Computational Fluid Dynamics
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Lecture – 07
Navier-Stokes Equation and its Simplified Forms

We will continue where we left in the previous lecture. We were discussing Navier-Stokes equations that is we discussed part of constitutive relations for a Newtonian fluid, it is where we left in the previous lectures. We are going to continue from there and complete Navier-Stokes equations and energy equation in this lecture.

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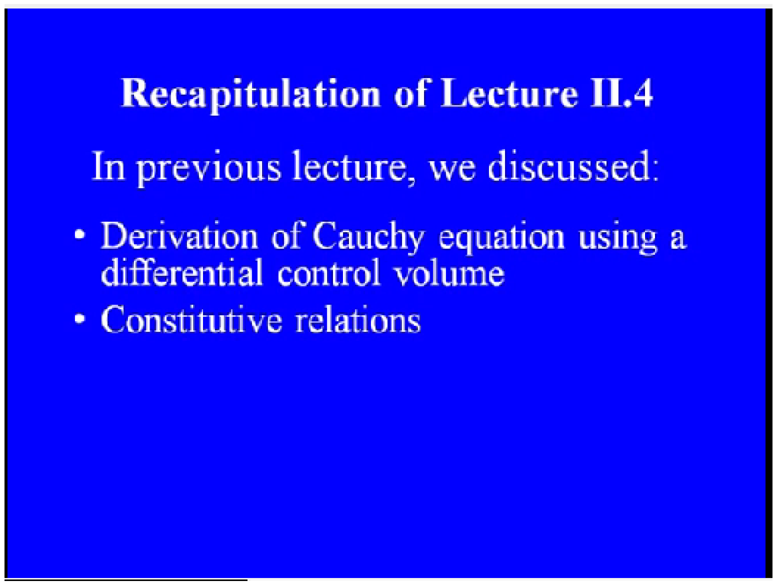

So summary of what we discussed in the previous lecture, we derived the Cauchy equation using a differential control volume and then we started off with constitutive relations for different types of fluids and in this lecture, we would focus primarily on Navier-Stokes equations for Newtonian fluids and energy equations.

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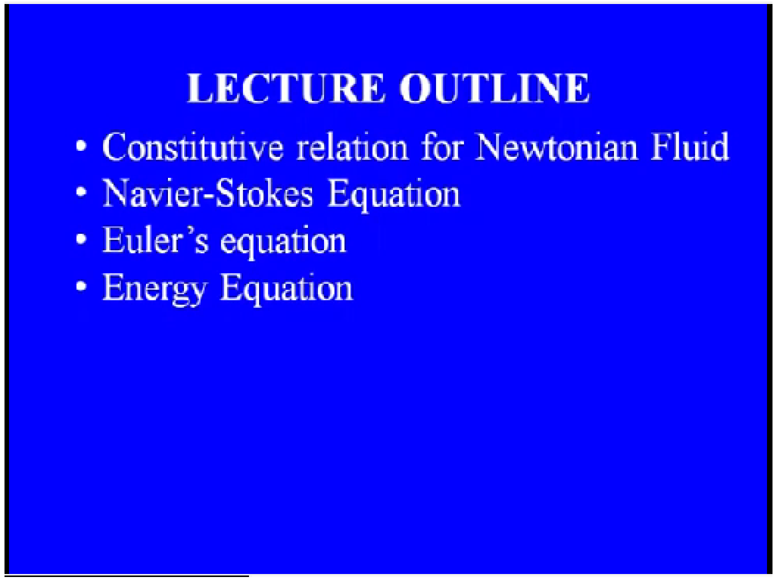

Outline of this lecture. We will complete the constitutive relation for a Newtonian fluid and then would substitute this in Cauchy equation to obtain the celebrated Navier-Stokes equations. We will have look at few simplified forms, all other equations one of the most prominent amongst them and then we would move on to the derivation of energy equation for a fluid body. So this is where we were that we wanted to find out a constitutive relation for a Newtonian fluid for which stress and strain rate relationship is linear.

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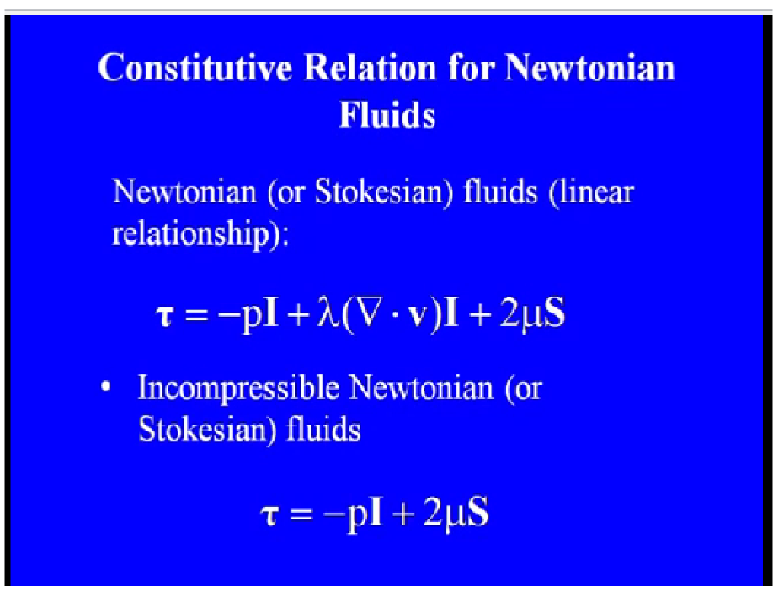

Which is given by tau=-pI lambda + divergence of vI+2 mu S, where S is our strain rate tensor. How do we obtain this equation, this would be discussed in bit more detail.

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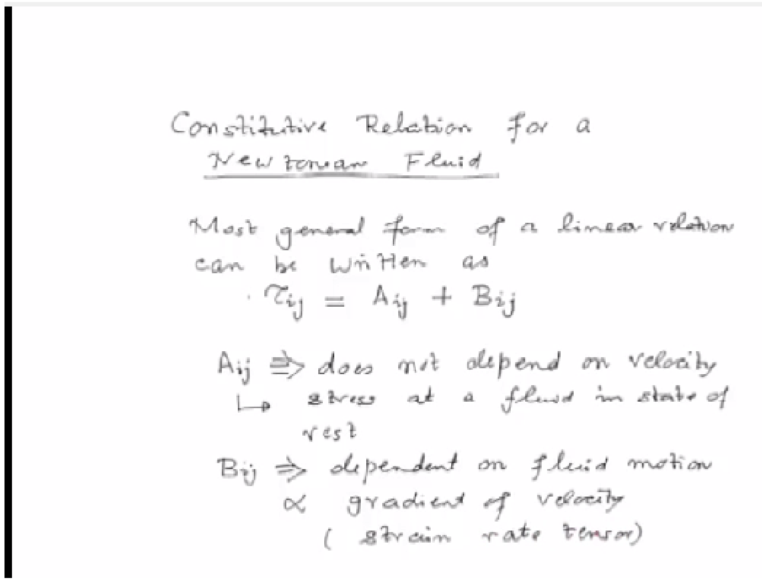

Now let us have a look at the constitutive relation for a Newtonian fluid. Now in the case of Newtonian fluid, we say that the stress tensor must be linearly related to strain rate tensor. So the most general form of this relationship of a linear relationship can be written as tau ij=Aij+Bij. Now we have broken it in 2 parts. So here Aij, this part does not depend on velocity. So it basically corresponds to the stresses at rest, that is the stress at a fluid in a state of rest.

And Bij, this is dependent on fluid motion. Basically, it would be proportional to the gradient of velocity or more particularly, it is proportional to strain rate tensor. So if you recall at the end of the previous lecture, we talked about the stresses in a fluid at rest.

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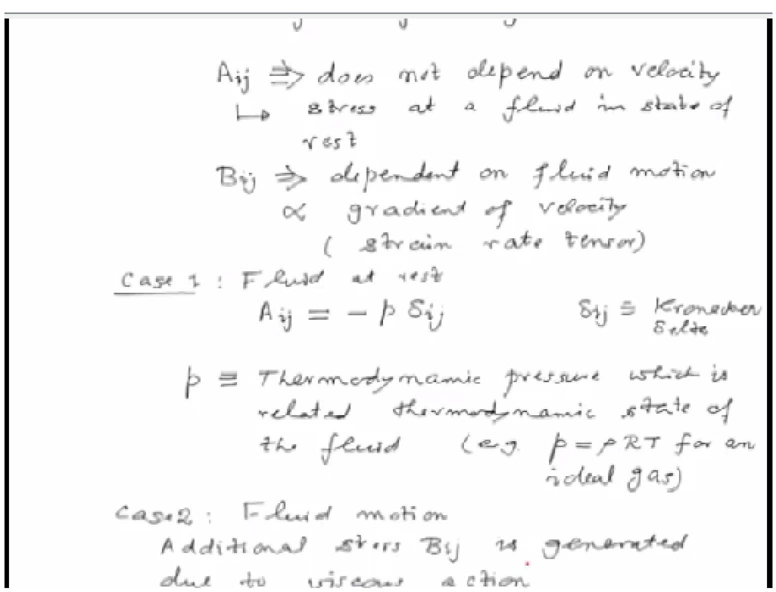

So if you look at that particular case what we call case 1, this fluid at rest and we can easily identify that our Aij is basically -p times delta ij, where delta ij is Kronecker delta which is a second order isotropic tensor and p is our thermodynamic pressure which is related to thermodynamic state of the fluid. For example, p=rho RT for an ideal gas. Now when we are dealing with fluid motion, in our case 2, fluid motion an additional stress term, additional stress Bij is generated due to viscous action.

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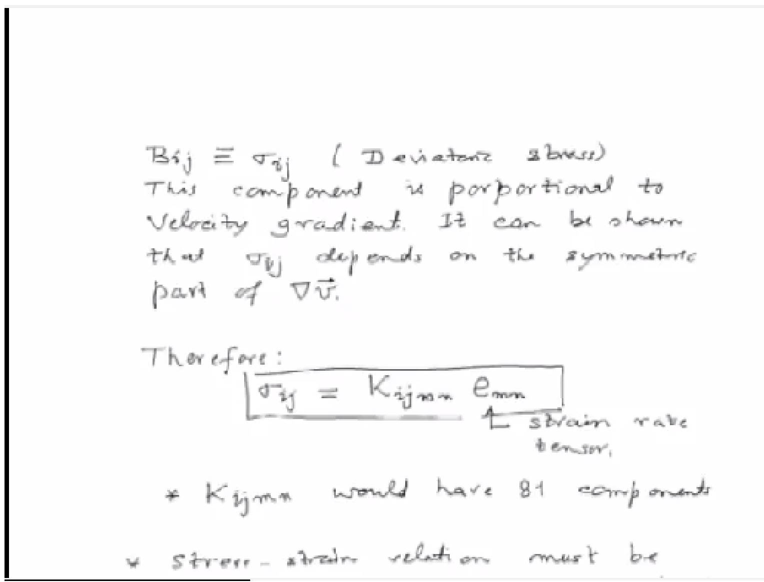

Now this additional term Bij, we commonly refer to it as Deviatoric stress tensor and we use a symbol Sigma ij for it. Now this component is proportional to velocity gradients. In fact, we can easily show that Sigma ij depends on the symmetric part of gradient of velocity because the anti-symmetric part which represents the rotations, they do not contribute to stress generation. Hence we can express Sigma ij as a product of a fourth-order tensor, let us call it as Kijmn*emn where emn is our strain rate tensor.

Now as far as tensor K is concerned, in general, it would have 81 components which basically represent or depend on the thermodynamic state of the fluid but this should satisfy specific requirements. For instance, the stress-strain relationship must be independent of rotation of the coordinates. So stress-strain relation must be independent of rotation of coordinate axes.

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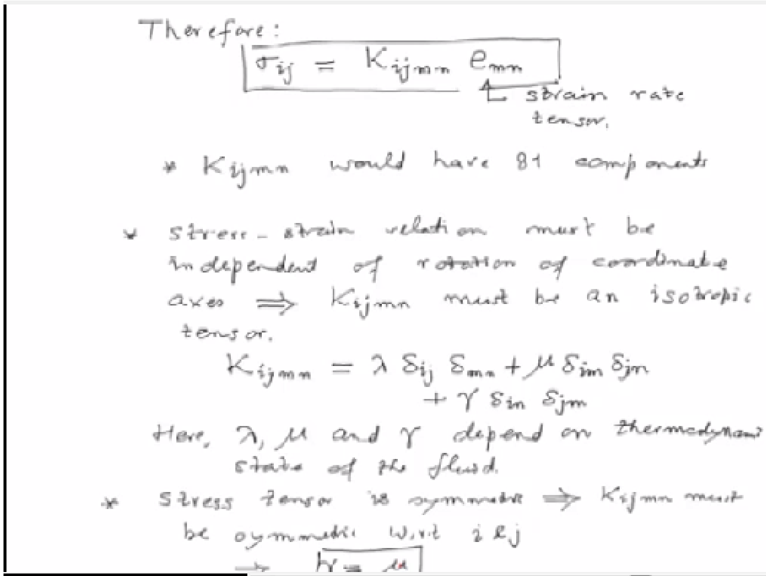

And this requirement implies that is Kijmn must be an isotropic tensor. So if that were the case, then we will have a lot fewer components in K. In fact, this K can be expressed as a product of 3 constants and so we cannot isotropic tensor delta ij. So Kikmn, this is given by lambda*delta ij delta mn+mu*delta im Delta in+gamma*delta in delta jm. Now here this, we have got 3 constants, lambda, mu and gamma, these depend on thermodynamic state of the fluid.

We have got another requirement that our Newtonian fluid is isotropic and our stress tensor is symmetric for isotropic medium. So stress tensor is symmetric which requires that is Kijmn must be symmetric with respect to i and j analysis and this is possible only if, gamma=mu. So this requirement reduces the number of material properties which we require by 1.

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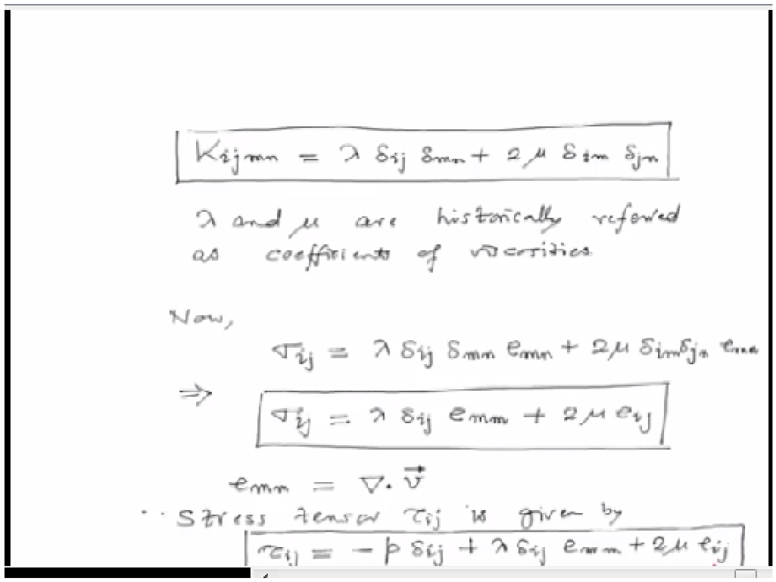

So now what we are left with is only 2 metal properties that is our Kijmn=lamda*delta ij delta mn+2 mu delta im delta jn. So now these to material constants, they have got specific name, lambda and mu or historically referred to as, referred as coefficients of viscosities. Our task would be to see what happens if we substitute this expression for Kij into sigma ij.

So our stress tensor sigma ij would be given by, sigma Ij=lambda*delta ij delta mn emn+2 mu delta im delta jn emn and this expression simplifies to sigma IJ=lambda*delta ij emm+2 mu eij. Now we can clearly say that this emm, it is nothing but divergence of the velocity vector and eij is our usual strain rate tensor. So now we can substitute for sigma ij in expression for tau ij and our stress tensor tau ij is given by -p delta ij+lambda delta ij emm+2 mu eij.

So this is generic relationship between the stress tensor and strain rate tensor for a Newtonian fluid.

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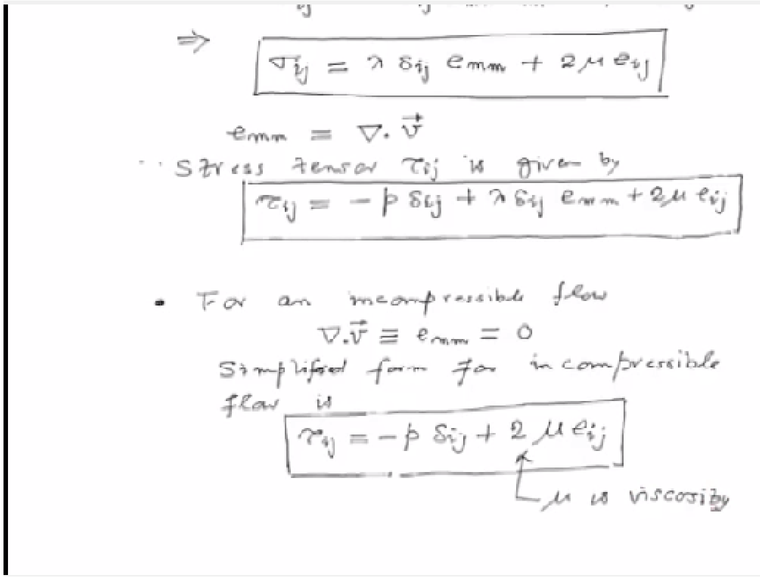

Now we will see what further simplifications we can have. If our fluid were incompressible or the fluid flow were incompressible. So for an incompressible flow, divergences of the velocity vector which is what this emm represents, this is = 0 and we get a simpler relationship, so this simplified form for incompressible flows as tau ij=-p delta ij+2 of mu eij that is we have got a single material constant which relates this stress tensor to the strain rate tensor.

This is our usual coefficient of viscosity. So mu is fluid viscosity. So this is what we had that the expression which we have obtained tau=-pI+lambda times divergences of v*I+2 mu*S and for incompressible Newtonian or a Stokesian fluids, we had tau=-p*I+2 mu*S.

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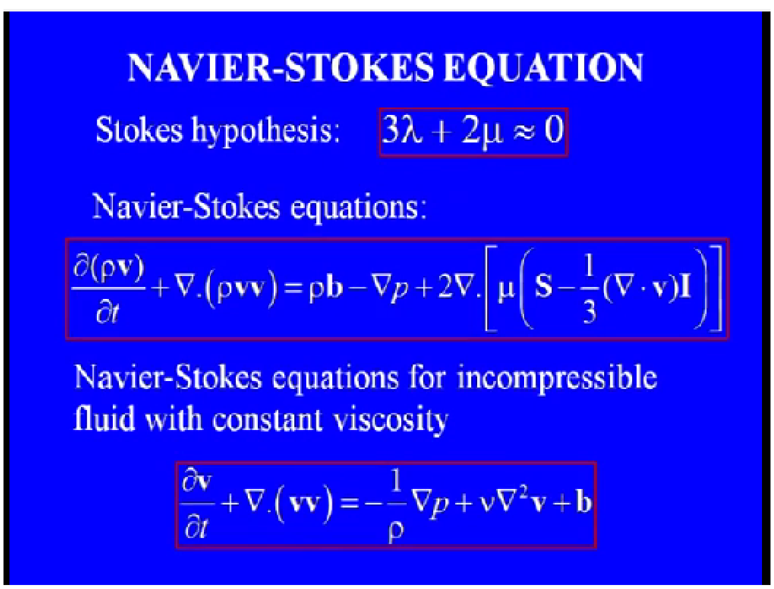

Can we obtain a similar relationship or simpler relationship for general or compressible Newtonian fluid, that is what our task would be.

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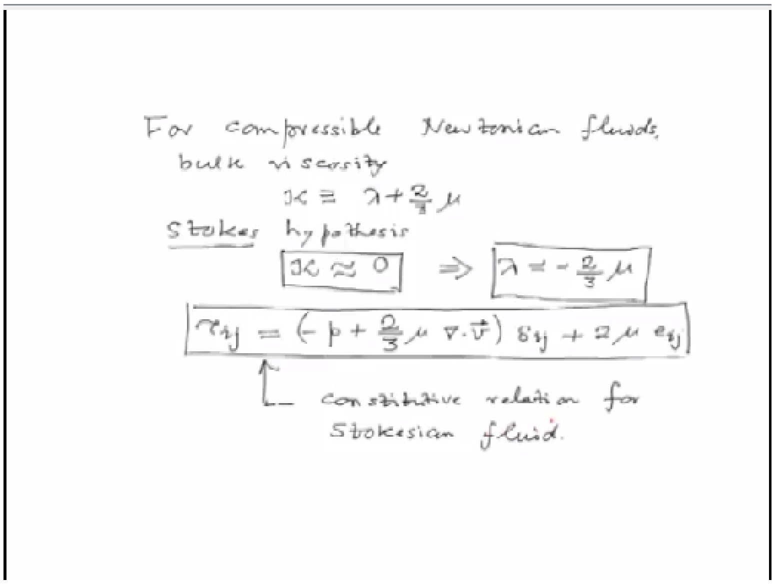

So for a general fluid what was observed by Stokes is, for compressible fluids or rather compressible Newtonian fluids, we have got a term which is called bulk viscosity which is given by kappa=lambda+2/3 mu and this was observed by Stokes to be very, very small. So the Stokes' hypothesis is this kappa is almost = 0.

This immeasurable quantity but is measurement of kappa would require very large density gradients which occur only in shockwaves and for majority of engineering fluids, for instance air and water, we have observed this Stokes hypothesis works pretty well. So kappa=0, this implies that our lambda=-2/3 mu. So with Stokes hypothesis, now we have got only 1 material constant or material property which is there in stress-strain relationship.

And now we can express our stress tensor tau ij = -p+2/3 mu divergence of velocity*delta ij+2 mu eij. Now since this relationship which we obtained was based on Stokes hypothesis. So this is also called constitutive relation for Stokesian fluid and if we substitute this expression for tau ij, in a Cauchy equation, the resulting equation is referred to as Navier-Stokes equation.

So this is what we get if we substitute for tau ij using Stokes hypothesis, we get a general form of

Navier-Stokes equation given by del rho v/del t+ divergences of rho vv=rho b-gradient of p+2 divergence of mu*S-1/3 divergence of v*i. Now this equation was independently derived by a French gentlemen Navier and the British gentleman Stokes, that is why these equations are referred to as Navier-Stokes equations.

We can have various simple forms or simplified forms of these equations.

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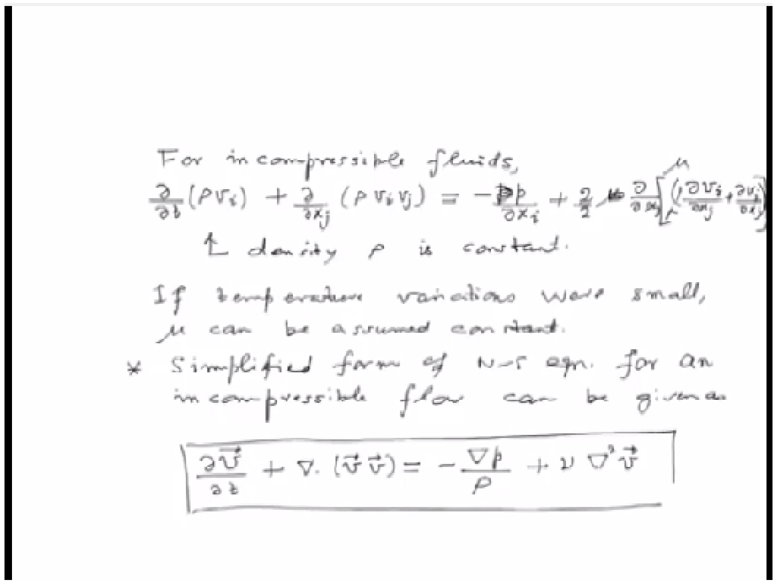

We have only seen 1 form for incompressible fluids and that simple form is given by del/del t rho vi+del/del xj of rho vivj=-del p/del/xi+2 mu del/del xj del vi/del xj+del vj/del xi/2. Now if we assume, if you want we can divide by density on both the sides or let us leave it as such by noting here that density rho is constant. Now if temperature variations were small, mu can be assumed as a constant and simplified form of this NS equation for an incompressible flow can be given as del v/del t+divergence of vv=-gradient of p/rho+mu*Laplacian of v.

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μ can be assumed constant.

* Simplified form of N-S eqn. for an incompressible flow can be given as

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \nabla\cdot(\vec{v}\vec{v}) = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v}}$$

* <u>Euler's Equation</u>

For high speed flows (e.g. around an aircraft), away from solid surface viscous effects are negligible.

$$\tau_{ij} = -p\,\delta_{ij}$$

$$\boxed{\frac{\partial (\rho\vec{v})}{\partial t} + \nabla\cdot(\rho\vec{v}\vec{v}) = \rho\vec{b} - \nabla p}$$

← Euler's equation

For Newtonian fluids, we can have few other simpler forms and other simpler form is what we call for inviscid fluid which leads to a so called Euler's equations. Now this Euler's equation is specifically valid for high-speed flows. For example, around an aircraft, okay. Now away from the solid surfaces that viscous effects can be neglected or negligible and hence our stress tensor is just that due to pressure.

And we get a very simple form for this equation, viscous terms drop out and what we get is del rho v/del t+ divergence of rho vv=rho b-gradient of p. So this simplified form was derived by Euler and that is why this is called Euler's equation and it is very widely used in numerical flows simulation for flows over aircraft and winged bodies. There is another case of interest which is extreme of that of the Euler equation. Suppose what we had is for low Reynolds number flows.

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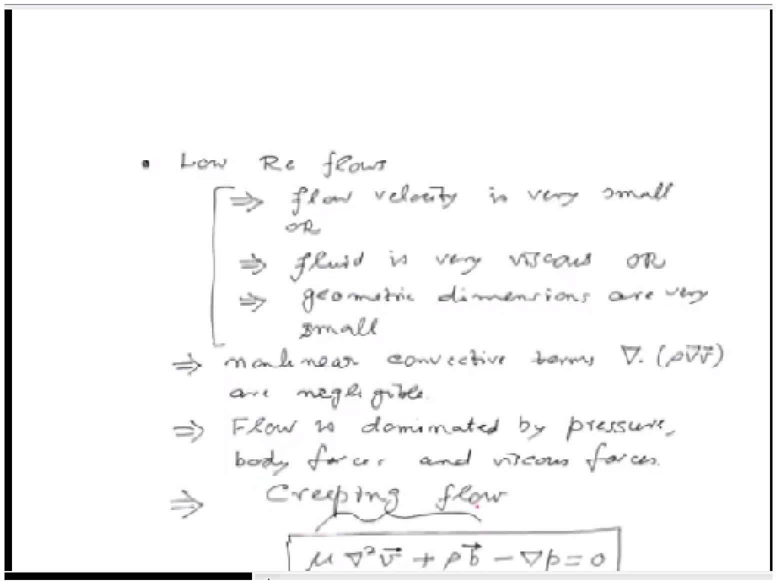

Now Reynolds number could be low because of what, that flow velocity is very small or the second case is, the fluid is very viscous or the third condition could be geometric dimensions are very small. So this last case could occur if we are dealing with electronic devices or the flow-through biological systems, we have got very small length scales in capillaries.

So in such situations in either of these situations, what we would have is our non-linear convective terms, terms which are represented by this divergences of rho vv, these are negligible and the flow is dominated primarily by, flow is dominated by pressure, body forces and viscous forces. This particular flow is referred to as creeping flow and we have got a very simple form which Navier-Stokes equation takes in this case, mu*del square of v+rho*b-gradient of p=0.

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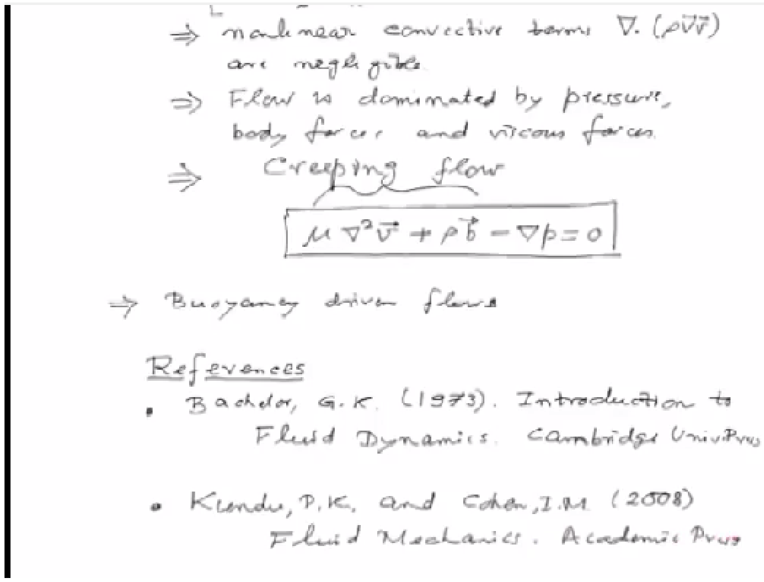

There are various other simpler forms, the most important functions I would likely refer to the ones where we have got buoyancy driven flows in which case the density variations can be ignored everywhere except in the momentum equation by including a term which accounts for density variations. For this and various other cases, please refer to the following book.

So just for the sake of recap of what we have discussed so for in connection with our conservation laws, good references could be the books by Bachelor, this book published in 1973 is one of the classic books, Introduction to Fluid Dynamics by Cambridge University Press. A more, a recent book is that by Kundu and Cohen, this most recent edition was published in 2008 by Academic Press. The book is simply titled as fluid mechanics published by Academic Press.

So these 2 references will give fairly detailed derivation of the governing equations of fluid flow which we had discussed so far.

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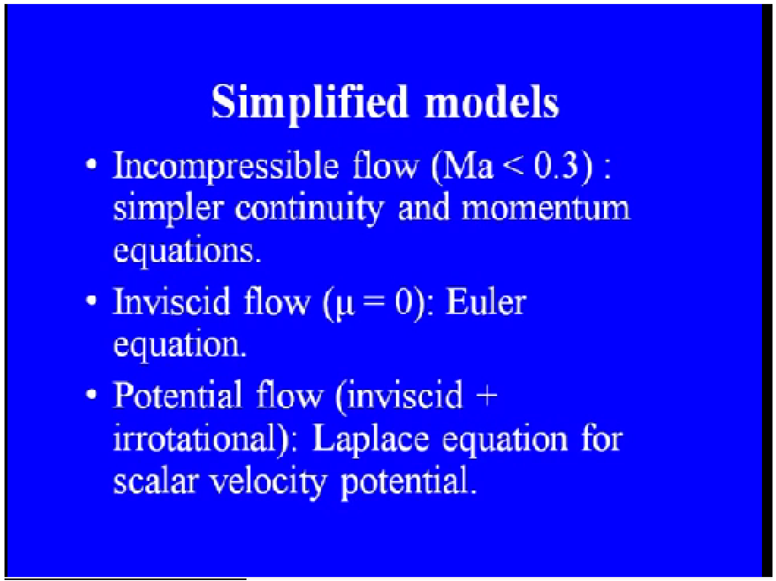

I would just like to mention few other simplified cases that we have already seen incompressible flow which leads to simpler continuity and momentum equations. We have already seen inviscid flows which is or approximations, very useful approximations for the case of high-speed aerodynamics and these are given by Euler equation.

We can also have another possibility that if you are dealing with high-speed flows away from the solid surfaces where we can ignore not just the viscosity, that is to say our flow is inviscid if in addition it is also what we call irrotational, that is the vorticity present is very, very small or negligible. We can represent our velocity as a gradient of a scalar function which is called velocity potential.

And in that case, we got a further simplification that instead of solving full equations, one for continuity and free momentum equations, we can have a single equation, single this scalar equation which is called Laplace equation for scalar velocity potential. So in numerical stimulation, this case becomes a lot easier, we need to just solve for this scalar velocity potential, take its gradients and we would get all the velocity components.

We can invoke the Bernoulli's equation to get the pressure field and so on. So this simplified and potential flow model is also very commonly used in some hydrodynamic flow stimulations as well as in aerodynamics flows. In the next lecture, we will cover or discuss in detail the energy equation.

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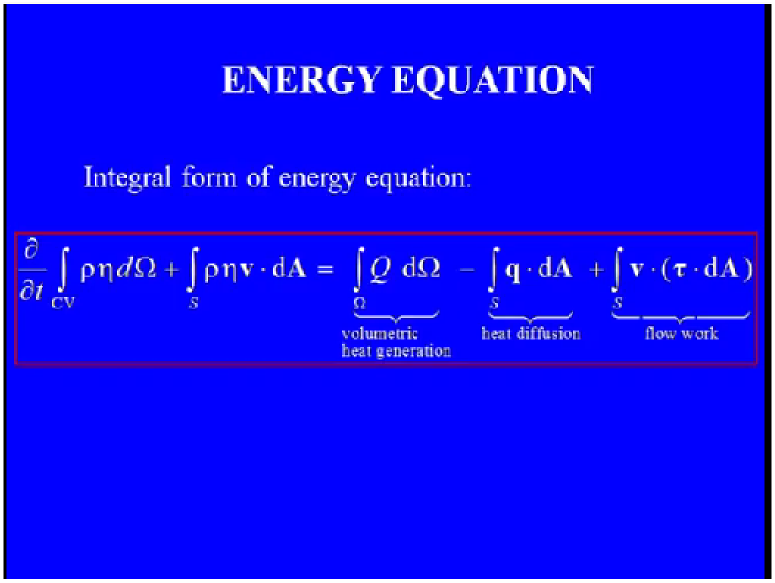

You will start from the first law of thermodynamics and we would obtain integral form of energy equation which is just for the sake of summary, we have given here del/del t of cv rho eta d omega+ surface integral of rho eta v.dA=q*d omega, this is a volume integral, -q.dA+v.tau.dA. On the left-hand side this eta that this represents the total energy per unit mass of a fluid which consists of internal energy as well as kinetic energy.

The potential energy has been incorporated in this flow work on the right-hand side. So the first term is temporal variation, the second one is conviction of eta with velocity field. The first term on the right-hand side represents volumetric heat generation due to volume sources in the fluid medium. The second term q.dA, q is q of surface flexes, so that represents what we call heat diffusion through the control surfaces.

And the last term represents the flow work which involves the work which the fluid must do against the stresses which incorporate the effect of both surface forces as well as body forces. Now how do we obtain this equation starting from the first of thermodynamics, that is what we will discuss in detail in the next class and would obtain this integral form as well as appropriate differential forms for the energy equation.