MECHANICS

Prof. Anjani Kumar Tiwari

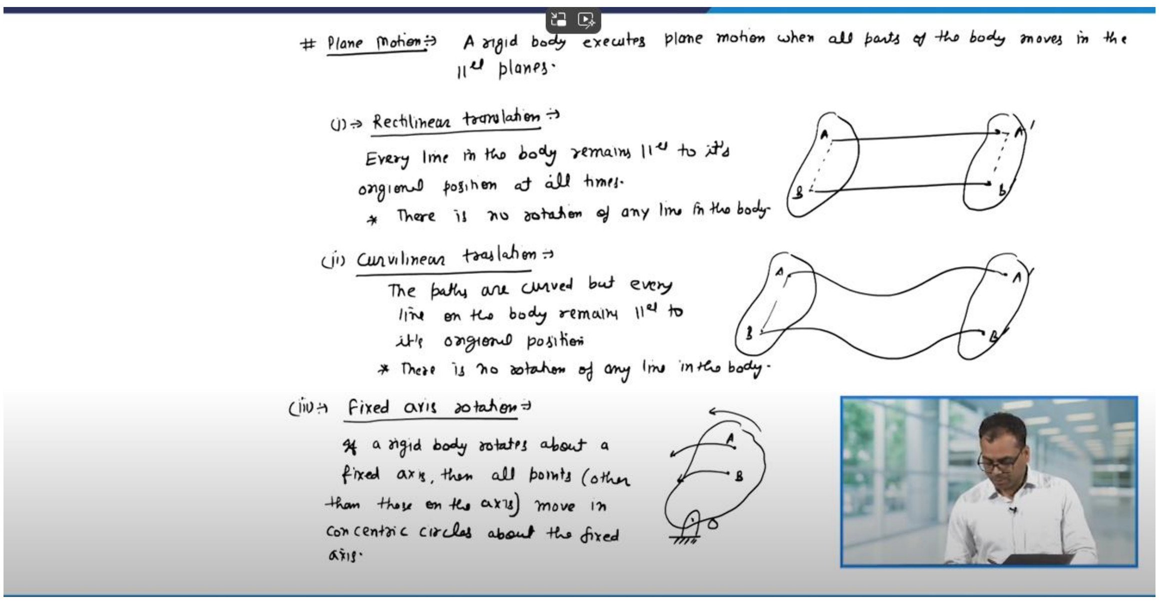
Department of Physics

Indian Institute of Technology, Roorkee

Lecture: 52

Translation and rotation of rigid bodies

Hello everyone, welcome to the lecture again. Today, we are going to discuss the translation and rotation of the rigid bodies in the context of the plane motion. Therefore, let us first discuss the plane motion. A rigid body executes plane motion when all parts of the body moves in parallel planes.

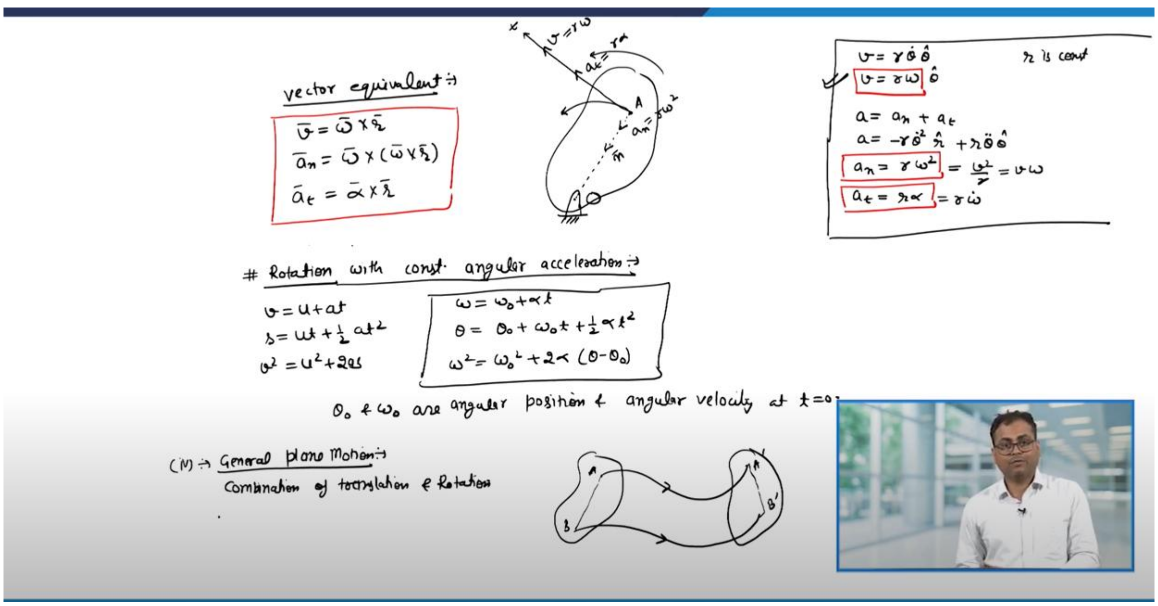


Now, this plane motion can be divided into several categories. For example, we can have rectilinear translation. So, here in a rigid body, it moves in the straight line, okay. So, every line in the body, so let us consider a line and let us say this point is A , this point is B , then every line in the body, it remains parallel to its original position at all times, okay.

So, this line AB will remain so, let us say $A'B'$ then AB is parallel to $A'B'$. So, here there is no rotation of any line in the body. Now, we can also have curvilinear translation. Let us say we have a rigid body and let us say we have two points AB and this rigid body moves in some curve path, but at all time these points AB which are now $A'B'$ remains parallel to AB then it is curvilinear translation.

So, here the paths are curved but again every line on the body remains parallel to its original position and again there is no rotation of any line in the body. Now, we can also have the fixed axis rotation and this happens when the body rotates about a fixed axis. For example, let us say I have a rigid body and let me fix some point on the body.

Let me call it O . Now, let us say I have a point A and a point B on this body, then when this body rotates about O , point A rotates in a circle and point B also rotates in a circle about O , okay. So, if a rigid body rotates about a fixed axis, then all points of course other than those I have fixed in the body. So, here in O , let me write down other than those on the axis move in concentric circles about the fixed axis. And we of course know how to analyze the fixed axis rotation. Because it is in the circle.



So, let me just summarize those results that we have obtained earlier. We know that $v = r\dot{\theta}\,\hat{\theta}$ because r is constant for circular motion and this I can also write as $v = r\omega\hat{\theta}$, the magnitude is $r\omega$.

This is the velocity and the acceleration a can be written as let us say $a = a_n + a_t$ where a_n is the acceleration along the $r - direction$ and a_t is the acceleration along the $\theta -$

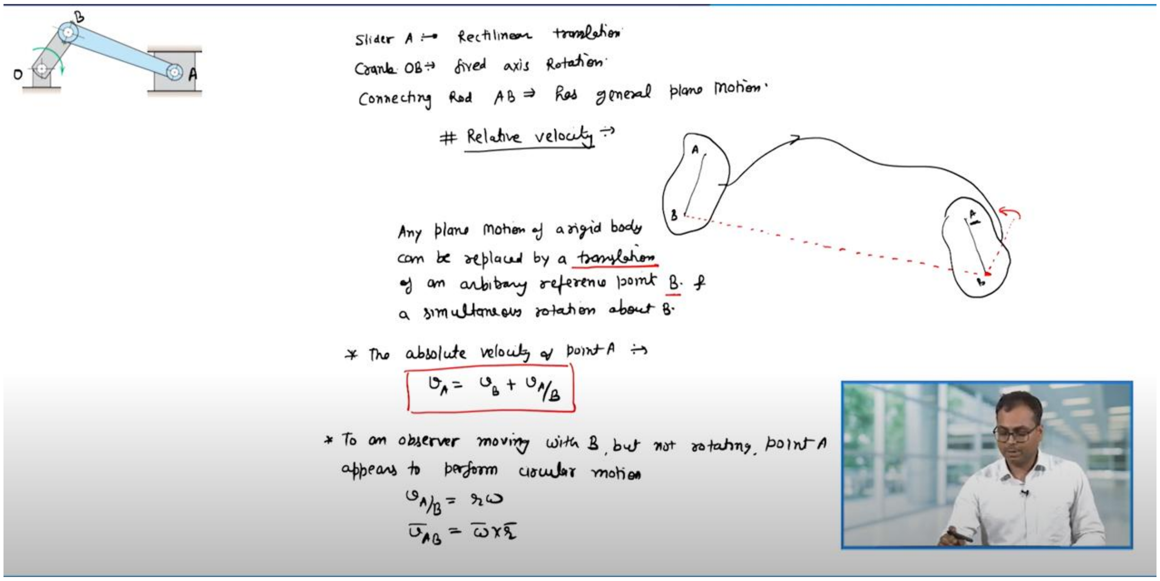
$direction$. So, this is $a = -r\dot{\theta}^2\hat{r} + r\,\ddot{\theta}\,\hat{\theta}$. So, here $a_n = r\omega^2$, $\dot{\theta} = \omega$. So, I can just write down ω^2 or I can rewrite it in different form $a_n = \frac{v^2}{r} = v\omega$ etc.

And a_t is the acceleration along the tangential direction is $r\ddot{\theta}$, $\ddot{\theta}$ is α. So, I can write down $a_t = r\alpha = r\dot{\omega}$, okay. Here, ω is the angular velocity and α is the angular acceleration a_n is the acceleration along the $\hat{r}$ direction and a_t is the acceleration along the $\hat{\theta}$ direction.

Let us look at the direction. So, we have this rigid body. We have fixed a point and this rigid body is rotating about the fixed point O. We have taken some point A. This point A is going to make a circular motion around O. This is the direction of n. This is the direction of t. The velocity $v = r\omega$. The acceleration along the t direction is $a_t = r\alpha$ and the acceleration along the n direction will be $a_n = r\omega^2$. Because it is fixed axis rotation, there will not be any velocity along the n direction. Now, these equations I can also write down in the, you know, vector form. So, vector equivalent $\vec{v} = \vec{\omega} \times \vec{r}$, okay, acceleration along the n direction was $r\omega^2$ can be written as $\overrightarrow{a_n} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ and a_t or the acceleration along the tangential direction will be $a_t = r\alpha$.

So, $\overrightarrow{a_t} = \vec{\alpha} \times \vec{r}$. Now, let us look at the rotation with constant angular acceleration. We already know the equation for rectilinear motion, which is $v = u + at$, $s = ut + \frac{1}{2}at^2$ and $v^2 = u^2 + 2as$. Similarly, we have the equation for constant angular acceleration which is $\omega = \omega_o + \alpha t$, $\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$ and $\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$. Here this θ_o and ω_o they are the angular position and angular velocity at $t = 0$. Now, we have seen the translation and the fixed axis rotation of the rigid body. In general, we can have the combination of them and it is called the general plane motion.

So, general plane motion, it is the combination of translation and rotation. So, let us say I have a rigid body, let me fix two points on the rigid body A and B, then in general plane motion this body moves and simultaneously it can also rotate. So, therefore, this is the combination of translation and rotation.



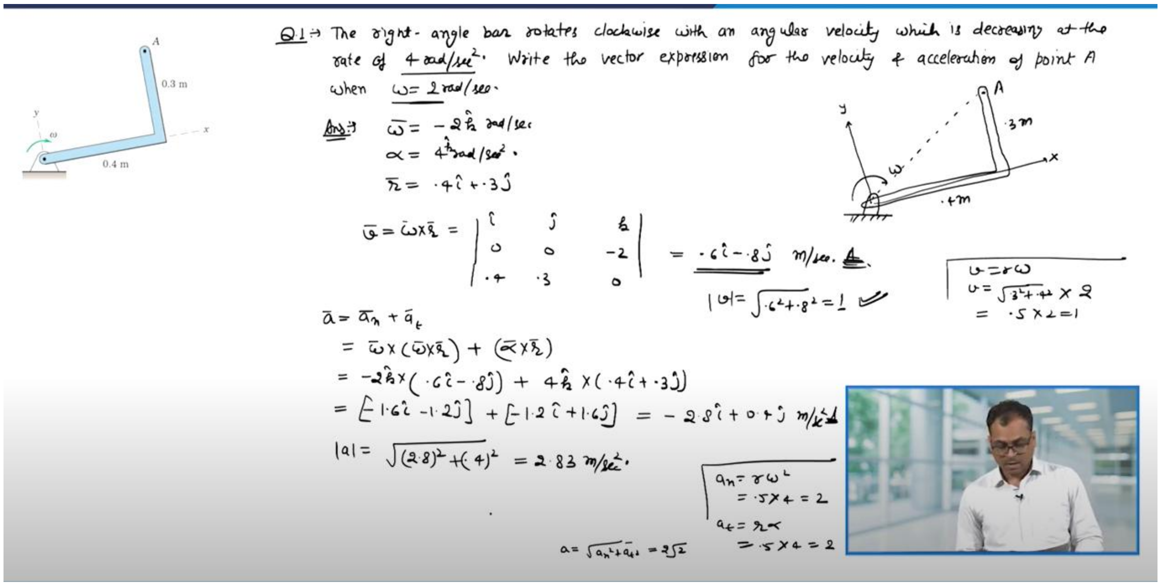
Now, let us understand various kind of motion by an example. So, this is the schematic of a reciprocating engine. Let us say this point is A , this point is B and this one is O . Here, the crank OB makes a fixed axis rotation about O . Because of that, the slider A moves in the rectilinear direction. So, slider A , this performs a rectilinear translation. The crank OB performs a fixed axis rotation about O . And the connecting rod AB , this performs a general plane motion because it is also moving and it is also rotating, a general plane motion.

Now, let us understand the relative velocity. Let us say I have a rigid body and this rigid body is making a general plane motion ok. Let us say I have two points A and B on the rigid body, then the same points AB after time t are like that. So, we have seen that any plane motion is the sum of the translation plus a fixed axis rotation.

Any plane motion of a rigid body can be replaced by a translation of an arbitrary reference point. Let us say that reference point here is B . And a rotation and a simultaneous rotation about B . So, what do I mean by that? So, we had this general plane motion. The motion can be analyzed by having two kinds of motion.

One is just linear translation of this point. So, this is the translation of the arbitrary point. Here, I have taken B as the arbitrary point and the simultaneous rotation about B . So, about B , you can see that point A was earlier here and now this has rotated. Now, the absolute velocity of this point A will be, so $v_A = v_B + v_{\frac{A}{B}}$.

Now, if there is an observer who is moving with B , but not rotating, then he will see as if point A is making a circular motion. So, let me also write it down to an observer moving with B but not rotating. To given, point A appears to perform a circular motion. In that case, the $v_{\frac{A}{B}} = r\omega$ or in the vector form $\overrightarrow{v_{AB}} = \vec{\omega} \times \vec{r}$.



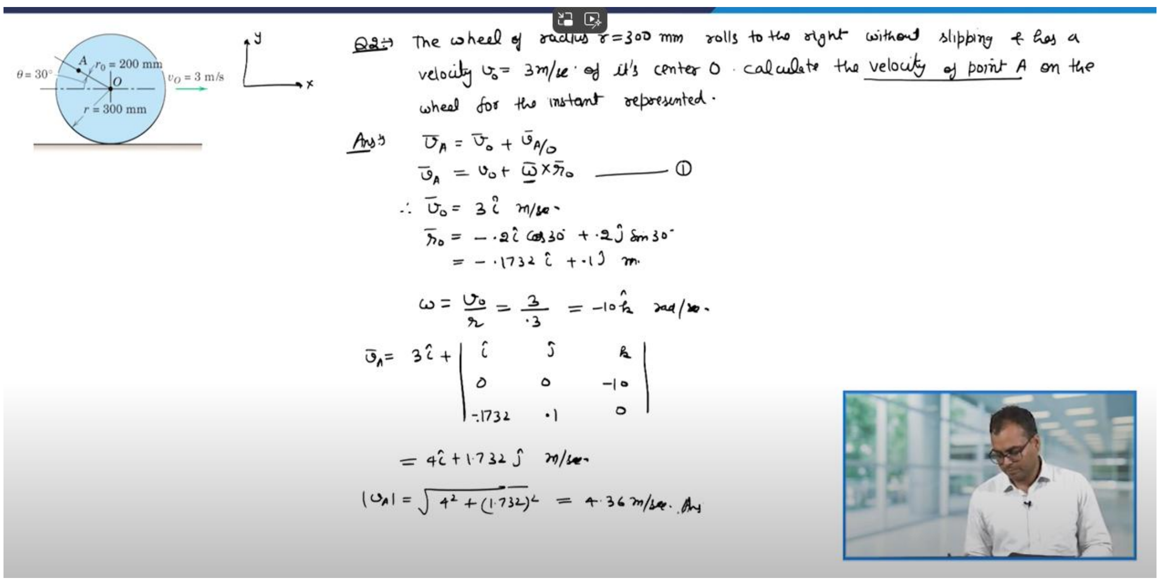
Now, let us look at some of the problems based on these concepts and the first problem statement is the right angle bar rotates clockwise with an angular velocity which is decreasing at the rate of4 $\frac{rad}{s^2}$, write the vector expression for the velocity and acceleration point A when $\omega = 2\ \frac{rad}{sec}$. So, here we have this bar.

And we have to find out the velocity and acceleration of point A . It is given that the x-axis is like this and the y-axis is like that and the bar is rotating in the clockwise direction. So, therefore, $\omega = -2\hat{k}\ \frac{rad}{sec}$. Now, α is also given, $\alpha = 4\ \frac{rad}{\sec^2}$ and for point A , we can write down the position vector, $\vec{r}$. So, $\vec{r}$ for point A will be this is $0.4\, meter$, this one is $0.3\ meter$. So, therefore, $\vec{r} = 0.4\hat{\imath} + 0.3\hat{\jmath}$.

Now, the velocity and acceleration can be find out because $\vec{v} = \vec{\omega} \times \vec{r}$ and this will be $\hat{\imath},\ \hat{\jmath},\ \hat{k},\ \omega = -2\hat{k}$. So, $0,\ 0, -2$ and $r\ is\ 0.4,\ 0.3\ and\ 0$ and $\vec{v} = 0.6\hat{\imath} - 0.8\hat{\jmath}\ \frac{m}{sec}$. , okay. Now, the acceleration has two parts. So, this has the component in the $n - direction$ and also the component in the $t - direction$, okay, $a_n\ and\ a_t$ and we know how much is a_n, $\overrightarrow{a_n} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ and $\overrightarrow{a_t} = \vec{\alpha} \times \vec{r}$, okay. Remember this formula here, $a_n\ and\ a_t$. So, let us put the values, $\omega = -2\hat{k}$, $\vec{\omega} \times \vec{r}$, we have already calculated.

It is $0.6\hat{\imath} - 0.8\hat{\jmath}$ plus $\vec{\alpha} \times \vec{r}$, and $\alpha = 4 \frac{rad}{\sec^2}$ and that will also be in the $k - direction$, okay. So, $\alpha = 4\hat{k}$ into r, $\vec{r} = 0.4\hat{\imath} + 0.3\hat{\jmath}$ and this comes out to be $\vec{a} = [-1.6\hat{\imath} - 1.2\hat{\jmath}] + [-1.2\hat{\imath} + 1.6\hat{\jmath}] = -2.8\hat{\imath} + 0.4\hat{\jmath} \frac{m}{\sec^2}$. Now, let us look at the magnitude of v. So, the magnitude of $v = \sqrt{(0.6)^2 + (0.8)^2} = 1$ and this you can also get by using $v = r\omega$.

So, $v = r\omega$. r is the distance which is $r = \sqrt{(0.3)^2 + (0.4)^2} = 0.5$ into ω. ω is given. It is $\omega = 2 \frac{rad}{sec}$ and this is $v = 0.5 \times 2 = 1$. Similarly, the magnitude of the acceleration $a = \sqrt{(2.8)^2 + (0.4)^2} = 2.83 \frac{m}{\sec^2}$. This also you can get directly by using a_n and a_t. So, $a_n = r\omega^2$ which is equal to r we have find out it is 0.5 into $\omega = 2$. So, $\omega^2 = 4$ that is $a_n = 0.5 \times 4 = 2$ and $a_t = r\alpha$. So, r is again 0.5, $\alpha = 4$. So, that is $a_t = 0.5 \times 4 = 2$. Now, acceleration $a = \sqrt{a_n^2 + a_t^2} = 2\sqrt{2}$ and that is again equal to roughly 2.83 $\frac{m}{\sec^2}$.



Now, let us look at another problem statement. The wheel of radius $r = 300\,mm$ rolls to the right without slipping and has a velocity $v_o = 3 \frac{m}{sec}$ of its center O. Calculate the velocity of point A on the wheel for the instant represented. Now, in this question, we have been asked to find out the velocity of point A and velocity of point A, I can find out using the concept of relative velocity. So, $\vec{v_A} = \vec{v_o} + \vec{v_{\frac{A}{O}}}$. Now, point A with respect to O is making a fixed axis rotation. Therefore, $\vec{v_A} = \vec{v_o} + \vec{\omega} \times \vec{r}$.

Okay. Let us say this is the $x - axis$ and this one is the $y - axis$. Therefore, $\vec{v_o} = 3\hat{\imath} \frac{m}{sec}$. Now, $\vec{r_o}$, I can find out from the figure, from the geometry. It is given that $\theta = 30°$ and $r_o = 200\,mm$.

Therefore, it will be $\vec{r_o} = -0.2\hat{\imath}Cos30° + 0.2\hat{\jmath}Sin30°$. And this is $\vec{r_o} = -0.1732\hat{\imath} + 0.1\hat{\jmath}\ m$. Now, let us find out the angular velocity ω of the wheel. So, $\omega = \frac{v_o}{r} = \frac{3}{0.3}$. And its direction will be in $-\hat{k}$ because it is rotating in the clockwise direction. So, therefore $\omega = -10\hat{k}\ \frac{rad}{sec}$. Now, we have ω, we have r_o, we have v_o. So, therefore, let us put everything in equation number (1). So $\vec{v_A} = 3\hat{\imath} + \vec{\omega} \times \vec{r_o}$. So, $\hat{\imath}, \hat{\jmath}, \hat{k}$, 0, 0, $-10\ and\ -0.1732,\ 0.1\ and\ 0$ and this gives you $\overline{v_A} = 4\hat{\imath} + 1.732\hat{\jmath}\ \frac{m}{sec}$ and its magnitude $v_A = \sqrt{4^2 + (1.732)^2} = 4.36\ \frac{m}{sec}$. With this, let me stop here. See you in the next class.

Thank you.