalpy right which is the constant in here because it is adiabatic it does not change. So, if you look at this equation if you have to solve it may be does not sort of look that scary now. So, now, these is essentially what I have reduced the Euler's equation to be. So, if I look at this this is essentially . So, this is essentially my Euler's equation so.

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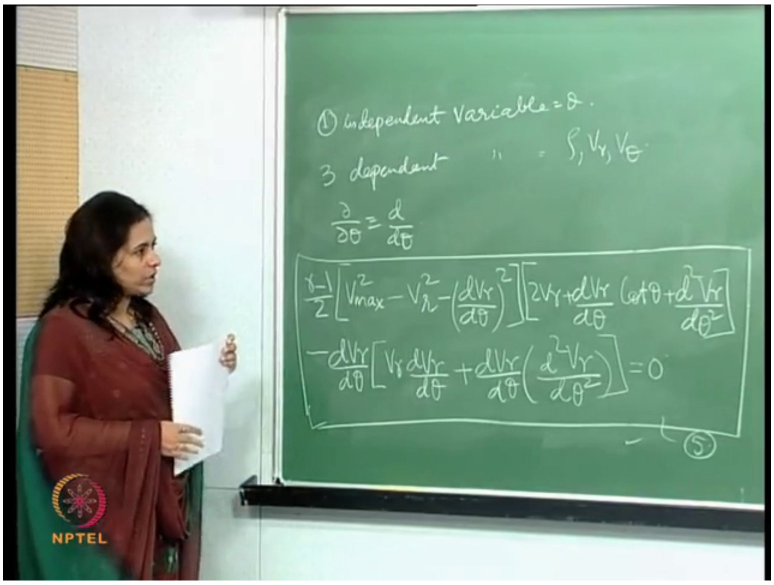

We have just one independent variable which is theta and we have 3 independent variables we have; sorry; 3 dependent variables right which are which are rho V r and V theta ok.

So, therefore, so, in therefore, what we can do is basically we can write these as simple derivatives we can write these as simple derivatives what I and I incorporate all of that into my Euler's equation right which I use over here. So, what I get here is something like this basically all I am doing is combining all these equations. So, that equation. So, this equation out here let us call this as five. So, this equation out here is essentially the Taylor McColl equation that I talked to you about at the beginning. So, this is the Taylor McColl equation for axisymmetric conical flow and this is the equation that we will try to solve for our axisymmetric conical flow ok.

Let us stop here. So, of course, this is there can be no there is no close form solution for this we need to solve this numerically we will see how. So, essentially. So, now, at least we have got an equation which with which we can work with. So, we have made no linearization no assumption nothing we just gone from basic governing equations for conical flow and got this nice expression and see how we can solve this for a kind of the flow we talked about which is conical axisymmetric flow.

Thanks.