Fluid Dynamics And Turbo Machines.
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Part C.
Module-1.
Lecture-2.
Differential Analysis.

Good morning and welcome back to the 2nd lecture of the 3rd week of this course on fluid dynamics and Turbo machines. In the first lecture on the 3rd week of the 3rd week where we are dealing with the differential analysis we have looked that the derivation of the mass conservation and momentum conservation equations by solving which we can get the velocity field as opposed to overall quantities like force and thrust or torque which you get in the case of an integral analysis. So for the ad the end of the last lecture what we got is a particular form of the momentum equations. We will take a look quick look at that.

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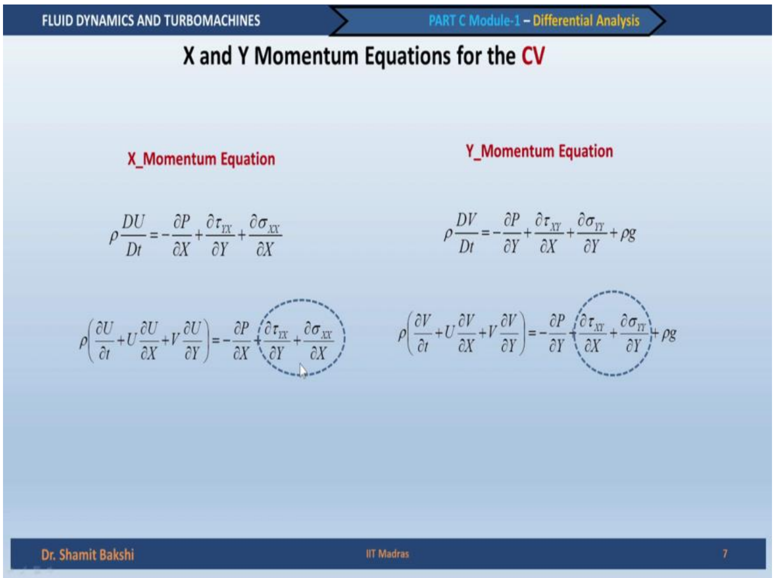

So the X and Y momentum equations for the CV, the X momentum equation look like this, so it has on the left-hand side the fluid acceleration multiplied by density and on the right-hand side we have the pressure gradient and the stress gradient. So, 2 types of stresses appears here, the shear stress and the normal stress. In the, okay, so this can be further written as, the first component as the local acceleration and the 2nd component of the acceleration as the convective acceleration.

So the total acceleration is composed of 2 components, one is the, one consider the time-dependent velocity, another consider the convective component of the velocity. So one, this is the expression for that and for Y momentum equation similarly we got this expression. Here again the gradient for P is with respect to Y because we are considering the force balance or the momentum balance in the Y direction, momentum conservation in the Y direction and we have this acceleration in the Y direction.

Additionally we have the weight of the fluid in the control volume. So this is the expression by writing the full expression for the total acceleration. So what we are going to do today is to look at this to particular terms in this equations, that is the gradient of the shear stress and the gradient of the normal stress. How can we write these stresses with respect to velocity? In the first lecture we have seen that for a simplified case, we can use the Newton's law of viscosity for the Newtonian fluids and relate the shear stress with the rate of deformation. But those expressions in which we had derived are not very general and we cannot directly apply this here. We will see what form it takes for the case of a general two-dimensional control volume.

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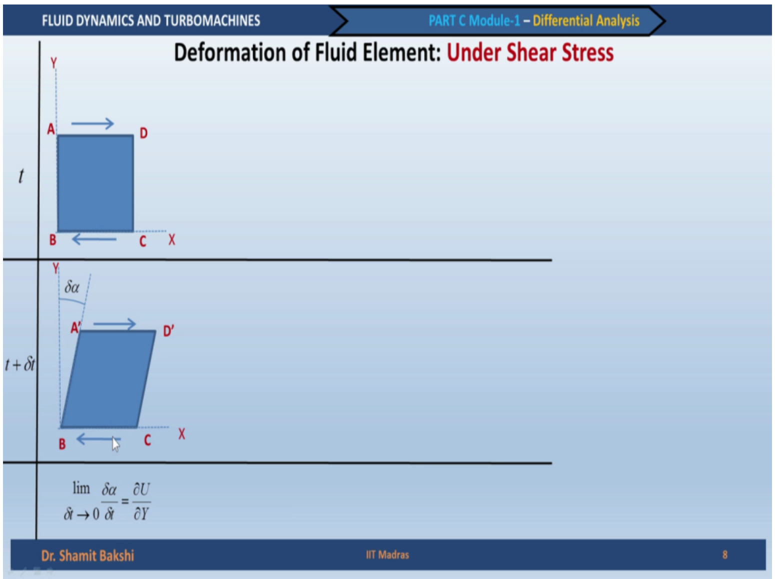

So we have to relate the stress, the relation between the stress and the rate of deformation is still valid but we have to see what is the mathematical expression for the deformation of a fluid element in case of a two-dimensional control volume. So for that let us say we consider a time T and we consider a 2-D control volume like we had considered in the last chapter when we dealt with the forces and also the momentum. So this is given as ABCD and now we consider that there are stresses acting on the top and bottom of this control volume or the fluid element. So at if, at a time point of time T, you have kind this kind of shear stresses acting on this element, what will happen at a time T plus Delta T? So at a time T plus Delta T this particular control volume will deform it will look something like this.

This is something which we have seen in the first chapter also and based on the calculations which we had made there we also got an expression for the rate of deformation. So Delta Alpha, so what changes, actually from the first stage to the 2nd stage, on application of the shear stress like this what changes is basically the angle of this edge of the control surface. So the volume of this element actually remains constant on application of this shear stress, it does not change, only the shape of the control volume changes. That means the length of these edges remains constant, only the angle, so this is only the angle this edge makes with the vertical axis, it changes on application of a shear stress as shown here.

So A moves to a new position A prime and B moves to a new position B prime, now if you consider these 2 times and we try to find an expression for the rate of deformation, we can, if we look at the time T plus Delta T, it is deformed and we have the deformation written in terms of Delta Alpha. So this Delta Alpha deformation has taken place over a time Delta T and we can find this rate of change, rate of deformation as, so we take a limit of Delta Alpha by Delta T when Delta T tends to 0 and we saw this in our first lecture, in our first week that this can written as a derivative of Dell U by Dell Y.

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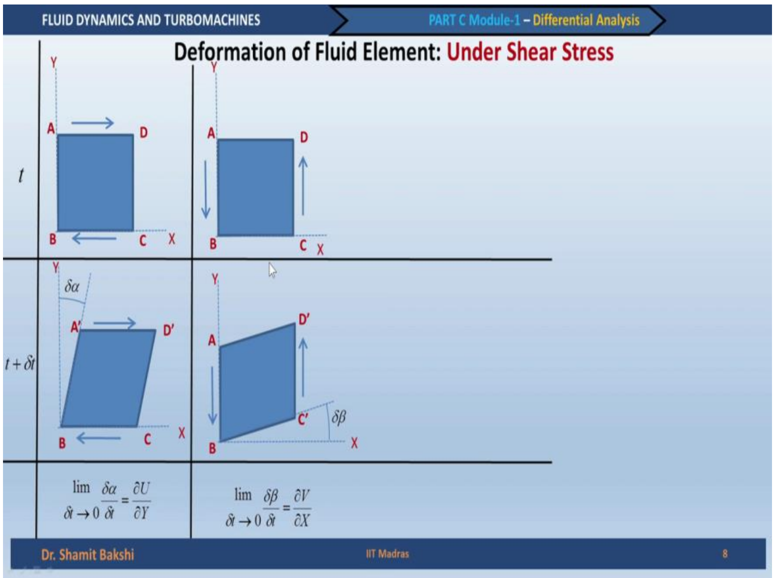

So now if we proceed in the similar way and we now apply, we take the same control volume and we apply a shear stress on the element ABCD in a Y direction now. So this is the direction of the shear stress, so on application of this kind of a shear stress, the deformation of the object or the elemental volume will be little different, so it is of this form. And now we have a deformation defined with respect to this angle is Delta beta and this is, this will give us now the rate of change of deformation of this element, like we did in the case of a deformation with respect to the force the shear stress is applied in the X direction even with the shear stress is applied in the Y direction we can find a similar expression for rate of change of deformation which will be given by Delta beta by Delta T at the limit of when Delta T tending to tends to 0.

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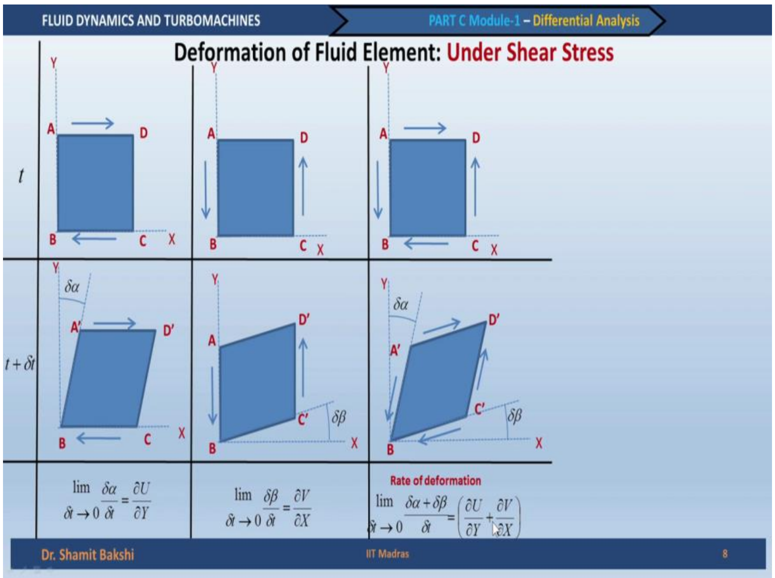

So under this condition we can define the rate of deformation as Dell V by Dell X. Now neither of these 2 cases demonstrated here alone represents a complete two-dimensional situation because in a two-dimensional situation the shear stress acts both in the X and Y directions, that is a combination of the first and the 2nd case. So if we look at a control volume or a fluid element in general it will be experiencing the shear stress is as shown in this figure. That means it has shear stresses acting in Y direction at the same time it has shear stresses acting in the X direction. Now what happens, it will be interesting to see what happens to this fluid element on application of these 2 shear stresses actually 4 shear stress is applied in both directions. So, if we look at such an element, it will undergo deformation like this.

So it will, the this edge that is BC of the control volume will make an angle is Delta beta like it has done in this case and this edge, that is the initial edge AB will go to A prime B position make in an angle Delta Alpha with the Y axis. So you can consider this to be a combination of the first case and 2nd case. So this is the, this is a more generalised situation for a two-dimensional control volume that it can deform, both the edges can actually change angle, so it can deform in both directions. This is a more general situation and when we are deriving a general equation momentum equation, we have to take a general expression.

So if we combine these 2 we can get an expression for the rate of deformation now for a two-dimensional element which is experiencing the shear stresses in both directions as Delta Alpha plus Delta beta by Delta T when Delta T tending to 0, at that limit we can write this as the rate of deformation, this is also the strain rate on the fluid element due to the shear stress. So this gives us quite a clear way of how to express the deformation of a fluid element under the action of shear stress. Just remember that we have not yet considered the normal stresses which had also appeared in the momentum equation which we derived before.

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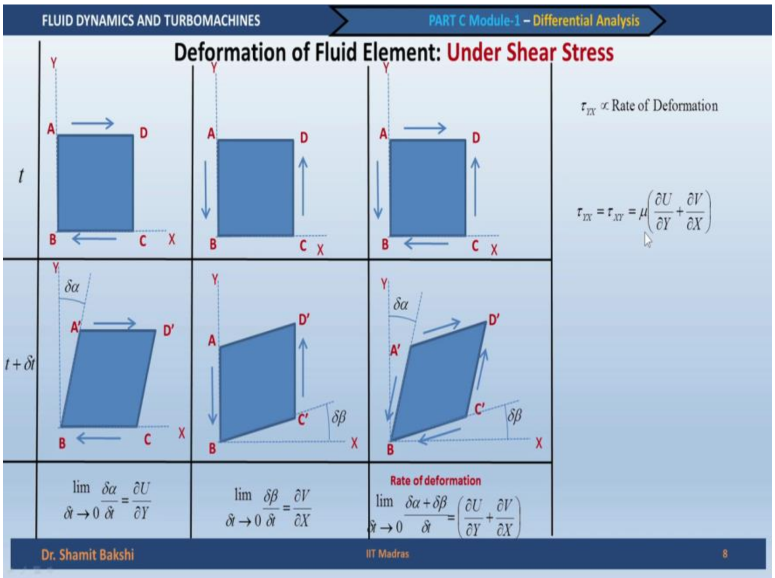

So now let us see this only gives us a expression for rate of deformation due to shear stresses, so that means now by using Newton's law of viscosity that which states that the shear stress is proportional to the rate of deformation and it is equal to the viscosity into the multiplied by the rate of deformation. So by utilising that law, the Newton's law of viscosity we can now relate the shear stress with the velocities. So Tao YX is proportional to rate of deformation, so we can write Tao YX, of course this has the same value because these 2 are, this condition is symmetric, there is no net rotation of this fluid element, so tao YX is equal to Tao XY and that can written as mu viscosity multiplied by the rate of deformation. So this is an expression, this is how we express the shear stress in terms of velocity. This can be utilised into, utilised in our previous expression for the X momentum equation.

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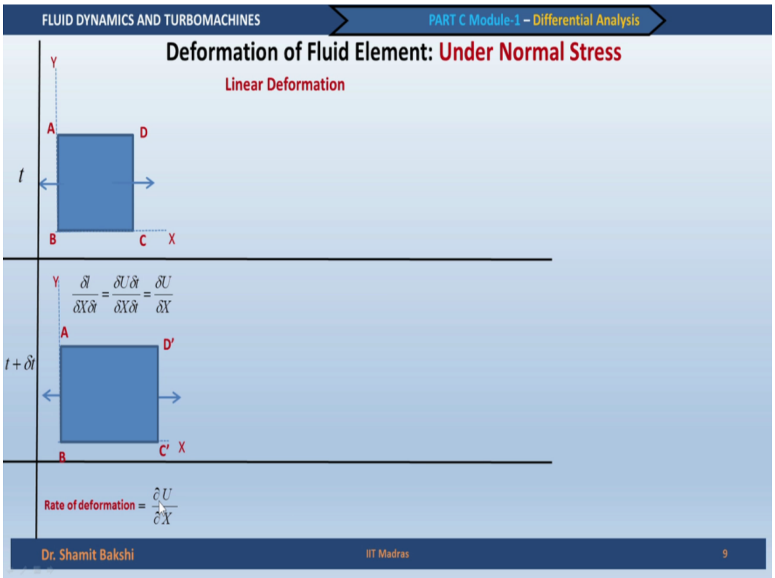

Before doing that, let us see what is the deformation of a fluid element under normal stresses. So in the case of normal stresses the situation, although it appears simple but it is quite complicated because we will see the results as we go forward. So under normal stresses we can divide the deformation into 2 components, one is the linear deformation which is the simpler part and let us look at that part to begin with. So again we consider a time instant T and we consider a volume element like this ABCD as before but now we only consider that the stresses, the X directional stresses, normal stresses are writing on this fluid element, there is no shear stress acting on this fluid element. So, only normal stresses in the X direction is acting on this fluid element, it will cause first of all a linear deformation, it may cause under certain circumstances which we will see very soon, it can cause a linear deformation.

So what it means, it goes, as we go to time T plus Delta T, it takes a new shape and what is that shape, the shape is like this, that means the fluid element has been stretched in the X direction. DC has now moved new position D prime C prime, now if you see this actually means, when we just consider, just even consider the X directional normal stress, we see there is a change in volume which the fluid element goes through as it moves from time T to T plus Delta T which was not the case in the case of the angular deformation under the shear stresses. So it only change the angle of the edges but it did not change of volume, here there is a change in volume. You can imagine that this change in volume can only be there if because this change in volume also means that now this control volume as the density, if the density is constant, it is difficult to achieve this or it is not possible to achieve this change in volume. We will see that very soon but before getting into that we get an expression for this rate of deformation, rate of linear deformation.

So what is the rate of linear deformation? As we had done in the case of the angular deformation, the rate of linear deformation is the change in the length divided by original length and the elemental time because they are talking about rate of change of deformation and not just the percentage change. So Delta L by Delta X and the whole divided by Delta T actually. So this delta L is actually difference in length along X axis between this element, that is the difference in length between AD prime and AD. So if we, that can be easily obtained as the Delta U, the difference in velocity multiplied by Delta T and so now we can write this, if we cancel Delta T from the top and bottom, in the limit we can write this as Dell U, we can first of all write it as Delta U by Dell X and in the limit Delta T tending to 0 this can be written as, rate of deformation can be written as Dell U by Dell X. So this is basically the rate of deformation due to rate of linear deformation in the X direction.

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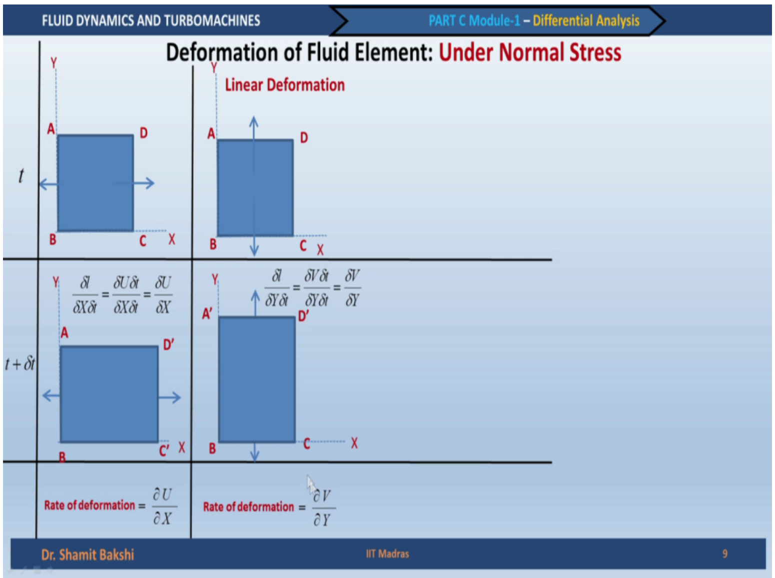

In the 2nd case let us consider a normal stress in the Y direction. So perpendicular to the face AD and BC now if we consider that, then we see that a similar thing happens, linear deformation happens in the Y direction, it gets stretched along Y axis, the X axis, as there is no stress or no force acting in the X direction, it remains as it is and again there is an increase in volume. So the volume increase as if you move from here to here, so the rate of deformation now can be again obtained like we obtained here, the change in length per unit original length, original length is now Delta Y per unit time, so in time Delta T. This Delta L is different from the Delta L defined before because this delta L is the change in length in the V direction which will be given by the velocity difference multiplied by Delta T.

So this can written as Dell V by Dell Y, like we did in the case of, even in the case of angular deformation. So rate of linear deformation along Y direction will be given as Dell V by Dell Y. A general element like we did in the last slide for the shear stress will experience both normal stress both in the X direction and Y direction, so let us see how it will look like.

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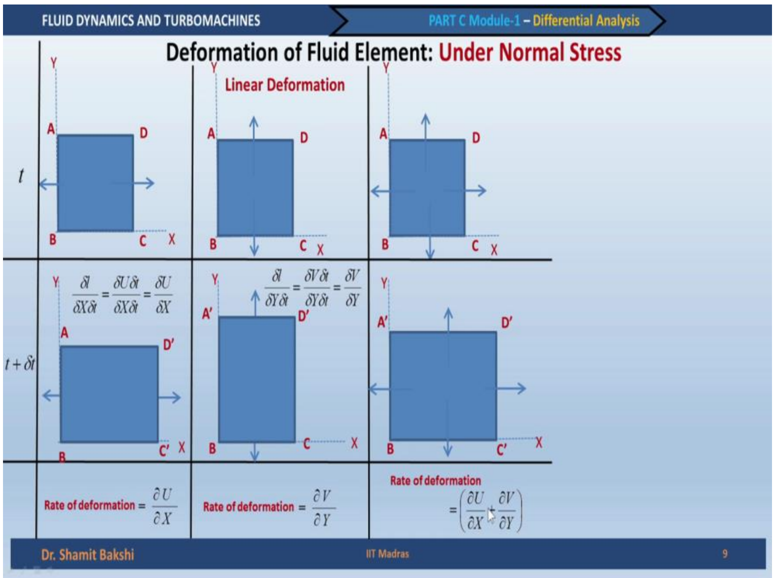

So it will look like something like this. So it has stresses acting in the X direction as well as in the Y direction and then if you consider all of them, then it expands both in X direction and Y direction and if we consider that, then we find that the rate of deformation, the total rate of deformation of the fluid element is actually Dell U by Dell X plus Dell V by Dell Y, as we have considered an incompressible flow here, this thing actually becomes zero. So for an incompressible flow which has been considered for this derivation, Dell U by Dell X is equal plus Dell V by Dell Y is equal to 0. So the linear deformation for an incompressible flow is actually zero. That means, that is also what we were trying to explain before that this deformation can only happen is the fluid or the flow is a compressible.

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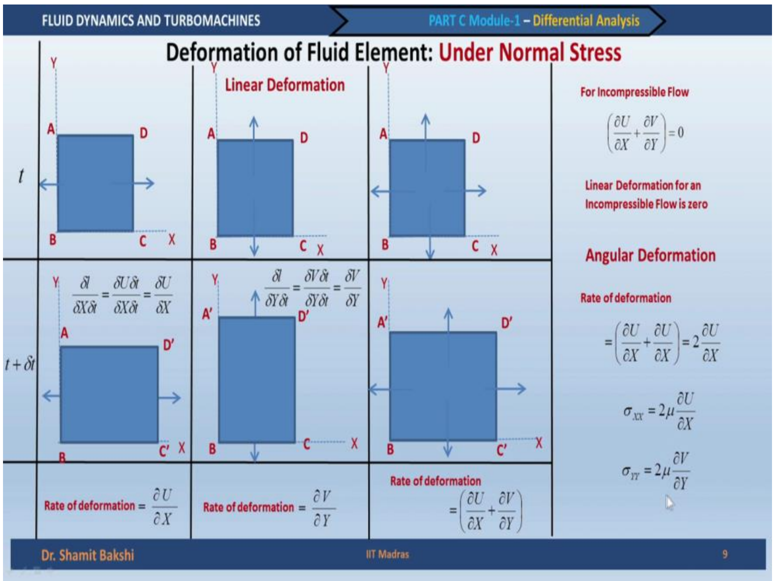

That means, because it changes there is a change in volume. If there is a change in volume on application of the stresses, then the volume, this volume change, the total change in volume is only possible if the density varies. If the density, so the density, if for example if you go from here to here, if the density decreases, then the volume can increase by keeping the mass constant but that is, so that is only admissible when you have an incompressible flow, admissible when you have a compressible flow. For an incompressible flow this is not possible. So this part we do not have to bother because we are considering incompressible flow.

What we have to take into consideration is the angular deformation under normal stress. Now this is a thing which is the most difficult part, we will not go into more details into this because this involves a very difficult to derivation. So what we just try to see if you if you look at normal stresses, although it is difficult to imagine that under the normal stress or let us go to a simpler situation like this, the X directional normal stress, so this stress itself can actually result in angular deformation because if we look at this fluid element, this fluid element on this surface is subjected to a normal stress but if we take a, if it is a square element if we take a diagonal of this fluid element, it will experience a shear.

So the normal stress can induce a shear in the fluid element which you can, you might be familiar with that with respect to our analysis of strength of materials where we try to define the state of stress in a material using a more circle. So even under these normal stresses acting on an element, fluid as well as the solid element, there will be shear stresses developed in the element and the shear stresses, stress if this is just a square element and this is unidirectional normal stress, then the shear will be maximum at 45 degree from the direction of application of the normal stress. And as we know in the case of a fluid, the fluid will always flow on application of a shear. So that is why this normal stress induced shear will cause a flow and so that will result in an angular deformation of this element. Without going into the details like we did here, we can directly write the expression of the rate of deformation here.

The rate of deformation is not very different, the expression for the rate of deformation is not very different if we consider our angular deformation for under the action of shear stress. So the shear stress induced the normal stress result in an angular deformation and that rate of deformation, the expression for it can be written in the form as below. This is, this can be derived from in details but we do not go into the derivation but we just observe the similarity of this expression with the shear induced deformation. So this, when we did the derivation of the shear induced angular deformation, it came out as Dell U by Dell Y plus Dell V by Dell X, now we are talking about, that means Sigma XX only applied here.

So if we replace the Y with the X and the V with the U in the expression of the shear induced rate of deformation we get the expression like this, Dell U by Dell X plus Dell U by Dell X. So it is actually 2 Dell U by Dell X. Now considering this as a rate of deformation, the shear, the normal stress Sigma XX can be written as twice mu Dell U by Dell X from the law of viscosity, from the Newton’s law of viscosity. So we see that not only shear stress but the normal stress acting on the fluid can be expressed in terms of the velocity gradient. Now essentially we have got the 2 expressions which were required to get, to replace the stress term appearing in the momentum equation. That means the expression for shear stress in terms of velocity and the expression for the normal stress in terms of velocity.

We can plug-in this velocity gradient into momentum equation and then we should be able to get an equation completely in terms of velocity. Let us do it and see how does the final equation look like. So before going into that, we can also see Sigma YY similar to Sigma XX can be written in this form as twice mu Dell V by Dell Y.

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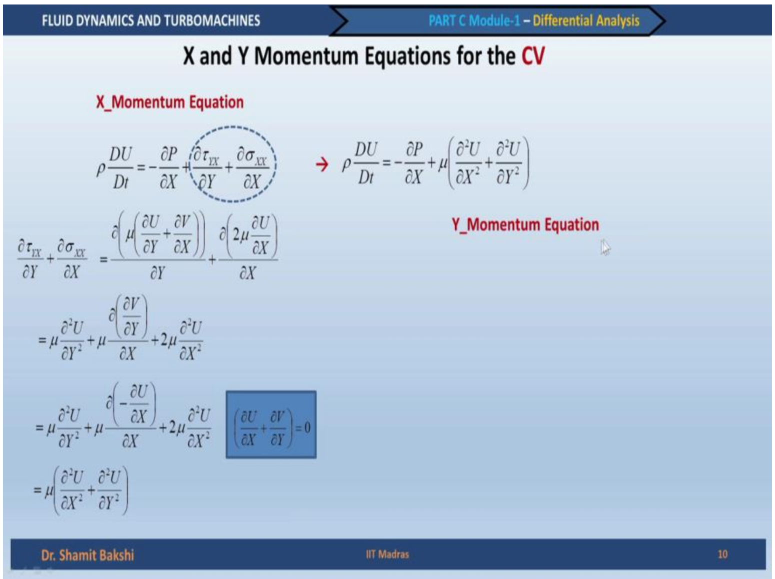

So considering the expression for the shear stress now, we can write the X momentum equation as, so this was the form of the X momentum equation up to which we derived it, it has, it had the terms or derivatives in with the of shear stress as well as of the normal stress, so we can write this now, this part can be now replaced with the velocity, the shear stress and the normal stress expressed in terms of velocity gradient. So this term we take up separately and we see how it looks like. So the first term that is Dell Dell Y of Tao YX, so Tao YX, if you remember is basically the rate of angular deformation due to the shear stress which is given as mu multiplied by the rate of angular deformation which is mu into Dell U by Dell Y plus Delta V by Dell X.

On the other hand Sigma XX as we had derived in the last slide looks like twice mu Dell U by Dell X, so now we can simplify this expression. We can write the first term, if you just take a derivative of course we consider mu as constant here, so otherwise we can write it as Dell Dell U of mu Dell U by Dell Y. If we consider mu is a variable quantity, then also we can derive it but we are assuming for the derivation the viscosity is constant. By considering a constant viscosity way get the first equation, first term in this expression as this Dell 2 U Dell Y 2, the 2nd one as Dell Dell Y of Dell V by Dell X actually but we have interchanged these 2 things, this is allowed as a rule of partial differential, partial different station that we can write this as Dell Dell X of Dell V by Dell Y instead of Dell Dell Y of Dell V by Dell X.

So we just exchanged this for a particular reason, we will see very quickly. And then this one comes out to be twice mu Dell 2U Dell X 2. So this is the 2nd face, now we can look at this expression carefully, if you look at this expression, what we get is, this is nothing but minus Dell U by Dell X because we are considering a 2-D incompressible flow, so Dell U by Dell X plus Dell V by Dell Y is equal to 0, so Dell V by Dell Y will be minus Dell U by Dell X. So, just plugging in the value of, plugging in that values here so that we can get a more simplified equation, we can simplify this further 2 mu into Dell 2U Dell Y2. So this is our final expression which will be, which can be now plugged into the original X momentum equation. Which is given, so this will be now given an as this.

So this is our final form of momentum equation, as you can see here, the unknowns here are in terms of the velocities and pressure. So we have gotten equation in terms of velocities and pressure and so we can now think of getting the velocity field. We have got it for U velocity, so similarly we can get it for V velocity also from the V momentum equation. Please note that even in this equation the V velocity appears because the total derivative of the U velocity has a convective term which is dependent on V.

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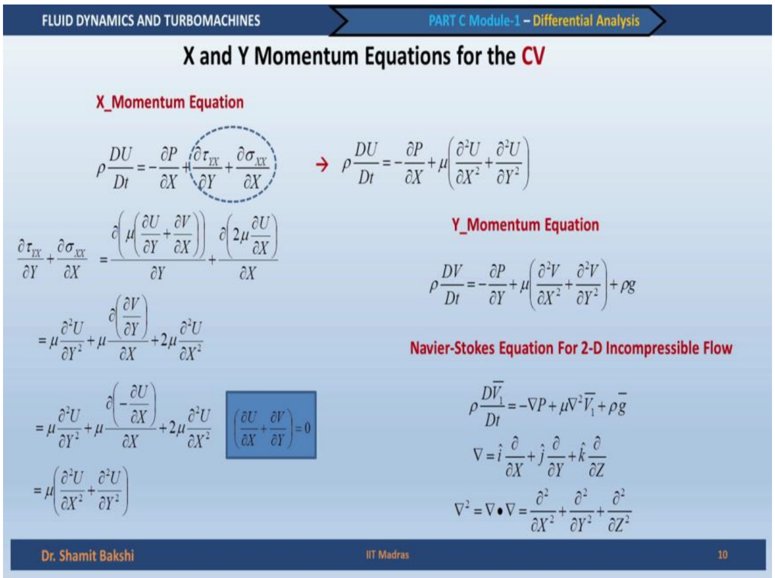

So in Y momentum equation, now similar to this we can get an expression like this by replacing the shear stress term appropriately we get mu into Dell 2V Dell X 2 + Dell 2V Dell Y 2 and rho g is the additional weight component here, so basically this gives us the Navier-Stokes equation for a 2-D incompressible flow in its full form. So this equation as you can see will allow us to solve for the velocity field and the pressure field. Of course you can make a note of the fact that we have 2 equations here, X momentum equation and Y momentum equation and we have 3 unknowns, the unknowns are U, V and the pressure. So the 3^{rd}

equation of course needs to be enforced is the continuity equation. Although it is not directly in terms of pressure, these 3 equations, the continuity equation or mass conservation equation, X momentum equation and Y momentum equation has us to solve for the 3 unknowns.

These 3 independent equations coming from the mass conservation and X and Y direction momentum conservation can help us to solve the full flow field in terms of the velocity at each point in the flow field and the pressure at each point in the flow field. So, that was our objective for differential analysis and we have got to that particular point. Now if you look at this equation, we can club these 2 equations together and write it in a vector form. It is quite useful to write these equations in a vector form because of several reasons. The first reason is that this equation now is a general, very generally equation applicable to a an incompressible flow with constant viscosity.

It has all the if you consider a 3-D velocity field also, that thing is embedded here because velocity has been entered as the velocity vector in both the expressions here. And acceleration due to gravity has been expressed as a vector quantity here, of course it has only one component which is the magnitude of which is g. So this is how we can write in a general form, this is basically, nabla which is also known as gradient, so this gradient of pressure, of course this is a vector quantity, is a vector operator, so this is in terms of vector calculus we are writing the expression, so this is a vector operator, either this Greek symbol is nabla and in vector calculus it is known as gradient, gradient of the scalar quantity, pressure is a scalar quantity.

So this can written as I hat, that is the unit vector in X direction multiplied by Dell Dell X, so this is an operator, so this gives the vector gradient of any scalar quantity like what is given here. Nabla square of the grad square here is actually Dell multiplied by Dell but not simple multiplication, it is a scalar multiplication, Dell dot Dell. So Dell square is basically Dell dot Dell and is expressed as this. So if you see now, if you plug in these things here, okay if you plug in the operator gradient or grad into this expression and Dell square into this expression, you get the same equation. Now this expression for grad or Dell square is written in terms of Cartesian coordinate system and you can write this equation in different other coordinate system.

For example if you are solving a flow through a channel, it is useful to, use a coordinate system like which is demonstrated here, the Cartesian coordinate system whereas if you are solving a flow through a circular pipe, then it is easier to take a cylindrical coordinate system and by writing equation in this vector form makes it coordinate system independent. That means this equation is actually for any coordinate system if you appropriately replaced the value of grad and Dell square into this equation. The expression for grad and Dell square in a cylindrical other spherical coordinates them are different but the Naviar-Stokes equation is the same. So this is a very compact way of representing the mathematical equations, in this particular case the Naviar-Stokes equation.

So this brings us to the end of the 2^{nd} lecture in the 2^{nd} lecture what we did is basically we have looked at the stress and stress, how the stress relates to the strain rate or the rate of deformation and we have seen separately how the shear stress relate to the rate of deformation and how the normal stress relates to the rate of deformation and we have got an expression of these stresses in terms of velocity field plugged them into the momentum equation which we had derived in the first lecture at the end of the first lecture. After plugging it there, we got a complete form of the momentum equation, the X momentum equation and Y momentum equation. Now we have the complete set of equation which can give us the velocity field in a fluid, in a flow.

By solving this equation means mass conservation and momentum conservation equation we can get the velocities and the pressure at any point in the flow. At the end of the lecture we also demonstrated how to write the equation in a compact and coordinate free manner using the vector notations. Thank you we will continue our discussion on this and we will have some demonstration on the application of the equation which are derived here in the 3^{rd} lecture of this week. Thank you.