Computational Fluid Dynamics

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Lecture – 04
Mass Conservation: Continuity Equation

Welcome back to the next lecture module 2, Mathematical Modeling. Let us first have a recap of what we had planned for this module. We discussed the conservation laws of the physics and we discussed notations and mathematical parameters. We had planned to cover the derivational governing equations of fluid flow and in today's lecture, focus would be on continuity equation that is what we call mass conservation equation.

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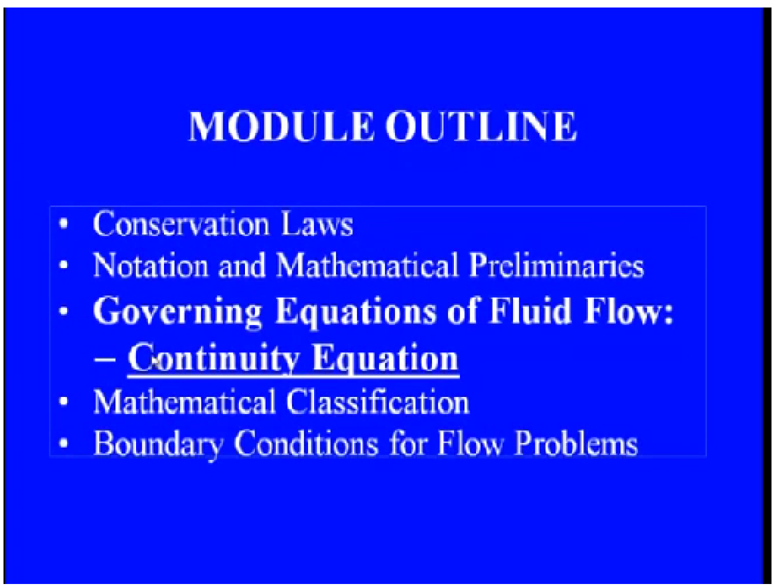

The next 2 topics, broad topics for mathematical classification and boundary conditions for flow problems which we would be taking in subsequent lectures. Now let us have a recap what we discussed in the last lecture.

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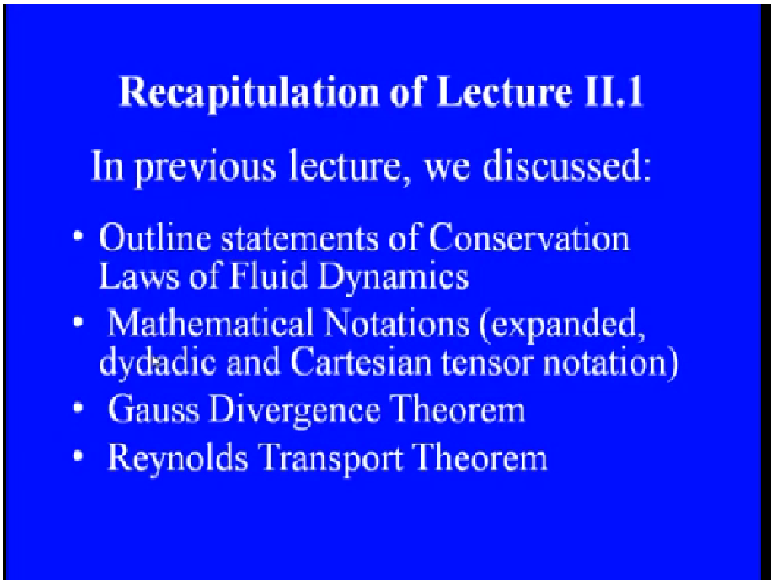

We had discussed the outline statements of conservation laws of fluid dynamics. We briefly outlined what we mean by the conservation of mass for the continuum system. Then we discussed Newton's second Law of motion which gives us what we call momentum conservation. We also talked about energy conservation and the constitutive relations which we require for the modeling of different materials.

Then we discussed different types mathematical notations which are commonly used in CFD. We saw in a given reference frame, how do we write our equations for a given physical law in expanded form which would give us an equation linked to a specific reference frame, Cartesian polar or cylindrical polar. We also looked at what we call coordinate free-form or direct notation and we looked at Cartesian tensor notation and different conventions which we use for use of this very simple and concise Cartesian tensor notation to represent our mathematical equations.

Then we discussed 2 theorems, the first one is Gauss Divergence Theorem which we said we are going to use primarily for changing volume integrals into surface integrals and vice versa, and the last theorem which we discussed what we call Reynolds Transport Theorem which allows us to change the coordinate axes plus what we call a control mass system into a control volume-based representation and vice versa.

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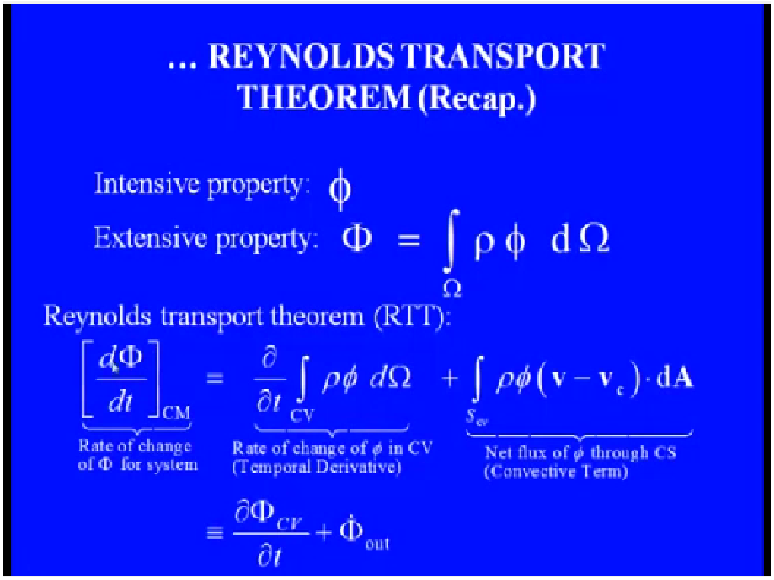

We will have brief recap of our Reynolds Transport Theorem because this is the one which we are going to use very often today and in the next few lectures. Let us say that we have got an extensive property, capital phi. Capital phi could be mass, it could be momentum, or it could be energy and the corresponding intensive property, that is property per unit mass that we denoted by small phi. So capital phi is defined as the volume integral of rho phi d omega.

Now the statement Reynolds Transport Theorem was that rate of change of phi for the system that is d phi/dt for this control mass system that is the system whose mass is fixed, it consists of 2 parts when we represent it in terms of the integrals for the control volume, the first one is the time derivative of the content of capital phi in the control volume that is rate of change of phi in the control volume.

This term you also represent as temporal derivative and we can simply say we can denote it as del capital phi subscript CV/del t+ we have to add what we call the net flux of phi through the control surface which is represented by an integral over the control surface of rho phi v-vc. dA where vc is the velocity of a chosen reference frame. So the second term on the right-hand side essentially represents what we call net flux of phi through cs. We also call it as convective term and we can use it and by notation for it, we can say this is capital phi. Out.

So essentially what Reynolds Transport Theorem say that we got a quantity capital phi, it is time rate of change for a control mass system, is equivalent to temporal derivative of phi for a control volume plus the net efflux rate of phi of the control surface and this is the one which we are going to use in the derivation of continuity equation, derivation of momentum equation and the derivation of energy equation subsequently.

Now in today's lecture, we would focus on our mass conservation equation which gives us what we frequently call continuity equation in fluid mechanics and it is an outline of today.

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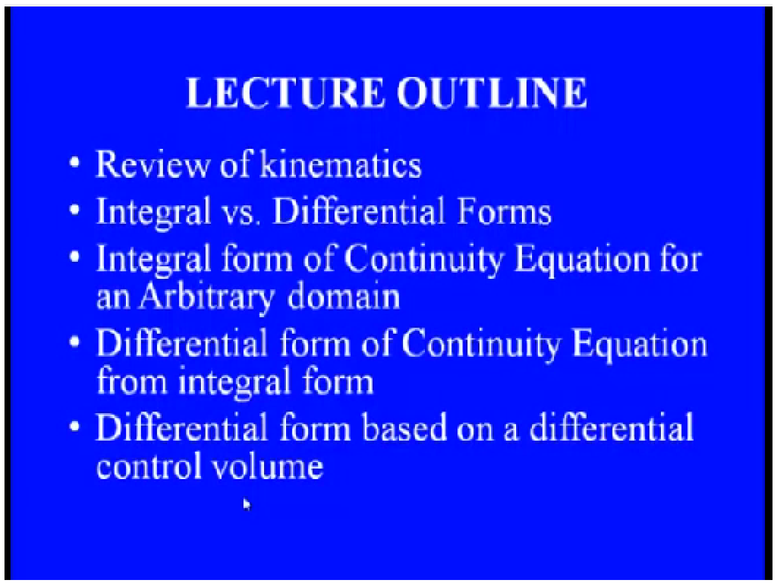

We will first have brief review of kinematics as the different notations of terms which we use in kinematic description of the flow. So various terms would be useful in representation of our conservation laws. Then we will have look at integral versus differential forms. We will derive in today's lecture as well as in subsequent lectures, integral form as well as differential forms, where do we need both of these forms, we will discuss briefly about these 2.

And thereafter we would obtain the integral form of continuity equation for an arbitrary control body or what we call arbitrary domain and next, we would use mathematical jugglery to obtain differential form of continuity equations, starting from the integral form. We will also obtain differential forms based on its simple differential control volume and in this way, we would put a full stop at today's lecture.

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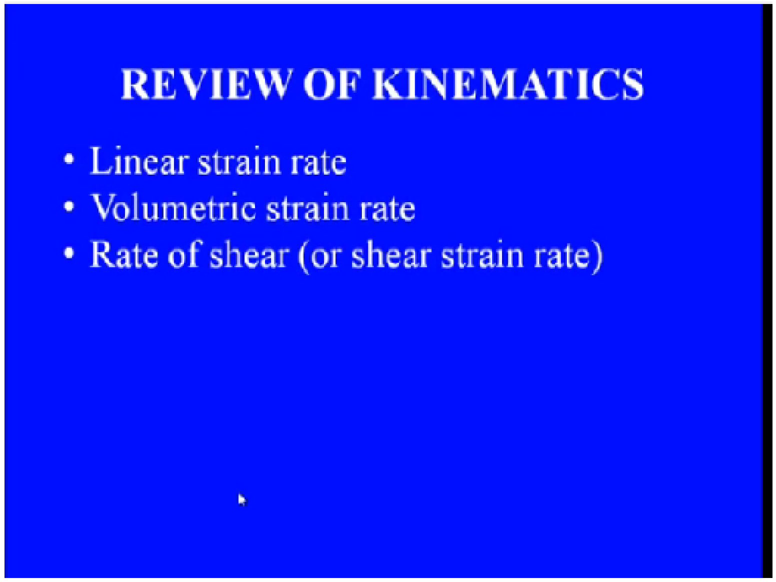

Now let us start with a review of kinematics. We will have a look at how we define linear strain rate, volumetric strain rate and rate of shear. Now before I proceed for definition of these terms, let us have a look at one particular derivative which is referred to as material derivative.

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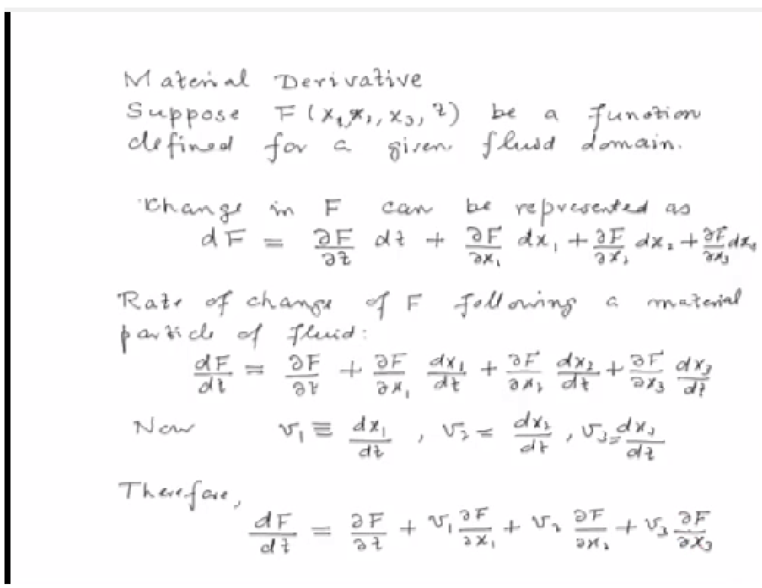

How do we define a material derivative and what we mean by it. Material derivative is useful in the fluid mechanics in the context of what we call Eulerian description of a flow. Suppose we have got any filled quantity which is function of our x1, x2, x3 spatial coordinates and time. A function defined for a given fluid domain. Now how do we find out the rate of change of F with respect to time.

So now we can use the basic law of calculus and change in F can be represented as the because this differential change in small dF, this is del F/del t that is partial derivative of F with respect to time * the small time increment dt+del F/del x1 dx1+del F/del x2 dx2+del F/del x3 dx3. So now what would be the rate of change of F with respect to time. Change of F following a material particle of fluid, this would be given by dF/dt and this would simply become del F/del t, the dt goes off, +del F/del x1 dx1/dt+del F/del x2 dx2/dt+del F/del x3 dx3/dt.

But now let us recall, the way we have defined our velocities. So the velocity component v1 is actually dx1/dt. Similarly, v2 is dx2/dt and v3=dx3/dt. Therefore, our df/dt, this becomes del F/del t+v1 del F/del x1+v2 del F/del X2+v3 del F/del X3. Let us write it a bit more concisely using our tensor notation.

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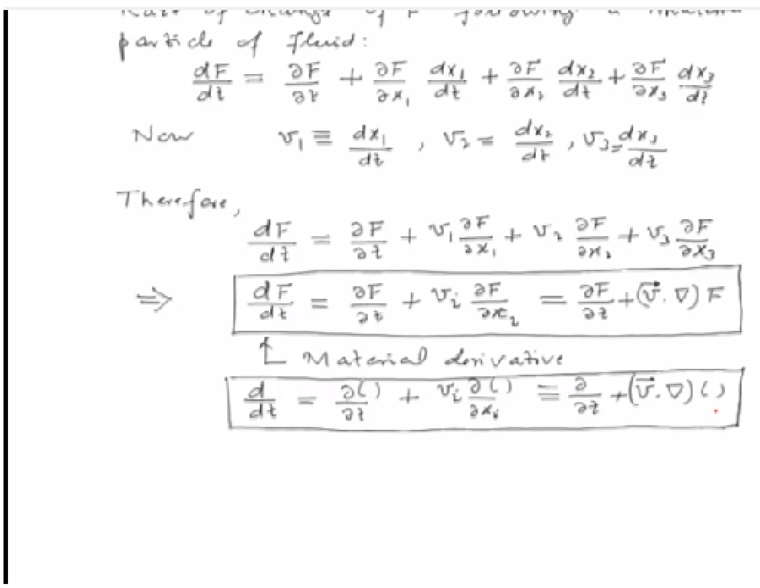

So dF/dt= del F/del t+vi del F/del xi or in vector form we can express it as del F/del T+v. del F/ So that is the expression for the rate of change of F following a material particle and that is why we call this as material derivative. We can extract this derivative operator, d/ dt and this could be written as del/del t+vi del/del xi or equivalently as del/del t+v. del. So this is material derivative operator which essentially tells us the rate of change with respect to time for a material system.

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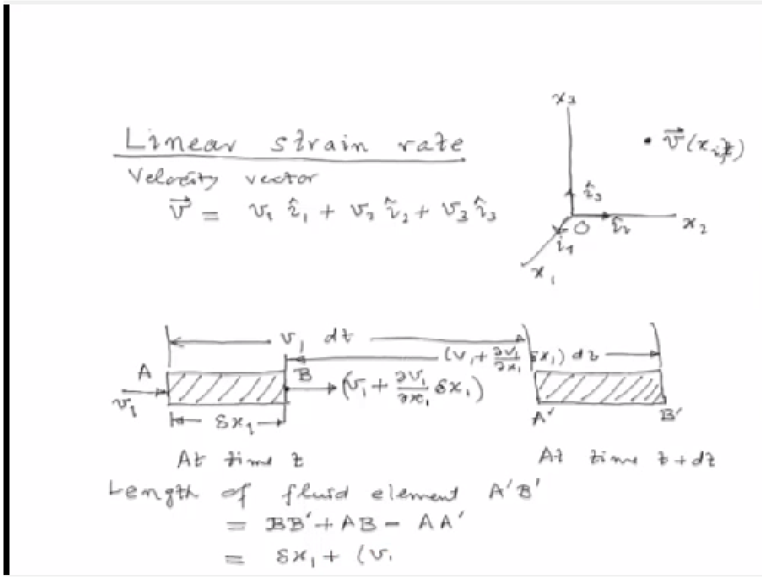

Next, let us have a look at the strain rates which we would frequently use in the description of the fluid motion and first let us concentrate on what we call, how would be define linear strain rate. It would be worthwhile to note that in fluid mechanics, we do not talk about a strain per se because that is meaningless for a fluid medium. So we always talk about in terms of the rate of linear strain or rate of serious strain.

Now how would we identify or how would we define this linear strain rate. Suppose we have chosen our Cartesian reference frame, x1, x2 and x3. Our velocity filled is given by v as a function of x and t. Now the velocity vector v can be expressed in component form, v1i1+v2i2+v3i3, where this i1, i2 and i3, these are unit vectors in x1, x2 and x3 directions respectively.

Let us just define the rate of linear strain in 1 direction, let us chose x1 direction and suppose we take a very simple linear element. So let us chose a small linear element of it in infinitesimal length delta x1, let it is ends be denoted as A and B. So x velocity of end A is called as v1, then we can use Taylor series expansion and obtain the velocity at point B which is situated at a distance delta x1 to the right as v1+delta v1/del x1 delta x1.

Now this is the position of this small linear element at time T. After small time increment, t+delta t, this element would have moved to the right under the influence of x velocity component. Now let us say the change position is A prime B prime. Now this A prime B prime that represents the changed element AB at time t+dt. How would we obtain the position of A prime, that is very simple. Difference between A and A prime would be given by v1dt. Similarly this difference between B and B prime, that can be easily obtained as v1+delta v1/delta x1 del x1*dt.

Okay now can be obtain what is the length of segment A prime and B prime. So length of fluid element A prime and B prime, this is simply equal to be B B prime+AB-A A prime, that is this is = delta x1+v1+del v1/del x1 delta x1*dt-v1dt.

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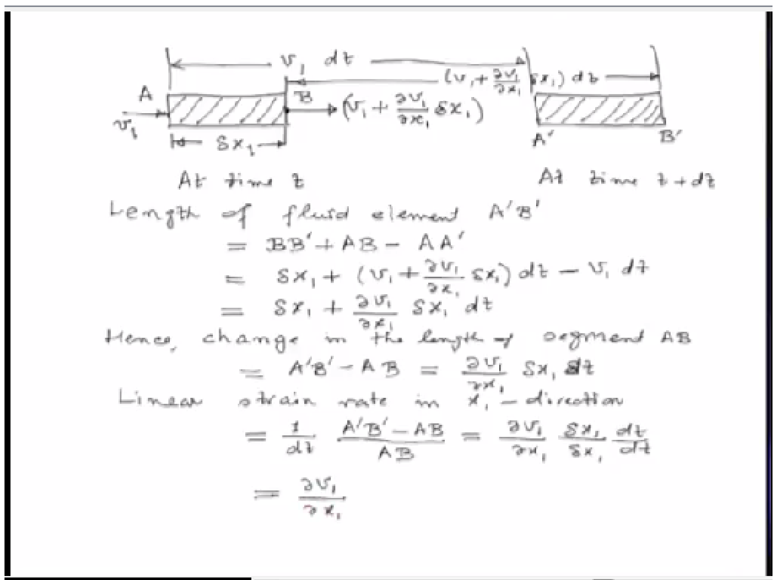

So if you simplify it, we get this is = delta x1+delta v1/delta x1 delta x1dt. So hence what is the change in the linear scale or length of the segment AB, change in the length of segment AB, this is = A prime B prime-AB=delta v1/delta x1 delta x1 delta t. So now we can define our linear strain rate. So linear strain rate in x1 direction which is given by the time rate of change that is 1/dt of change in length that is A prime B prime-AB/the original length AB.

So if you do that, you would essentially get delta v1/delta x1 delta x1/delta x1 dt/dt, it is nothing but delta v1/delta x1.

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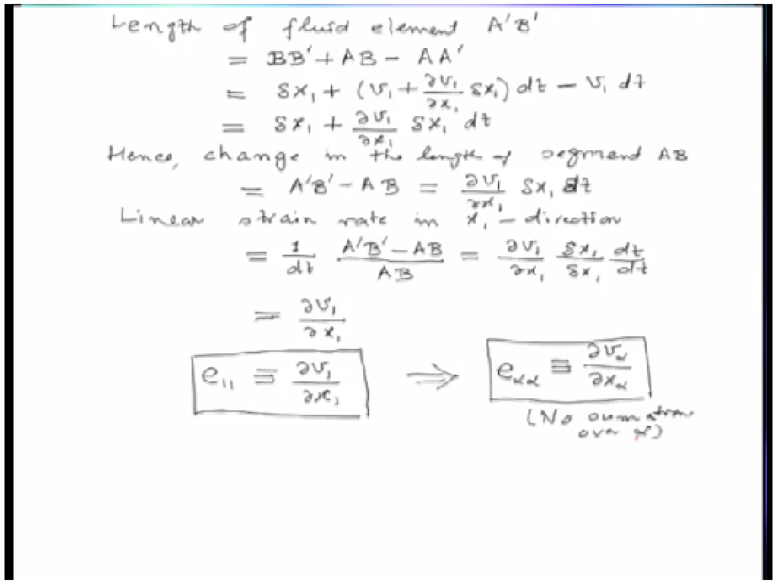

We can use a symbol small e to denote what we call strain rate tensor. So e11 can be expressed as delta v1/delta x1. Now this can be generalised and we can say that linear strain rate in any direction, coordinate direction xi that can be represented as in direction x alpha delta v alpha/over delta x alpha and here you would not use no summation over alpha.

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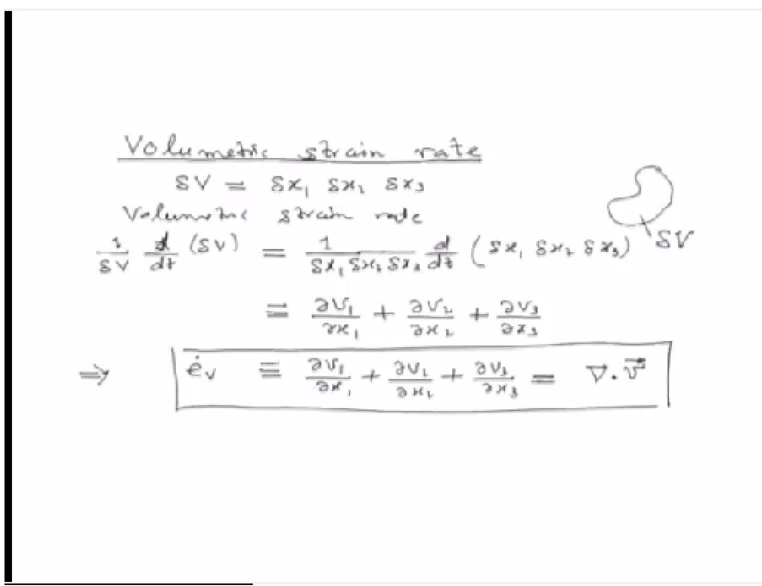

If you go back to your solved mechanics class, what we learnt that volumetric strain rate is nothing but the summation of the rest of normal strains in 3 mutually perpendicular directions. Suppose we take a small volume, let us call it as delta V, now this delta V can be given by delta x1 delta x2 delta x3. So how do you define volumetric strain rate. Volumetric strain rate that would be given by d/dt of delta v 1/delta V. So this is 1/delta x1 delta x2 delta x3 d/dt of delta x1 delta x2 delta x3.

Now we can expand the terms and we can easily see that this is equivalent to delta v1/ delta x1+delta v2/delta x2+deltav3/delta x3 or in other words if use a symbol e subscript v. to denote a volumetric strain rate, this is given by del v1/delx1+del v2/del x2+del v3/del x3 which we can easily identify as divergences of velocity field. Okay, so volumetric strain rate for any fluid element that is simply obtained by finding out what is divergences of V at a given position.

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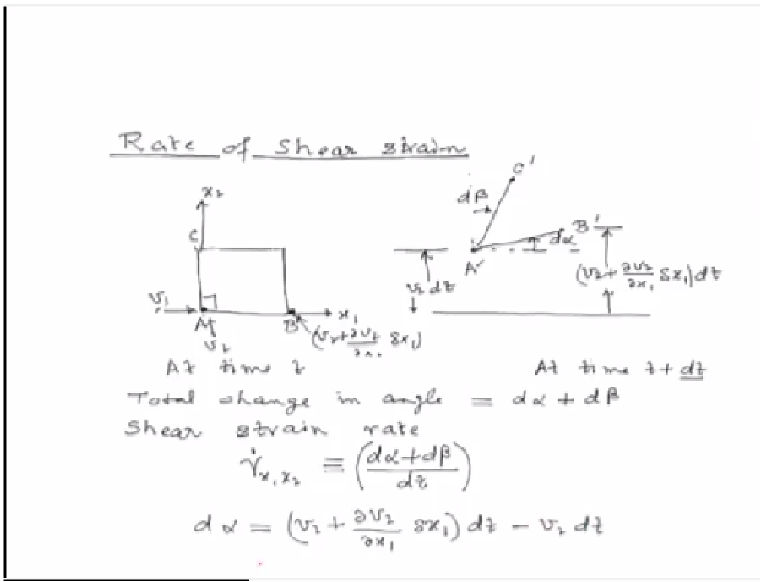

Next important quantity is what we call rate of shear or shear strain rate. For this, we have to consider small rectangular fluid element and find out what is the rate of change of the angle between 2 line segments. Suppose we call these segments as AB and AC which were initially perpendicular. What happens to these segments, what happens to the change in the angle as this fluid element deforms under the effect of velocity field. Suppose new position of these elements is given by A prime B prime and C prime.

So on the left-hand side, this is the position at time t. Now here we have got the position at time t+dt where dt is infinitesimally small time increment. Now we can identify 2 angles, the change with respect to the horizontal of the line segment A prime and B prime, let us call this small angle as D alpha and the second change from vertical AC which was initially vertical, now it has become inclined, let us call this angle as D beta.

So what is total change in the angle from 90 degree. So total change in angle, this is given by d alpha+d beta. So now we can define the rate of shear or shear strain rate, we normally use symbol gamma for shear strain and for rate, let us put a small dot over it, so gamma. x1x2 and this would be it initial set AB, it is along x1 direction and AC is along x2 direction. This would be defined as d alpha+d beta/dt. D alpha+d beta that represents the net change in the angle and it is time rate we can obtain by dividing it with respect to dt.

Now let us see how we can obtain d alpha and d beta separately and for that we need to have a look at the change in the positions, A shifts from A to A prime and this shift would be given by the v2 velocity component at point A, so it is v2*dt. How much would be the change in B, this change can be obtained if you look at the v2 velocity at B. So this will be given by v2+del v2/delx1 delta x1, this would be the velocity at point B.

So v1 and v2, at point B, we have got the velocities, v2 velocity would be given by Taylor series expansion as v2+del v2/del x2 delta x1. So we see that velocity, this multiplied by dt. So now can we find out what is d alpha. D alpha would be v2+del v2/del x1 delta x1*dt-v2*dt.

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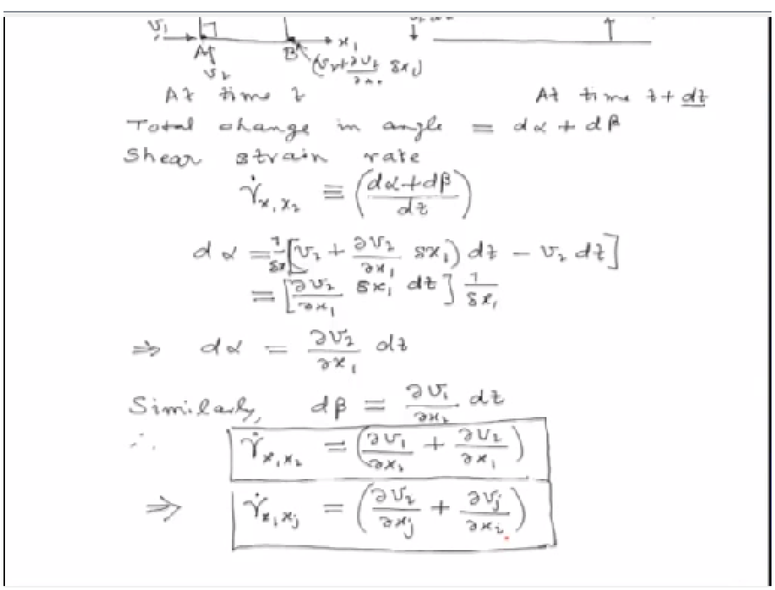

Or in other words, this is = del v2/del x1 delta x1*dt and factor d alpha would be, this is an angle, so we have to divide it by the arc length which is delta x1, this whole thing multiplied by

1/delta x1, so hereby we obtain d alpha= delta v2/delta x1 dt. Similarly, we can obtain the expression for d beta. So d beta would be given as delta v1/delta x2*dt. So therefore, gamma.x1x2, this becomes del v1/del x2+del v2/del x1.

In fact, we can now generalise it for any pair of the directions, that is we can write, gamma dot xixj as del vi/del xj+del vj/del xi. So this is expression for the rate of shear along 2 mutually perpendicular directions.

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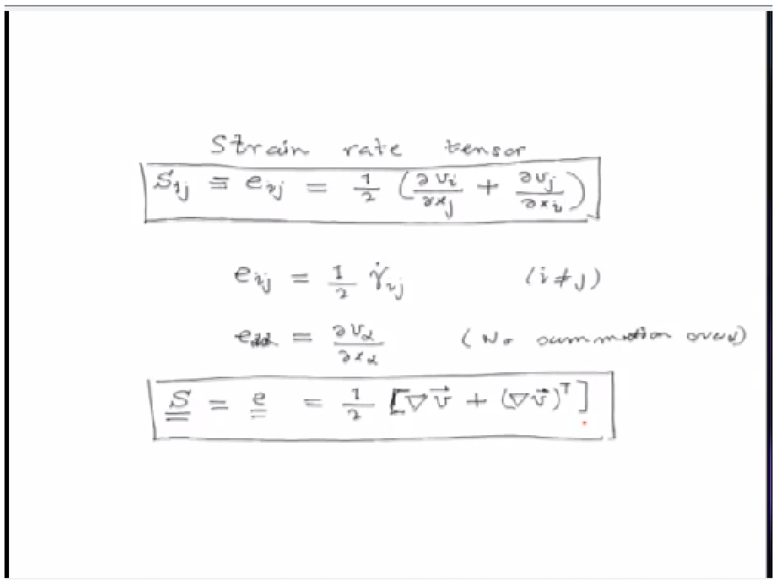

Next can we define what we call strain rate tensor. Looking at the 2 previous definitions which we had, we can define it as strain rate tensor. Now there are different symbols used, very often people use symbol capital S or some people prefer to use small e. So this can be represented as 1/2 del vi/del xj+del vj/del xi. So we can easily verify that expression for linear strain rate which we have defined earlier or the shear strain rate, these 2 are identical.

That is we can easily establish that eij=1/2 gamma ij., i is not equal to j and similarly with eii or e alpha alpha, this becomes del v alpha/del x alpha, no summation over alpha. So in the tensor form or direct notation, we can write S or e as 1/2 of gradient of velocity vector + a transpose of the gradient of the velocity. So this is what our strain rate tensor looks like in terms of the gradient of velocity vector.

Okay, next let us come back to a mass conservation equation or continuity equation. We know from the basic physics that mass of a system is conserved that is to say if capital M represents the mass of a system, it is time derivative has to be 0. Now how do you define mass for an arbitrary volume omega. So M is defined as volume integral of rho*1 d omega.

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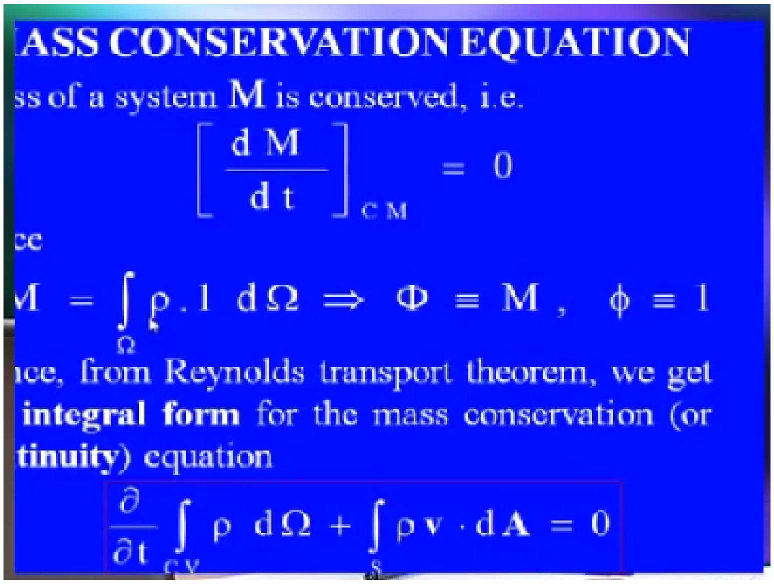

Now we can identify if we compare this definition of mass within intrinsic and extrinsic properties which we had introduced earlier in definition of Reynolds Transport Theorem. We can clearly see that phi, capital phi is same as capital M and small phi =1.

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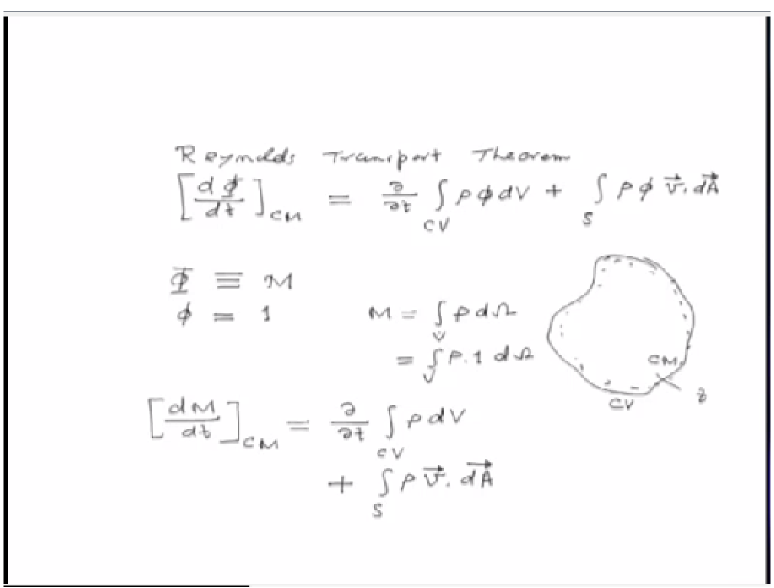

So if we go back to our Reynolds Transport Theorem, see Reynolds Transport Theorem for a quantity capital phi was d capital phi/dt for the control mass, this was given as del/del t of cv rho small phi dV+the surface integral of rho phi v.dA, this holds true for an arbitrary control volume and cm can be identified as the system which occupy this controlled volume at any time instant capital T.

Now we have already identified our capital phi is identical to M and a small phi that is our intrinsic property in this case becomes 1 because M is defined as rho d omega or we can say that this is rho*1d omega. So now let us put small phi =1 on the right-hand side and what we get a dM/dt for a control mass system, this = del/delt t of rho dV+ the surface integral rho v. dA. So now we have obtained the rate of change of mass for a control volume.

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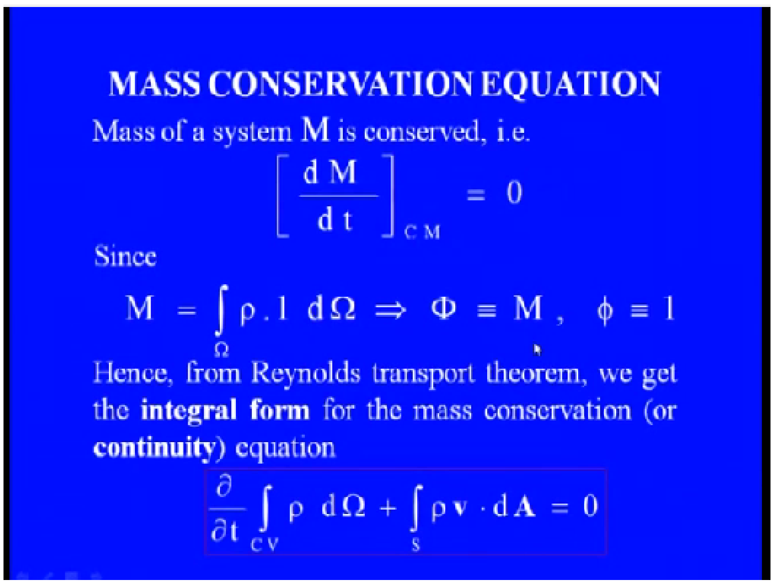

and we can now put it in the expression for this mass conservation equation dM/dt CM=0 and we get this particular form for mass conservation equation and this is what we call the integral form of mass conservation or continuity equation. It is given as del/del t or rho d omega/ CV+the surface integral rho v.dA=0.

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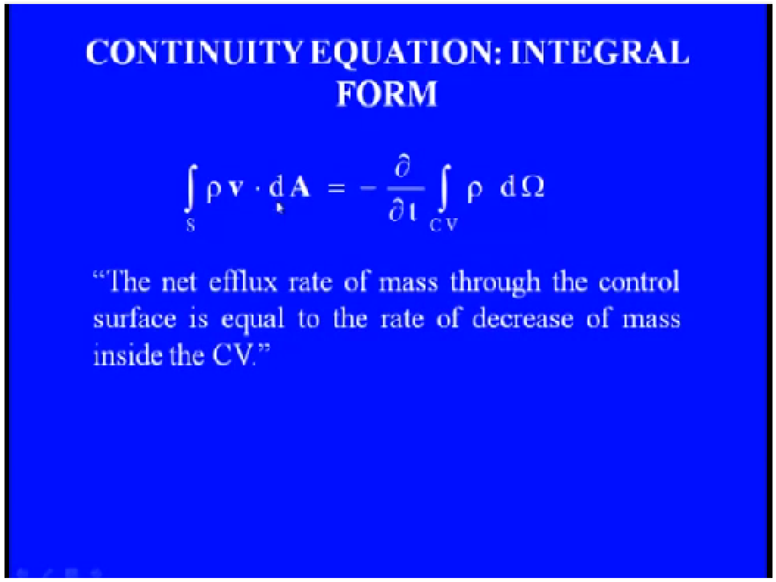

Now we can also just change it slightly and we can write it as the surface integral of rho v.dA=-del/del t the volume integral of rho d omega. So this form of continuity equation tells us that the net efflux rate of mass through the control surface is equal to the rate of decrease of mass inside the control volume. Now let us have a look at simplified forms for steady flow and for incompressible flows. What will happen to this mass conservation equation.

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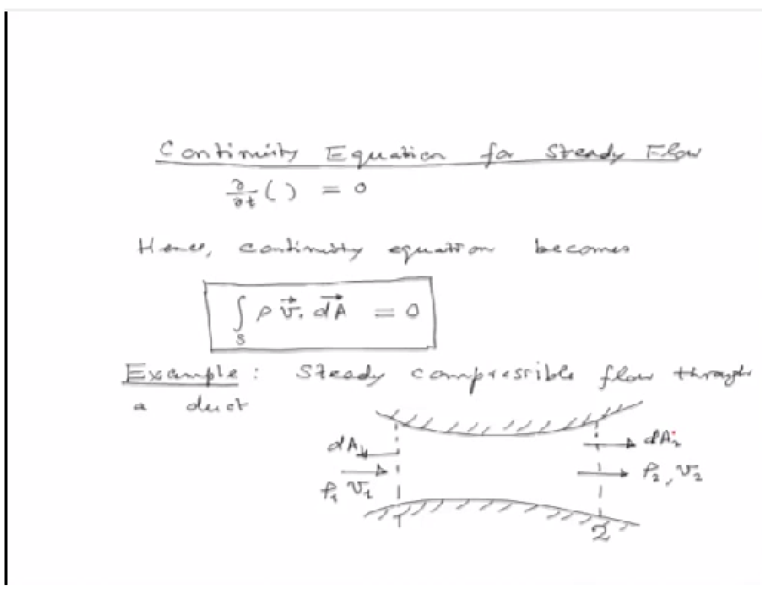

So continuity equation for a steady flows, so in this case del/del t of anything that is = 0. Hence our continuity equation becomes surface integral of rho v.dA=0. Now let us have look at a simple one-dimensional example of a steady-state compressible flow through a duct of variable cross-section. So a steady compressible flow through a duct. Let us identify station 1 as the inlet, velocity at station 1 is v1, density is rho 1 and at outlet which we can indentify at station 2, we have got density as rho 2 and velocity as v2 in x direction.

The hatched part that denotes the solid boundaries of the duct. Now let us apply continuity equation to this, what do we get. At the inlet, the area vector points in the direction opposite to that of the velocity and at the outlet, the velocity vector and the area vector, they are in parallel direction.

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So surface integral s rho v.dA, this can be expressed as integral over area A1 rho 1 v.dA+A2 rho v.dA. There will not be any contribution from the side portions because that is a solid boundary. Now this rho v.dA at station 1 that will be given as -rho 1v1 integral of dA/A1+rho 2V2dA2/A2. Please note this change negative sign comes because v1 and dA, these 2 v and dA vectors, they are antiparallel at station 1.

So we simply get -rho 1v1A1+rho 2v2A2 and from continuity this equation, this would be equal to 0. So this gives a simplified form for one-dimensional flows rho 1v1A1-rho 2v2A2. The next simpler case would be that of incompressible flows. So what we mean by incompressible flow. In the case of incompressible flow, density is constant that is to say it does not vary with time and hence our del/del t integral rho dV, this becomes all integrally 0.

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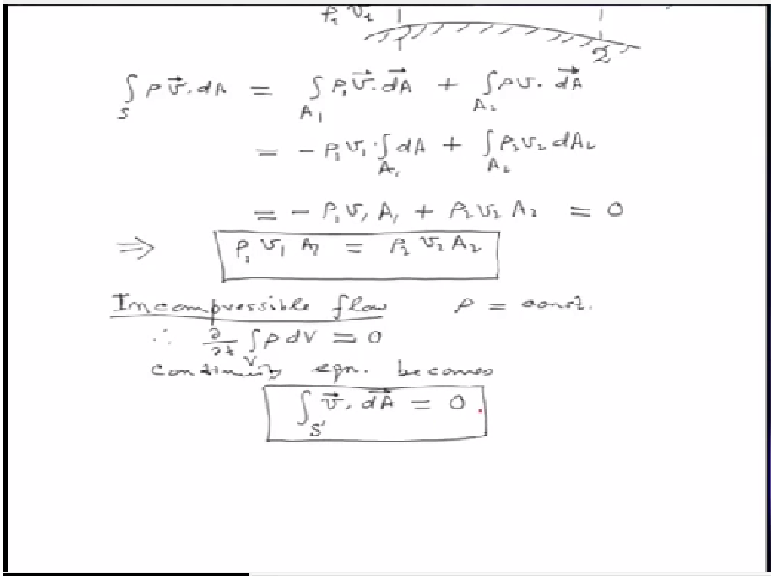

And the simplified form continuity equation becomes integral v.dA over the surface of the control volume=0. So now let us stop here as far as the integral forms are concerned. Now these integral forms are important because these we require in CFD if you are using finite volume method.

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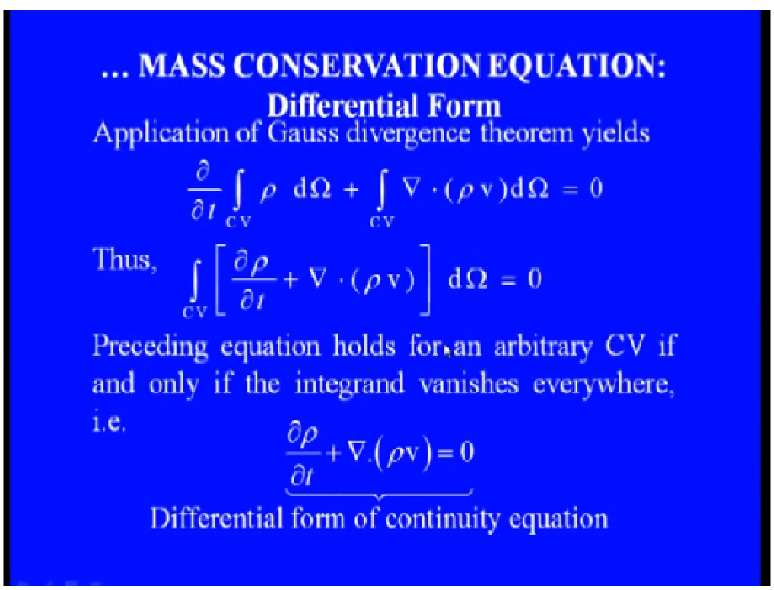

Next let us proceeds and derive the differential form because that is what we would need if you want to make use of finite element or finite difference methods. Now before we can proceed to differential form, let see how we can obtain the differential form starting from our integral form of continuity equation.

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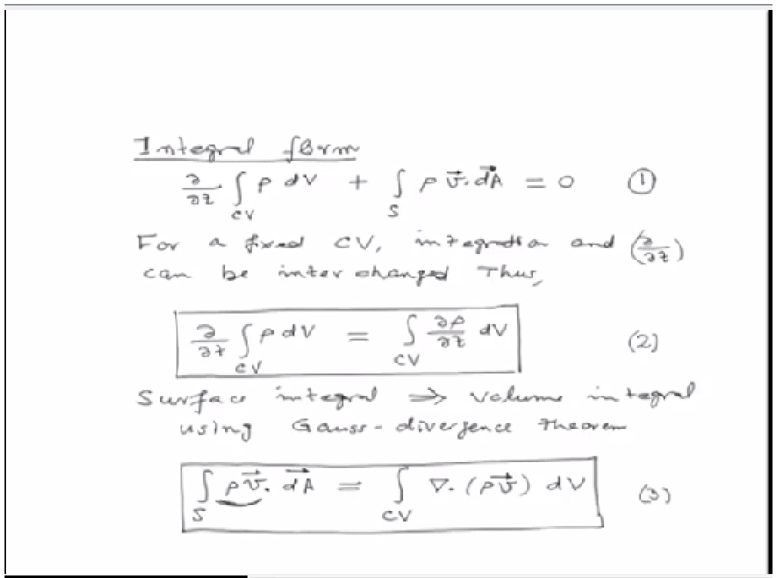

So what was your integral form. In integral form, we had 2 terms. First was a temporal derivative del/del t of rho dv+a surface integral. Now let us see what happens if we had a fixed control volume, that is a control volume which did not change in time. For a fixed control volume, integration and temporal derivative operator can be interchanged. So thus what we can write is del/del t of integral rho dv over the control volume, this is same as the volume integral del rho/del t/dV. This is one part.

Next one is surface integral. Now let us transform this surface integral into a volume integral using Gauss Divergence Theorem which we had looked at in the previous lecture. So the surface integral is rho v.dA. Now what is equivalent volume integral, this remember that whatever we have to the left of our dot operator, okay, we had to take divergence of that quantity. So it is simply becomes divergence of rho vdV. So that is it, this is all. Now we have transformed our surface integral into corresponding volume integral.

Next we are going to put these 2 expressions we have just derived into our integral form of continuity equation if you call it as 1, so let us put this expressions 2 and 3 in our integral form continuity equation 1.

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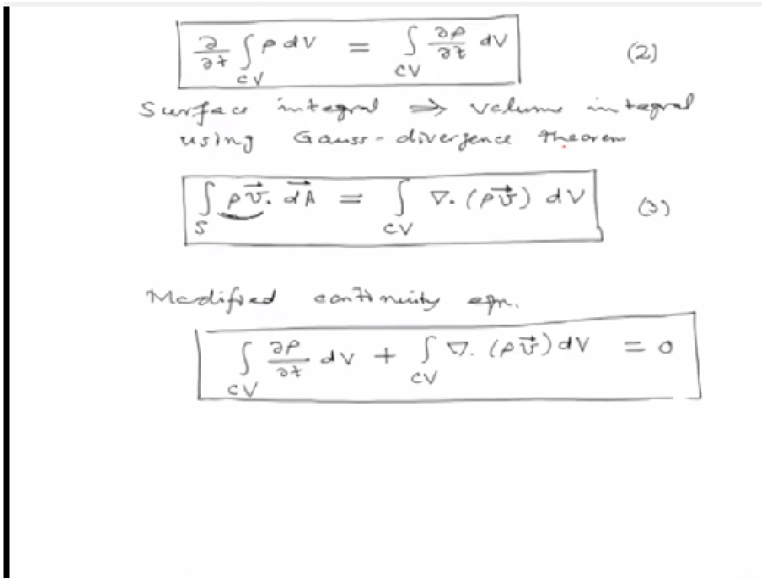

And we get modified continuity equation as, it becomes integral over CV del rho/del t dV+integral over CV divergence of rho vdV=0. Now what we have got on left-hand side are 2 integrals with the same control volume and hence we can easily couple these 2 together and we can write this as del rho/del t+divergence of rho V. This is what becomes our integral, integrate this function over the control volume CV and this is = 0.

Now please remember our choice of the control volume was arbitrary and this integral equation will hold good for an arbitrary control volume if and only if that integrand function that is the term in this square bracket is identically 0. If that were not the case, what a situation in which it can vanish. Suppose over half of the domain, we had del rho/del t+divergence of rho V that is equal to, this particular function is positive.

In other half, it is negative, equal and opposite in magnitude, so that is 1 possibility in which case this integral can vanish. Now in this case, what we can do, we can choose it another control volume which is now entirely in positive half; even then, this equation must hold good but we had assumed that del rho/del t+divergence of rho v were positive in that half.

So our assumption which we made earlier that is wrong, that this equation can hold good only if this term del rho/del t+divergence of rho v, this particular function were 0 everywhere in a control volume, okay. So that is what we say in mathematical language that the preceding equation will hold for an arbitrary control volume if and only if the integrand vanishes everywhere, that is del rho/del t+divergences of rho v=0 and this leads us to what we call differential form of continuity equation.

The first term is the time derivative of density and the second term is divergence of rho*v. So for compressible flows which particular equation represents what we call transport equation for density.

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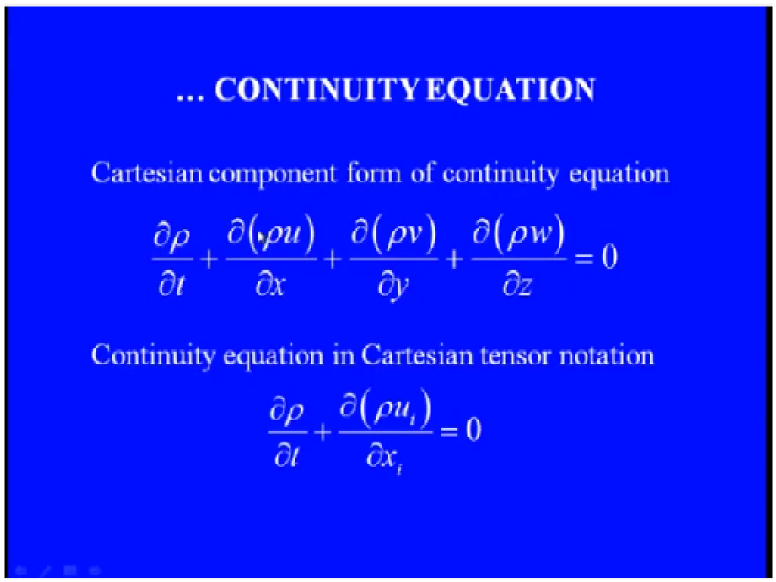

Now the previous vector equation can be written Cartesian component form as del rho/del t+del/del x of rho u+del of rho v/del y+del of rho w/del z=0 where u, v and w, they represent the velocity components in x, y and z directions. Now this particular expanded equation can also be written in concise Cartesian tensor notation as del rho/del t+del/del xi of rho ui=0, where ui represents the velocity vector in ith coordinate direction.

Now what are simplified forms which we can have of this differential continuity equation, similar to what we had had earlier.

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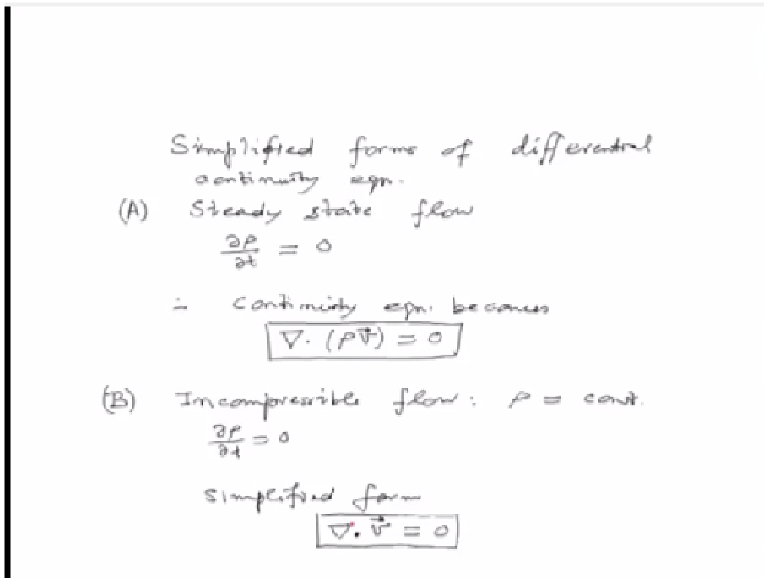

Okay, now let us have look at simplified forms of continuity equation, differential continuity equation. So let us have a look at the steady state flow. Now in this case, del rho/del t, this would be equal to 0, hence our continuity equation becomes divergence of rho v=0. The second simplification could occur if we assume density to be constant, that is the case of incompressible flow.

Incompressible flow, rho is constant. Now in this case, del rho/del t would again be 0 and rho can be taken out and we can divide by rho and we get the simplified form, divergence of v=0. Now in this case, the important point to note is the divergence of v, what does it represent. This essentially represents the volumetric strain rate and it is consistent with our assumption of incompressible flow.

Incompressible fluid cannot be compressed. So volumetric strain rate has to be 0 in that case. So that is what this continuity equation also represents but it also tells us some more things.

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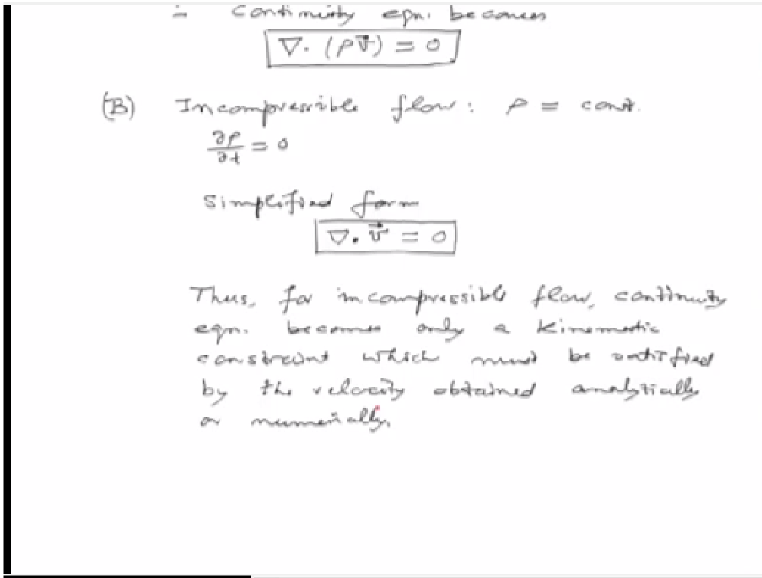

That in the case of incompressible flow, our continuity equation is merely a kinematic constant, that is for incompressible flow, the continuity equation becomes only a kinematic constant which must be satisfied by the velocity field obtained analytically or numerically. This will have important ramifications when we come to the numerical solution of Navier-Stokes equation for incompressible flow and that is something which we will have a detailed look at when we come to that topic.

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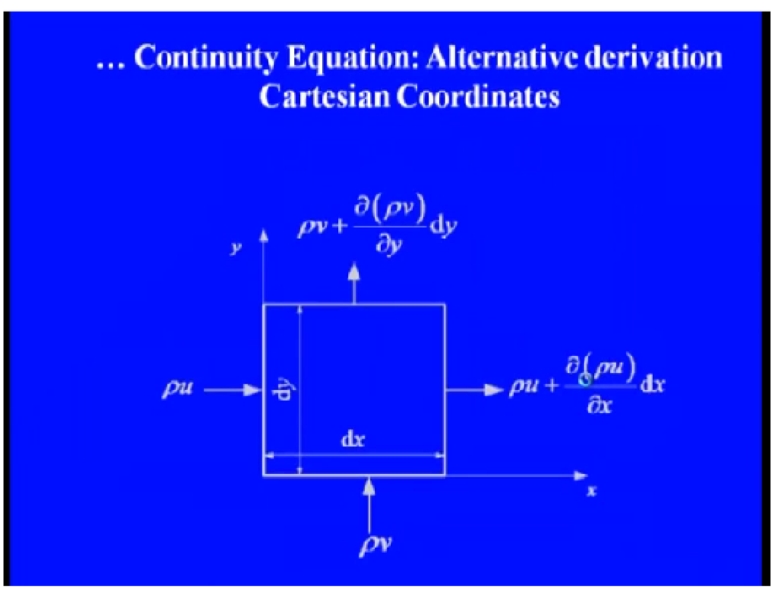

Now this differential form which we had obtained that was based on the integral form that we took a generalised integral form and used Gauss Divergence Theorem and we obtained our differential form. We can also obtain it started from first principle, so differential continuity equation (()) (53:31).

Okay, so now let us have a look at the Cartesian coordinate system, this is a small simple differential element of length dx and dy and let us see, let us have a look at clearly. What is the mass flow which is coming in at the left plane given by rho*u. Similarly the velocity at the left plane is rho u. This particular function rho u at the right plane can be given as rho u+del rho u/del x dx using Taylor series expansion. Same way, we can interpret this function rho v at the bottom plane and the top plane.

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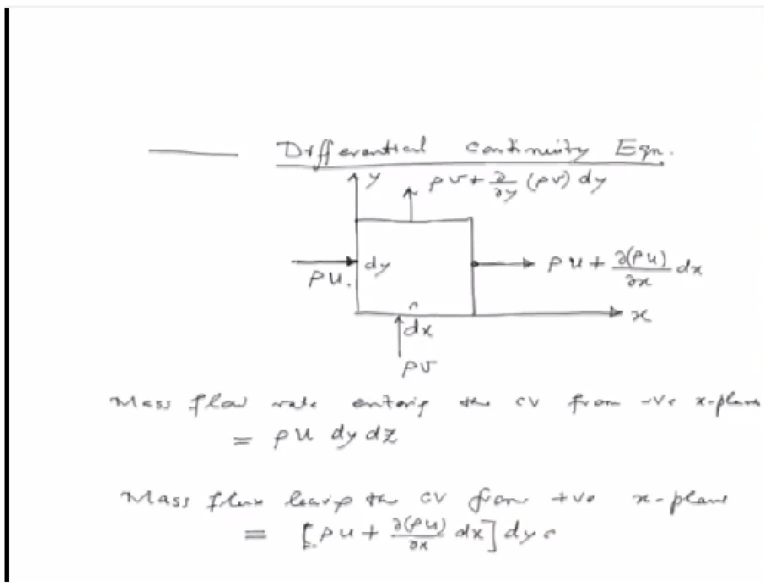

Now let us see how we can derive our differential continuity equation using the simple Cartesian element. So let us redraw the figure again. Rho u rho v give functions at negative x plane and negative y planes respectively. Their values at the centre of the positive x plane will be given by Taylor series expansion rho u+del of rho u/del x dx. Similarly, in the positive y plane, rho v+del/del y of rho v dy.

Now let us take one direction at a time. So what is the mass flow which enters from the let plane. So mass flow rate entering this control volume from negative x plane. This is nothing but rho*u and the area of this plane is dydz. Now mass flux leaving the control volume from positive x plane, that would be who u+del of rho u/del x dx*area dydz.

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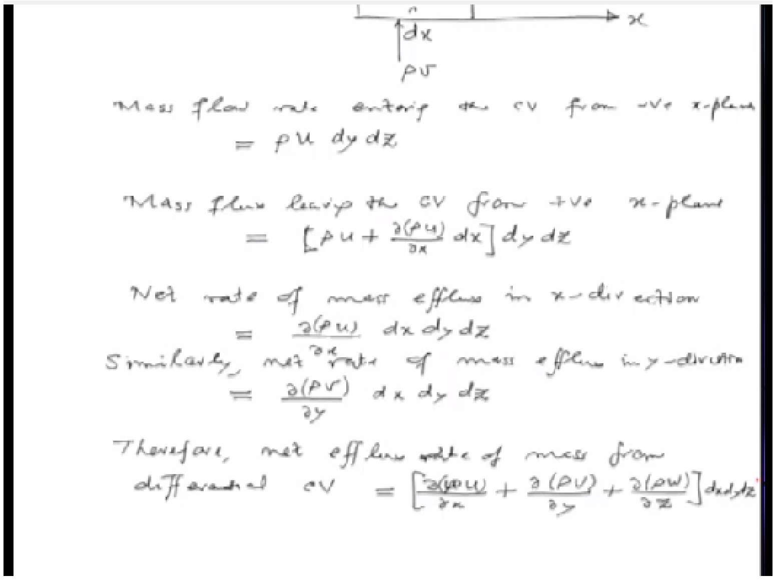

So can we obtain the net rate of efflux, that is net outflow, net rate of mass efflux in x direction, we will just simply subtract the inflow from outflow and if you do that, we get a very simple expression, del of rho u/del x dxdydz. The same exercise we can repeat with respect to y planes. So similarly net rate of mass efflux in y direction, this would be given by del of row v/del y dxdydz and similarly in z direction.

So now what is the net efflux from the control volume. So therefore net efflux rate of mass from differential control volume, this is = del u rho u/del x+del of rho v/del y+del of rho w/del z dxdydz.

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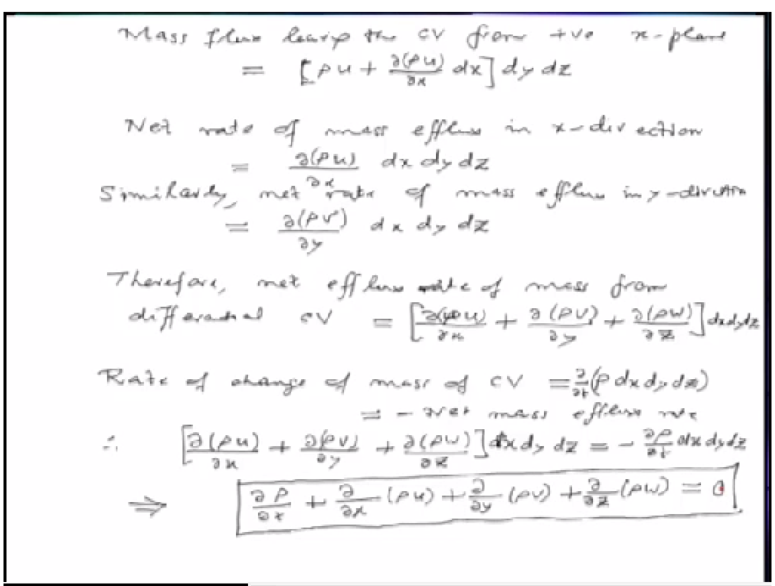

If you have to satisfy the continuity, this must be balanced by the rate of increase or rather rate of change in mass in the control volume. So a rate of change of mass of CV, this would be given by del/del t of rho dxdydz and this must be negative of the efflux rate. So if you combine these 2 expressions, we simply get del of rho u/del x+del of rho v/del y+del of rho w/del z*dxdydz=-del rho/del t dxdydz and thereby by rearrangement of the terms we get del rho/del t+del/del x of rho u+del/del y of rho v+del/del z of rho w=0.

And this precisely the differential form of continuity equation which we had obtained earlier. So this is another way in which we can obtain the differential form of continuity equation.

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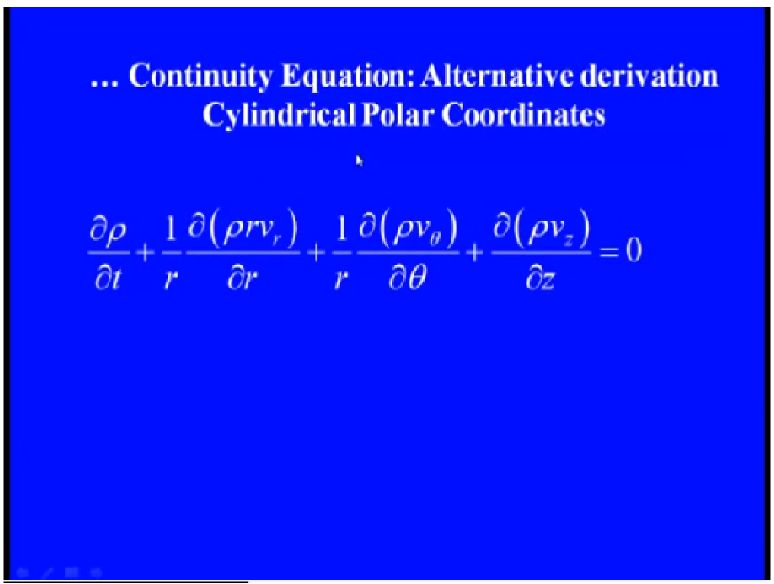

The same exercise can be repeated for cylindrical polar coordinate system and this I would leave as an exercise, the final equation as del rho/del t+1/rho del of rho r vr/del r+1/r del of rho v theta/del theta+del of rho vz/del z=0 where vr, v theta and vz, these are the velocity components in r theta and z directions respectively. So please take a small differential cylindrical polar control volume and complete this derivation following the steps which we used earlier with rectangular coordinate system.

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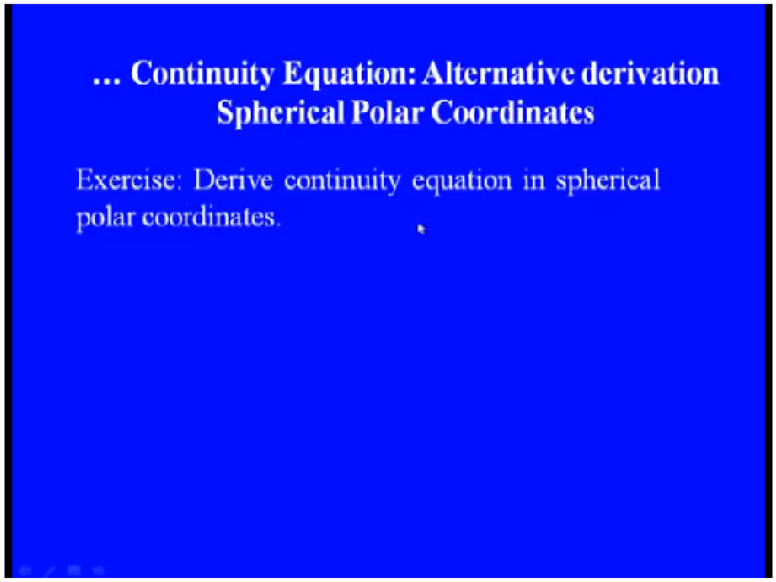

The next exercise would be derived continuity equation in spherical polar coordinate system. Once again chose a small differential element in spherical polar coordinates. In both of these polar coordinates, please be aware of the changes in area across different r positions in new derivations. That is where we are going to now put full stop to this lecture on continuity equation and in the next lecture, we will take up the derivation of momentum equation.