Dynamics and Control of Mechanical Systems
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Lecture - 02
Position and Orientation of a Rigid Body

Welcome to this NPTEL course on dynamics and control of mechanical systems. My name is Ashitava Ghoshal, I am a professor in the department of mechanical engineering and in the centre for product design and manufacturing and Robert Bosch centre for cyber physical systems Indian Institute of science Bangalore. In this course we will start with the representation of rigid bodies in 3D space notation and basic concepts. Let us look at position and orientation of a rigid body in 3D space.

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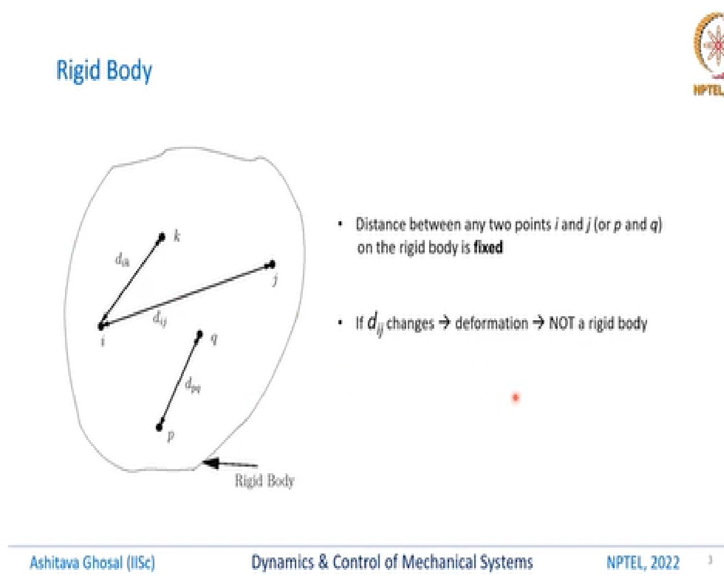

So, a rigid body can be denoted by this strange looking shape. So, basically this denotes an abstract rigid body. A rigid body can have several points. So, for example we can have a point i and a point j similarly point i and a point k and so on. We can always find the distance between two points let us say i and j this is denoted by d_{ij}. So, this is the normal distance which is used in everywhere it is a Euclidean distance.

So, the distance between any points two points i and j are say for that matter between p and q on the rigid body is fixed. So, that is one definition of a rigid body that if I pick any two points the distance does not change. So, there is no deformation. So, if d_{ij} changes as the rigid body moves then we say that it is not a rigid body. There is deformation happening in the rigid body.

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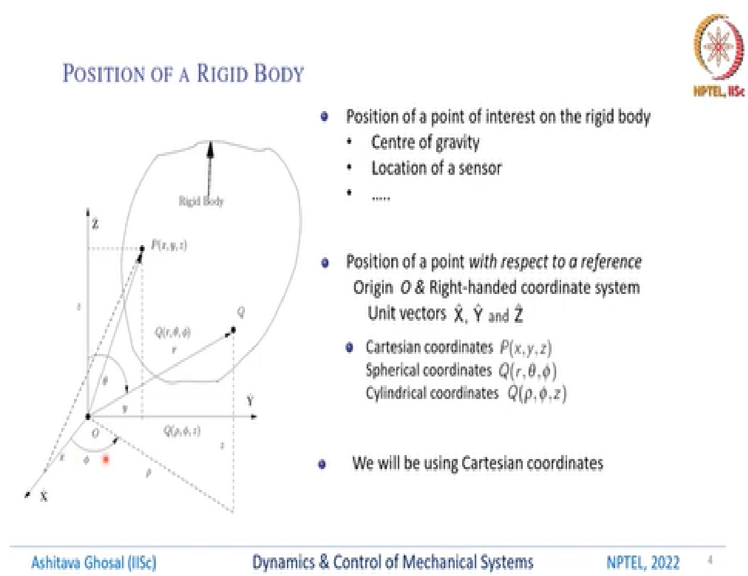

So, now let us look at the rigid body in 3D space. So, we have a reference coordinate system X Y and Z with an origin. There are many ways you can represent this rigid body. So, the position of a rigid body is basically the position of a point of interest on the rigid body. So, it could be the centre of gravity it could be the location of a sensor on the rigid body or it could be any other point.

So, the position of a point on the rigid body is again as mentioned earlier is with respect to a reference coordinate system or origin and a right-handed coordinate system $\hat{X}$, $\hat{Y}$, $\hat{Z}$. So, there are various ways of representing position of a rigid body you can have Cartesian coordinates. Cartesian coordinates are nothing but x, y and z the usual Cartesian coordinate. You can have spherical coordinates and you can have cylindrical coordinates.

So, in this course we will be mostly using Cartesian coordinates. So, let us look at Cartesian coordinates a little bit more detail so what is X, Y and Z of this point P. So, it is nothing but you draw a vector from o to P and you project the vector along the X axis along the Y axis around and along the Z axis. So, these projected distances are x, y and z . In the spherical coordinates you can have these three different angles.

So, you can have two different angles θ, ϕ and the distance to that point. So, spherical coordinates are given by r, θ, ϕ. cylindrical coordinates on the other hand are given by this radius ρ and some ϕ and z. So, we will be always or most of the time using Cartesian coordinates.

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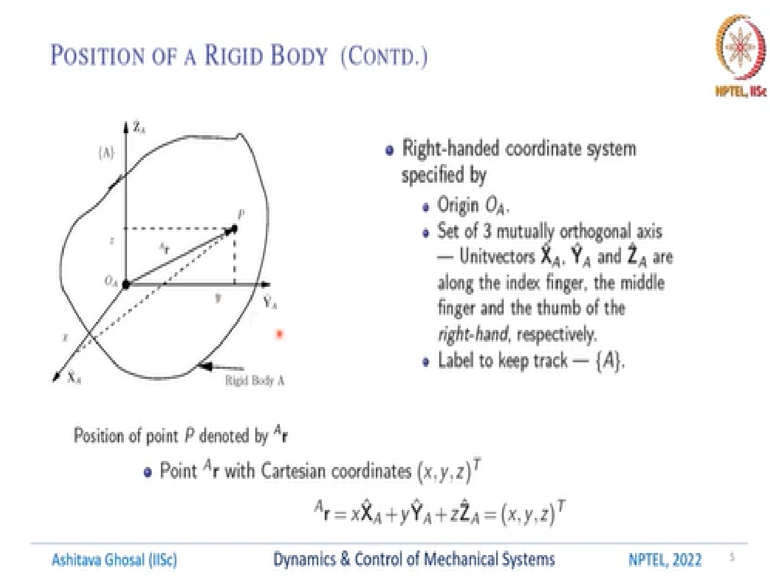

So, let us continue. So, let us see we have this rigid body in a reference coordinate system A and as I said we pick a point P on this rigid body and we find this vector Ar which is from the origin of a coordinate system which is denoted by O_A to P and then we have these three projections which is x, y and z. So, as I said we have a right-handed coordinate system specified by the origin O_A a set of three mutually orthogonal axis unit vectors $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$.

So, you can think of this along the index finger the middle finger and the thumb. So, again Z is X cross Y this is the right-hand system and as I said earlier, we have to label this reference coordinate system with A because we will have several such coordinate systems. So, the position of a point P is denoted by Ar in this rigid body and the position vector Ar can be denoted by 3 Cartesian coordinates x, y and z.

And mathematically this Ar can be written as x along $\hat{X}_A$, y small y along this $\hat{Y}_A$ unit vector and z along $\hat{Z}_A$. So, remember $\hat{X}_A$ in its own coordinate system is 1 0 0 likewise $\hat{Y}_A$ in its own coordinate system is 0 1 0 and $\hat{Z}_A$ in its own coordinate system is 0 0 1. So, if you expand this out you will get x, y and z as a column vector.

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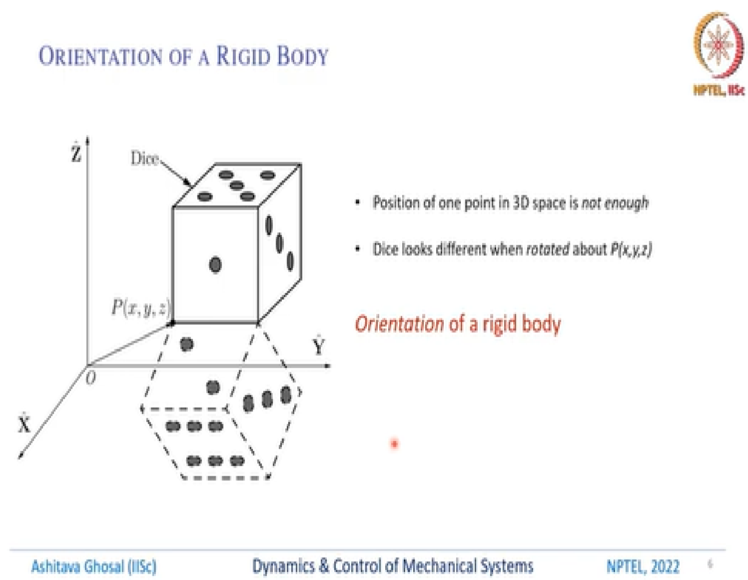

Now let us look at the very important notion of the orientation of a rigid body and why do we need this concept of orientation. So, let us look at a dice. So, you can see this dice has a dice has six faces. So, one such faces with 1 another one is with 5 and this one is 3. Now I can locate the corner of this dice by this vector which components x, y and z. But this is not enough to completely describe the dice.

So, for example this price could be rotated about this axis and this axis and then you can see different things. So, you will see that there is A_2

on this face, 3 on this face and 6 on the other face. So, remember in a dice opposite sides add up to 7. So, when you rotate it this 5 will now you will see as 2. Similarly, this one you will see a 6 whereas these three sort of remains the same this is still the face with the three.

But as you can see this dice looks completely different when you rotate about this line keeping this corner point fixed. So, just one point on this rigid body with components x, y and z is not enough. So, that is what I have said the position of one point in 3D space is not enough. So, the dice looks different when rotated about P x, y, z and what we need basically is something called the orientation of the rigid body. Just one point is not enough to completely describe the rigid body.

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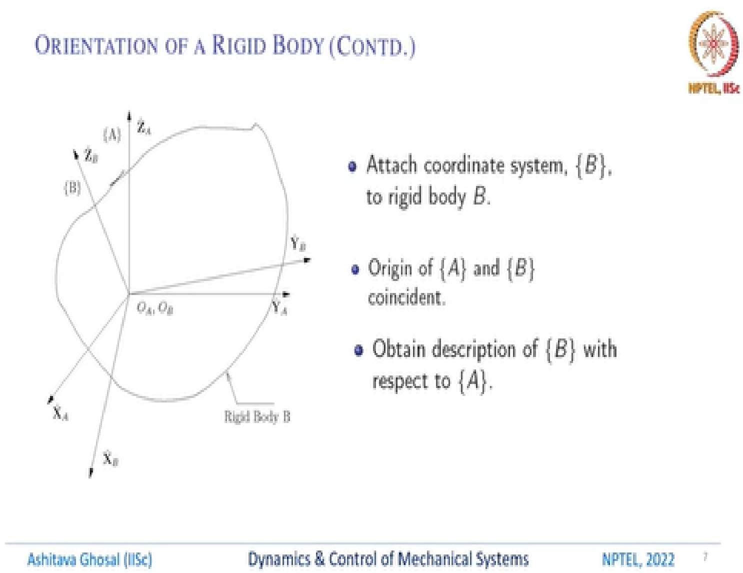

So, let us continue with this concept of orientation. So, let us do a little bit mathematically. So, we have this rigid body, and we have a coordinate system reference coordinate system $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ labelled as A with the origin O_A. And we have another coordinate system which is fixed to the rigid body which is $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ and this we are looking at orientation of the rigid body. So, we are not really interested in so the origin of the two coordinate systems can be at the same place.

So, again as I said we attach a coordinate system B to the rigid body and the origin of A and B are coincident. So, what we want to do is we want to obtain a description of this B coordinate system with respect to A. So, if you think about it so I have a rigid body and I have a fixed coordinate system and there is a coordinate system which is attached to the rigid body. So, as I look at this rigid body in different orientation.

So, basically if I can describe to you what is the B coordinate system with respect to the fixed A coordinate system that will tell me everything about the orientation of the rigid body.

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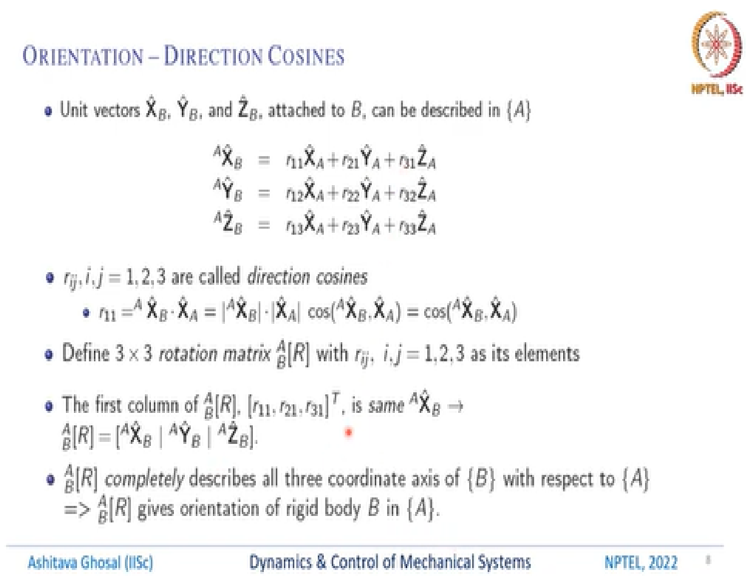

So, let us continue with this concept of orientation. How do we mathematically represent orientation? So, one very nice way is what is by using called direction cosines. So, we have this unit vector s $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ which are attached to the rigid body B and we want to describe this unit vector $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ with respect to the A coordinate system with respect to $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$.

So, basically, we can project $\hat{X}_B$ vector onto $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ exactly the same way as we projected the vector of a point to a point along $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ and we said that the coordinates were X, Y and Z. In this case we will see the coordinates are r_{11}, r_{21}, r_{31}. Likewise, if you project $\hat{Y}_B$ onto $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ we will call the coordinates as r_{12}, r_{22}, r_{32}. and for Z r_{13}, r_{23}, r_{33}. So, we will see later why this particular way these coordinates are labelled.

Why is it r_{11}, and now why is not this one not r_{12}, why are we calling it r_{21},? We will see that in a little while. So, this r_{ij}, are called the direction cosines. And what do we mean by direction cosines? Basically r_{11}, is nothing but the dot product of $\hat{X}_B$ with $\hat{X}_A$. So, the by definition of dot product $\hat{X}_B$ dot $\hat{X}_A$ in the same coordinate system is nothing but the magnitude of AX_B, magnitude of $\hat{X}_A$ in the cosine of the angle between the two vectors.

So, it is the cosine of the angle between $\hat{X}_B$ and $\hat{X}_A$ and we can define a 3 x 3 rotation matrix $BA[R]$ with all the elements r_{ij}, i, j=1, 2, and 3. So, what is the first column of this rotation matrix? So, first column of the rotation matrix will be r_{11} into 1 0 0, r_{21}into 0 1 0 remember Y axis in its own coordinate system is 0 1 0, Z axis in its own coordinate system is 0 0 1. So, the first column will be r_{11}, r_{21}, r_{31}.

What will be the second column? The second column will be the Y axis $\hat{Y}_B$ axis in this reference coordinate system A and the third column will be the $\hat{Z}_B$ axis in the reference coordinate system A. And this is one of the reasons why we deliberately labelled these coefficients be r_{11}, r_{21}, r_{31} like this because I want the first column to be $\hat{X}_B$ with components be r_{11}, r_{21}, r_{31}.

So, as I said this rotation matrix contains the X vector, Y vector, the $\hat{Y}_B$ vector, $\hat{X}_B$ vector and $\hat{Z}_B$ vector with respect to the A coordinate system. So, as I had argued earlier if I know these axes which are fixed to the rigid body with respect to a reference coordinate system then I know the how the rigid body is oriented with respect to the reference coordinate system. So, hence $BA[R]$ completely describes all three coordinate axis of {B} with respect to {A} and that implies $BA[R]$ is the orientation of rigid body B in A.

So, just let us go through it once more. So, what I am trying to do is I am trying to write this vector $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ which are fixed to the rigid body with respect to the A coordinate system. And just like any position vector in a rigid body when you project it onto the $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ axis it has three components. In the case of a position vector, it was x, y and z. In the case of this we are going to call it r_{11}, r_{21}, r_{31}.

Likewise, $\hat{Y}_B$ when projected onto $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ r_{12}, r_{22}, r_{32} and $\hat{Z}_B$ r_{13}, r_{23}, r_{33} . And when we organize all these coefficients in the form of a matrix which is B with respect to A and matrix with elements r_{ij} then each column of this rotation matrix is nothing but the at first column is $\hat{X}_B$, second column is $\hat{Y}_B$ and the third column is $\hat{Z}_B$.

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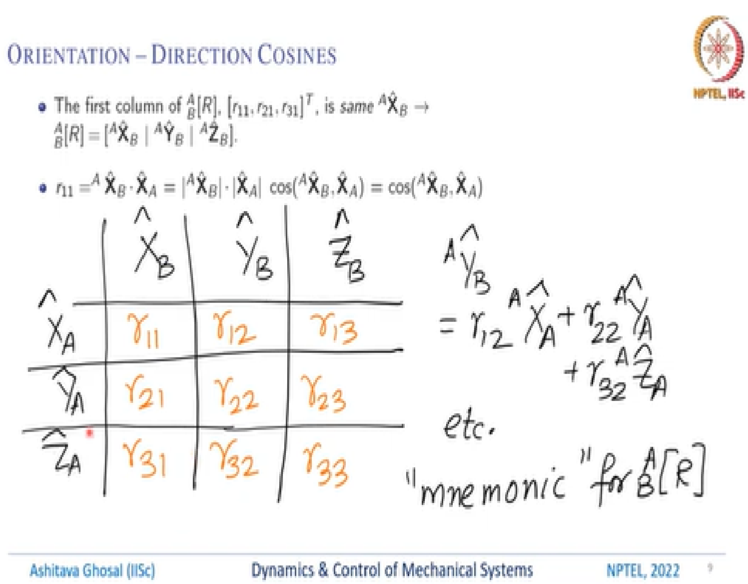

So, as we discussed in the last slide the first column of the rotation matrix $BA[R]$ has r_{11}, r_{21}, r_{31} and this is the same as the $\hat{X}_B$ axis in the A coordinate system and the rotation matrix first column is $A\hat{X}_B$, second column is $A\hat{Y}_B$ third column is $A\hat{Z}_B$. And as I had mentioned earlier r_{11} is nothing but the dot product of $\hat{X}_B$ with $\hat{X}_A$. So, it is the magnitude which is both of them are one. So, we are left with cosine of the angle between these two axes $\hat{X}_B$ and $\hat{X}_A$.

So, here is AQuick short way or a mnemonic to remember what the direction cosines are and what are the elements of the rotation matrix $BA[R]$. So, we put the $\hat{X}_B$ along this vertical line $\hat{Y}_B$ in this vertical column $\hat{Z}_B$ also vertical and the horizontal rows are $\hat{X}_A$, $\hat{Y}_A$ and $\hat{Z}_A$. So, what you can see is r_{11} is the dot product of $\hat{X}_B$ with $\hat{X}_A$, r_{21} is the dot product of $\hat{Y}_A$ with $\hat{X}_B$, r_{31} is the dot product of $\hat{Z}_A$ with $\hat{X}_B$ and so on.

So, r_{23} you can easily see it is the dot product of $\hat{Z}_B$ with $\hat{Y}_A$. And $A\hat{Y}_B$ this column vector is r_{12} along $\hat{X}_A$ axis, r_{22} along $\hat{Y}_A$ axis, r_{32} along $\hat{Z}_A$ axis. Later on, we will see that when we are rotating about either the $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ or the $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ axis these are two different kinds of rotations. We will you can see that the columns are not changing when you are rotating about $\hat{X}_B$, the first column will not change. Similarly, if you are rotating about $\hat{Y}_A$ the second row will not change.

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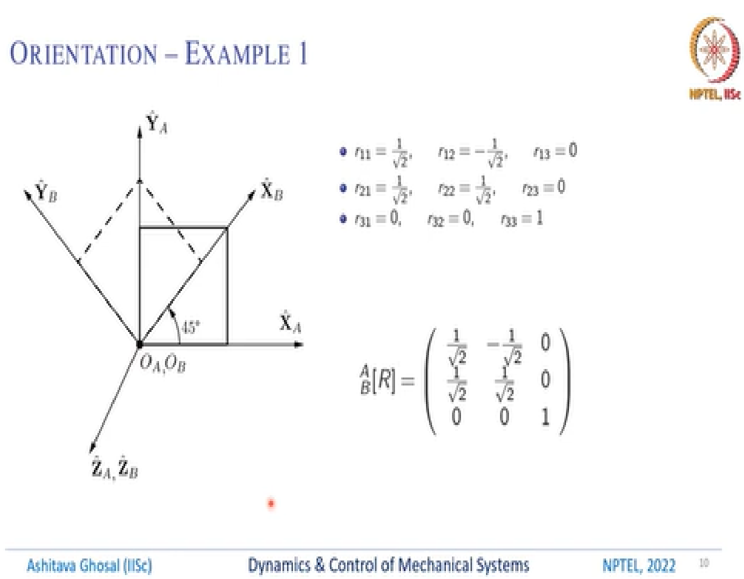

So, let us look at an example. So, I have, and this is a very simple example this is a planar example. So, I have a rigid body which is nothing but a square in this case. So, let us say I rotate this square about this Z axis by 45 °. So, this dark square will now become this dotted square. So, we have a reference coordinate system $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ and then we have a coordinate system $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ and origins O_A and O_B are at the same place.

And in this example $\hat{Z}_A$ and $\hat{Z}_B$ are at the same place. So, let us find out what are the direction cosines. So, how do I find direction cosines? So, basically, I find r_{11} is nothing but the dot product of $\hat{X}_B$ with $\hat{X}_A$. So, in this example this is rotated by 45 °. So, cos of 45 ° is 1 by root 2. How about r_{12}? We can see that it is - 1 by root 2. How r_{13}? What is r_{13}? It is the dot product of $\hat{X}_B$ with Z, X and Z.

So, since this is a planar rotation, we have this as r_{13} as 0. What is r_{21}? If you go back and see the slide earlier slide, r_{21} was the dot product of $\hat{X}_B$ with respect to $\hat{Y}_A$. So, that you can shown to be 1 by root 2, r_{22} is the dot product of $\hat{Y}_B$ with respect to $\hat{Y}_A$ so this is 1 by root 2. And how about the Y is r_{33}=1? Because it is the dot product of $\hat{Z}_B$ with respect to $\hat{Z}_A$. So, hence we can find the all the element of this rotation matrix for this simple planar case the square is being rotated about the Z axis.

And we can organize all these things in the form of this rotation matrix. So, the first element is r_{11} so this is one by root 2 second element is r_{21} which is also 1 by root 2 and the third element is 0. Likewise, the second column is the y axis with written in the A coordinate system. So, that isr_{12} is - 1 by root 2, r_{22} is 1 by root 2, r_{32} is 0 and the third column is the $\hat{Z}_B$ axis with respect to in the A coordinate system. So, now $\hat{Z}_B$ and $\hat{Z}_A$ are at the same place so we have 0 0 1.

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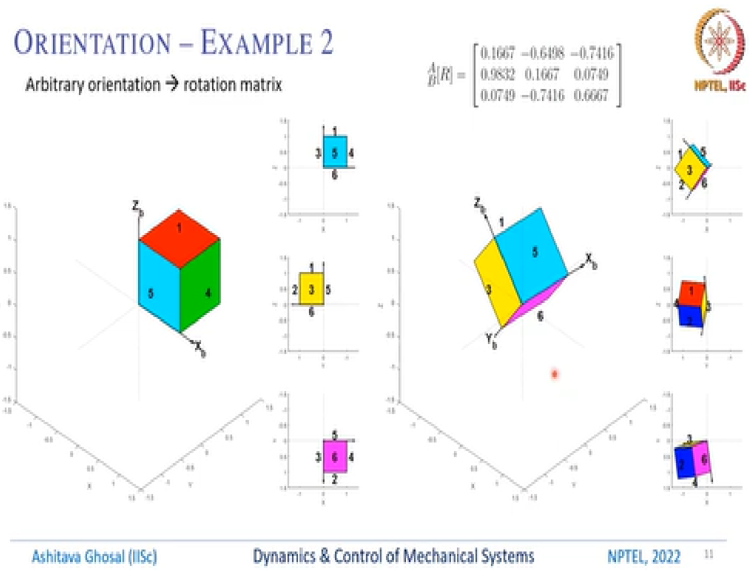

So, let us look at a slightly more interesting and harder example. So, we want to do how to find the rotation matrix for some arbitrary orientation. So, let us say we are given this rotation matrix, and this comes from this example. So, in several examples from now on we will use this cube. So, this cube is like a dice except now instead of marking 1, 2 and 3 we have coloured it different colours. So, 1 means the face with 1, 5 means the face with 5, and 4 means the face with 4.

So, this is also shown in this slide pictures. So, 3 is on the other side which is not visible. So, opposite to 1 is 6 which is here and 5 and the opposite to that will be 2 which is shown in this view. So, 1 and 6 5 and 2 and this is the third one. So, remember in a dice sum of the opposite faces add up to 7. So, now let us look at this cube at these dices in this given orientation. So, here what you can see is this $\hat{X}_B$ which was like this basically along this X axis, parallel to this X axis, Y $\hat{Z}_B$ parallel to the Z axis, X Y in this direction.

Now it has rotated. So, $\hat{X}_B$ is in this direction, $\hat{Y}_B$ is in this direction and $\hat{Z}_B$ is in this direction. So, now you do not see 1, 5 and 4 these faces you see 1, 3, 5 and little bit of so one is on the other side which you actually do not see and little bit of 6. So, the three views of these dice are given in this form. So, the task is that I want to find the orientation of this rigid body. So, again what you can see is because this is done using some software tool in MATLAB.

I can find out what is the dot product of $\hat{X}_B$ with $\hat{X}_A$, $\hat{X}_A$ is in this direction, $\hat{Y}_A$ is in this direction, $\hat{Z}_A$ is in this direction. So, I can find out the dot product of $\hat{X}_B$ with A and it turns out it is 0.1667. So, $\hat{X}_B$ with Y is 0.9832 so first column vector is the $\hat{X}_B$ with respect to the original A coordinate system. Second column is $\hat{Y}_B$ with respect to the original A coordinate system.

So, as you can see it is slightly harder to visualize or to compute but nevertheless the basic definition still hold.

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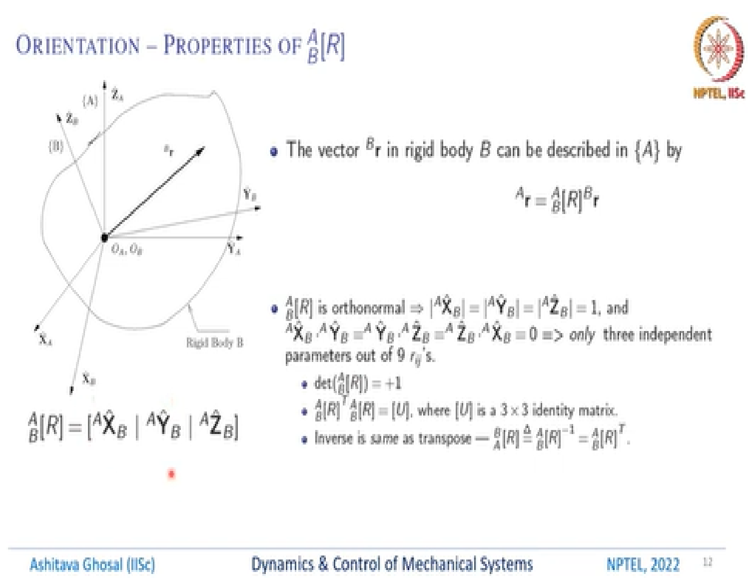

So, let us look at some of the properties of a rotation matrix $BA[R]$. So, as I said we have a rigid body, we have this A coordinate system which is the reference coordinate system, B coordinate system which is attached to the rigid body, the origins are at the same place O_A and O_B. And the first column of this rotation matrix is $\hat{X}_B$ with respect to A, second column is $\hat{Y}_B$ with respect to A, third column is $\hat{Z}_B$ with respect to A.

So, this vector which is locating a point on the rigid body which is Br we can describe this vector in a coordinate system and this is again well known you must have seen it in undergraduate that $BA[R]$ into Br will give you this vector in the A coordinate system. So, if you pre multiply a vector with a rotation matrix you change the coordinate system. The second important property of a rotation matrix is that each of these columns are unit vectors.

Why are the unit vectors? Because it is nothing but the $\hat{X}_B$ axis it is a unit vector except it is described in the {A} coordinate system. So, the magnitude of this each of this axis is still unit vector so the magnitude is 1 . So, $A\hat{X}_B$, $A\hat{Y}_B$magnitude $A\hat{Z}_B$magnitude are all equal to 1. It is also important to see that this $A\hat{X}_B$ and $A\hat{Y}_B$ are perpendicular to each other because remember this is how it was fixed.

The $\hat{X}_B$, $\hat{Y}_B$ and $\hat{Z}_B$ are fixed on the rigid body {B} but they are still a right-handed coordinate system they are still orthogonal to each other. So, we have three constraints here that the magnitude of this column vector is 1 and $A\hat{X}_B \cdot A\hat{Y}_B$ is 0, $A\hat{Y}_B \cdot A\hat{Z}_B$ is 0 and $A\hat{X}_B \cdot A\hat{Z}_B$ is also 0. So, we have three constraints here and then there are three constraints here so this is a 3 by 3 matrix. Remember there were nine r_{ij} 's, i *and* j were going from 1, 2 and 3.

So, there are nine direction cosines in this rotation matrix. But because of these 6 constraints there are only three independent parameters out of this nine r_{ij} 's. This is a very important concept that although the rotation matrix contains nine quantities only three of them are independent. The next important property of this rotation matrix is that the determinant of this rotation matrix is +1 and because the determinant is 1 you can show that the transpose up into this matrix.

So, $BA[R]^T \cdot BA[R]$ is an identity matrix. So, I am going to use U as an I as an identity matrix because I did not use I because I later run, in dynamics, we will denote inertia. So, $BA[R]^T$ into $BA[R]$ will be a identity matrix. So, because of these two properties the inverse is same as the transpose. So, remember any matrix A, inverse of A into A will be identity that is a property of a matrix.

But in this case, it transpose into that matrix is also identity so hence transpose is same as inverse. So, inverse of this rotation matrix will be denoted by $BA[R]^{-1}$. Also, physically what it means is instead of B with respect to A, we have A with respect to B that is what inverse means and that must be that is the same as $BA[R]^{T}$.

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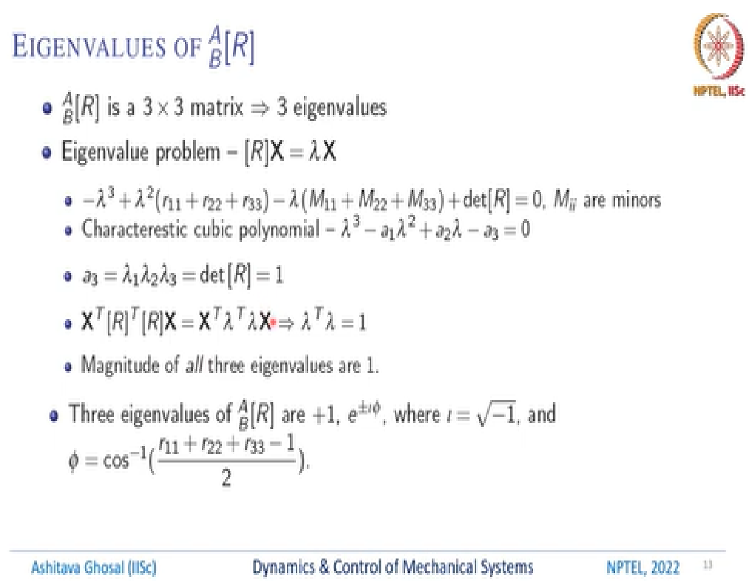

So, let us continue, let us look at what are the eigen values of this matrix. So, it is a 3 by 3 matrix hence there are three eigenvalues and what is the how do we find the eigen values. We state the eigenvalue problem which is R into X will be same as λ X. There is nothing new this is a standard eigenvalue problem for any matrix that A into X is same as λ into X where λs are the eigen values.

In this case if you expand this eigenvalue which is nothing but determinant of R X - λ X = 0. We can find what is called as the characteristic polynomial and the characteristic polynomial is given in this form. It is some - λ^3 + λ^2 into r_{11}+ r_{22}+ r_{33}- λ into this M_{11}, M_{22}, M_{33} are called the miners of this rotation of this matrix and the last term is the determinant of R. This is an expansion of the determinant of R - λ_i = 0.

So, the characteristic polynomial is cubic and can be written in this form - λ^3. So, this is not minus $\lambda^3 - a_1 \lambda^2 + a_2 \lambda - a_3 = 0$. So, a_1 is the sum of the diagonal elements, a_2 is related to the minus and a_3 is the determinant of this matrix R. So, a_3 is given by $\lambda_1 \times \lambda_2 \times \lambda_3$ that is equal to 1. So, this comes from linear algebra.

So, these are called the invariance and one of the invariants is the product of the eigen values. And we know this is also equal to determinant of this rotation matrix which we know is one. So, hence $\lambda_1 \times \lambda_2 \times \lambda_3$ the 3 eigen values the product is always 1. Let us continue little bit more. So, let us from this eigenvalue problem I can rewrite as following. If you take the transpose of left- and right-hand side.

And three multiply by X^T R transpose into R X will be same as R X transpose λ transpose λ into X. Now this is same as λ transpose λ which is 1 . Why is that? Because $R^T X$ is identity. Remember the inverse is the same as the transpose for the rotation matrix so this is identity. So, the left-hand side is $X^T X$ and the right hand side is $X^T \lambda^T \lambda X$.

So, both sides $X^T X$ in some sense can be removed and then we have $\lambda^T \lambda = 1$. So, what this shows is that the magnitude of all three eigen values are 1. And we also have that the three product of the three eigen values is also equal to 1. So, magnitude of each eigenvalue is 1, product of the three eigenvalues is 1 . So, there are very few ways which both of these can be satisfied.

So, one such way which is that the eigen values of $BA[R]$ are 1 and $e^{\pm i\phi}$ Why? Because see if the product of these three is 1 so I can have for example λ_1 is 1 λ_2 is half and λ_3 is 3 then also I will get 1. But then this one is telling me that the magnitude of each of the λs is 1 . So, I cannot have λ_3 as 3 or λ_2 as 1 by 3. So, the only way is that one of the eigenvalue is 1 and the other two are $e^{\pm i\phi}$.

So, what is $e^{\pm i\phi}$? It is nothing but $cos\phi \pm i\, sin\phi$. So, I here means $\sqrt{-1}$ and ϕ turns out that it is cos inverse of this. So, this can be proved mathematically if you go back and use all the properties of the rotation matrix and also go and expand this characteristic polynomial so this is well known fact. So, I am not going to go into the derivation of this.

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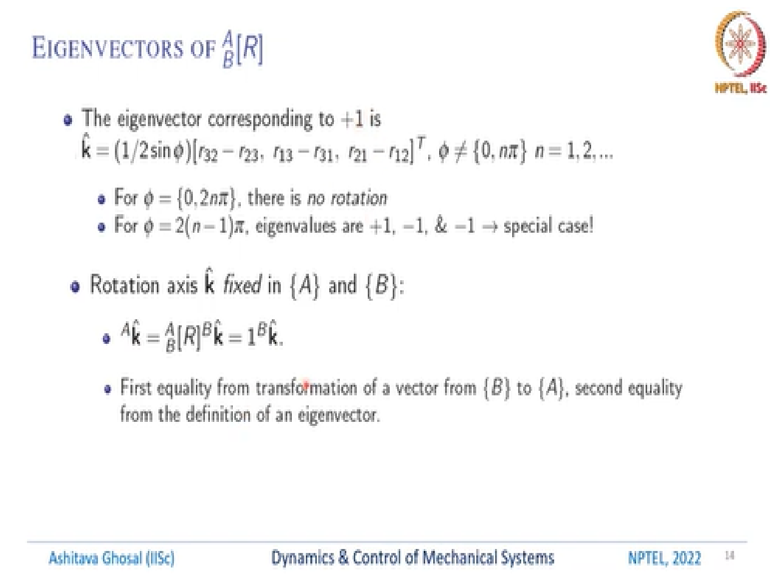

So, one of the eigenvalue is 1 so the eigen vector corresponding to 1 can be written in this form. So, let us call that eigen vector as $\hat{k}$ I am going to write the unit eigen vector. So, that can be written as 1 by 2 sin ϕ multiplied by a column vector which is r_{32} - r_{32}, r_{13} - r_{31}, r_{21} - r_{12}. So, the previous one we saw that the angle ϕ was related to r_{11}, r_{22} and r_{23}. Whereas here the eigen vector corresponding to 1 is related to the other elements of the rotation matrix which is basically r_{32}, r_{32}, r_{13}, r_{31}, r_{21} and r_{12}.

So, it turns out that this can be obtained, and I will show you little later that how we can obtain this. But from basic linear algebra any eigenvalue problems for a real eigenvalue which is 1 we can find what is the eigen vector. So, one important thing to notice here is this is one divided by 2 sin ϕ. So, ϕ cannot be 0 or n π so we will see some of this come little bit of complications later on. So, if ϕ is 0 or 2 n π there is no rotation.

So, ϕ means that there is a rotation angle if the ϕ is 0 there is no rotation. If you have ϕ is 2 n - 1 π so then the eigen values are + 1 - 1 and - 1, this is a special case. In this case you do not have $e^{\pm i\phi}$. So, this rotation axis which is k or this eigen vector which we obtained for this eigen value

1 is fixed in A and B and here is the proof. So, we can write a vector in a coordinate system and another vector which is given in the B coordinate system.

So, if I pre multiply by rotation matrix I get it in A coordinate system. But this $BA[R]$ into $B\hat{k}$ is also same as 1 into $B\hat{k}$ why ? because this is the eigen vector corresponding to the eigenvalue 1 λ. So, remember R X is same as λ X. So, that this is the eigen value problem. So, one side we have rotation of B to give you A and the other side it is the eigenvalue problem. So, first equality comes from the transformation of a vector from B to A and the second equality from the definition of an eigen vector.

So, hence $A\hat{k}$ is same as $B\hat{k}$ or in other words this unit vector $\hat{k}$ which is corresponding to the eigen vector corresponding to the eigen value 1 is fixed in both coordinate system.

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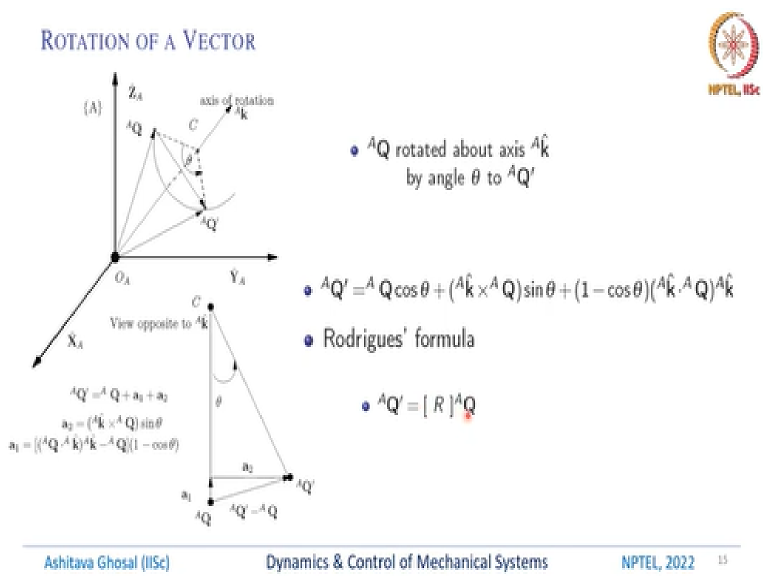

In the last slide I showed you that there is a vector $\hat{k}$ which is fixed in both coordinate system which is also called the axis of rotation. So, in this slide I want to show you that if I rotate a vector AQ about this axis of rotation so we want to find out what is the expression for AQ' which is the rotated AQ about this axis $\hat{k}$. So, in this picture we have a reference coordinate system A and from the origin we have a vector AQ which is rotated about $\hat{k}$ and it goes to AQ' and the angle subtended if you look from this side, is θ .

So, I want to find the expression for AQ' in terms of AQ k and θ. So, if you look at this picture 3D picture from this side opposite to the direction of the rotation axis you can see that AQ' can be given by AQ plus some vector along A_1 and another vector along A_2 . So, the question is what is A_1 and A_2 ? So, what you can see is A_2 will be perpendicular to $A\hat{k}$ and also to AQ . So, it is along a vector which is $A\hat{k}$ cross AQ and this angle sine θ comes because it is this projection. So, you can see that there is a sin θ which will show up A_1 is a vector which is from Q to the centre towards the centre. So, what you can see is it will consist of two parts, one is this AQ dot $A\hat{k}$ along $A\hat{k}$ - AQ . So, this is the direction that one and this 1 – cos θ comes because this entire magnitude cannot be taken.

We just want a small portion of it. So, this is in some sense like 1 and this is like cos θ. So, this is 1 - cos θ so we will get a term like this. So, AQ' will be basically AQ which is this vector + A_1 + A_2 . So, as I said we want to find out AQ' when it is rotated about $\hat{k}$ by an angle θ. So, AQ' is given by AQ into cos θ. So, you can see here you will have minus AQ into 1 - cos θ + some other terms will show up. θ So, hence you will be left with AQ cos θ then AQ so $A\hat{k}$ cross AQ into sin θ which is this along this A_2 and then the rest of it 1 - cos θ into $A\hat{k}$ into dot AQ along this k axis. So, this term is coming from here into 1 - cos θ and - AQ into 1 - cos θ + AQ will left with this term. So, this is a very well-known formula this is called the Rodriguez formula. So, basically what it is telling you is that if I have a vector which is rotated about another vector in 3D space.

I can write the location of the new rotated vector in terms of the original vector A and the axis of rotation and θ. So, this is a very, very famous formula and we will use this formula later on to derive rotation elements of the rotation matrix. This AQ' can also be written as a rotation matrix into AQ . So, this is just like any other transformation.

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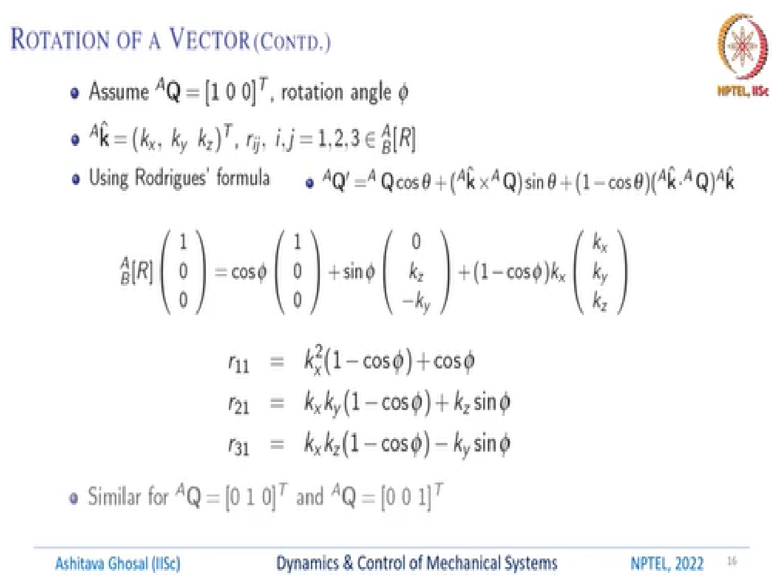

So, let us continue, let us assume that this AQ is 1 0 0. What is 1 0 0? It is the X axis and this is being rotated by an angle ϕ. So, what is the general form of this rotation axis is k_x, k_y, k_z as a column vector. I want to find out what is the rotation matrix $BA[R]$. Basically, I want to find out what is this r_{ij} . So, if you use this Rodriguez formula so we have $BA[R]$ into 1 0 0. So, it is that and basically, we have cos ϕ into 1 0 0 sin ϕ into cross product and 1 - cos ϕ k_x into this. So, this can be seen and if you simplify this you will get r_{11} as k_x square into 1 - cos ϕ + cos ϕ r_{21} is k_x k_y 1 - cos ϕ + k_z sin ϕ and r_{31} is nothing but k_x k_z 1 - cos ϕ - k_y sin ϕ. So, what is this $BA[R]$ into 1 0 0 that is nothing but $\hat{X}_B$? So, this is the first column of the rotation matrix, and these are the elements of the first column of the rotation matrix. So, hence the first column of the rotation matrix can be written in terms of k_x, k_y, k_z which is along the rotation axis and then angle ϕ which is the angle about which it is rotated.

Similarly, if you assume AQ is the Y axis or AQ is the Z axis, we can go back to the Rodriguez formula and then just apply the Rodriguez formula and we can find out what is the second column and the third column of this rotation matrix. But now we are finding this rotation matrix in terms of k_x, k_y, k_z and this rotation angle ϕ.

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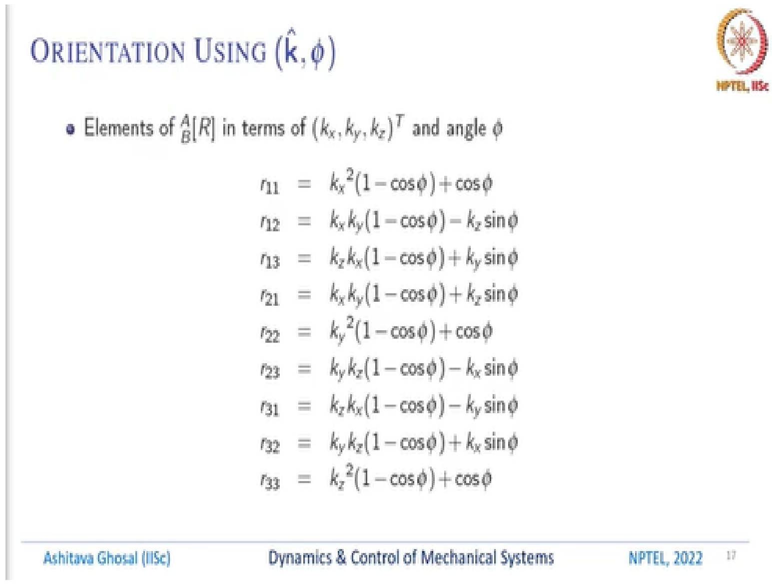

A little bit of algebra will tell you that these are the elements of the rotation matrix, r_{11} is k_x square into 1 - cos ϕ + cos ϕ r_{12} is k_x k_y - 1 - cos ϕ - k_z sin ϕ r_{13} is this and so on so r_{31} is this.

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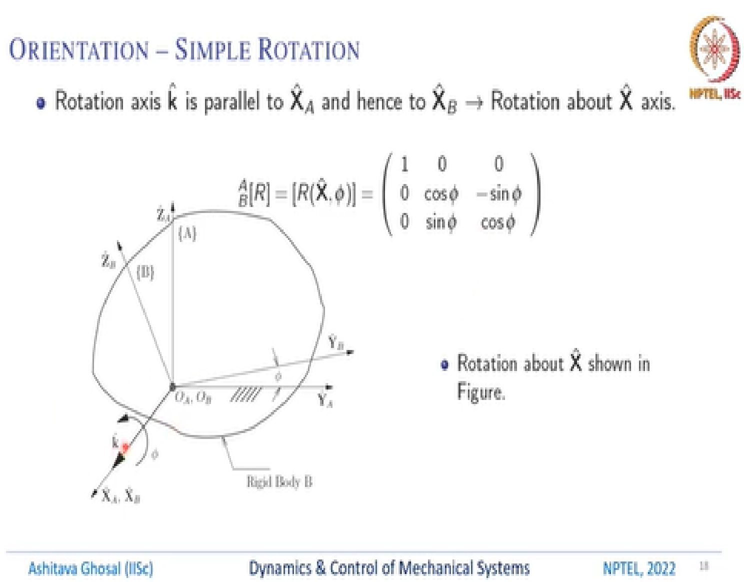

So, once we know this general form of a rotation matrix in terms of k_x, k_y, k_z and this rotation angle about the rotation axis k rotation angle ϕ. So, we can assume that k is parallel to $\hat{X}_A$ and of course hence it is parallel to $\hat{X}_B$. Because remember X k does not change between the two coordinate systems. So, the rotation X axis is fixed in both A and B and as a special case I am going to assume that this rotation axis is the X, Y axis.

So, let us see the picture. So, what we have is this rigid body my rotation axis is the X axis so hence $\hat{X}_A$ and $\hat{X}_B$ are along the same direction and we are rotating this rigid body by an angle ϕ about this k axis. So, again the origins are at the same place. So, this is the rigid body to which we attach the B coordinate system and A coordinate system is the reference coordinate system. So, what is the angle between $\hat{Y}_A$ and $\hat{Y}_B$?

It is ϕ it is the same angle between $\hat{Z}_A$ and $\hat{Z}_B$, only the X axis is aligned $\hat{X}_B$ and $\hat{X}_A$ are at the same place along the same direction. So, let us find out the rotation matrix which is $BA[R]$, I am going to denote this for a reason which we will see very soon by R into X, ϕ. So, basically it is a rotation matrix consisting of a rotation about the X axis by an angle ϕ and we can find the elements of the rotation matrix as 1 0 0 first column.

Why? Because X axis are at the same place so $\hat{X}_B$ and $\hat{X}_A$ are at the same place. This one is 0 cos ϕ sin ϕ and 0 - sin ϕ cos ϕ so is that correct. So, let us see the Y axis in with respect to $\hat{Y}_A$, $\hat{Y}_B$ axis with respect to $\hat{X}_A$, $\hat{Y}_A$ and $\hat{Z}_A$. So, $\hat{Y}_B$. $\hat{X}_A$ will be 0 because remember this is rotation about the X axis. So, $\hat{Y}_B$ dot $\hat{X}_A$ will be 0 which is what you see here $\hat{Y}_B$. $\hat{Y}_A$ is cos ϕ.

This is cosine of this angle, $\hat{Y}_B$. $\hat{Z}_A$ is sin ϕ and likewise for the Z axis 0 - sin ϕ, cos ϕ. So, what is what we see in this figure is a rotation about X axis and this is what the picture is.

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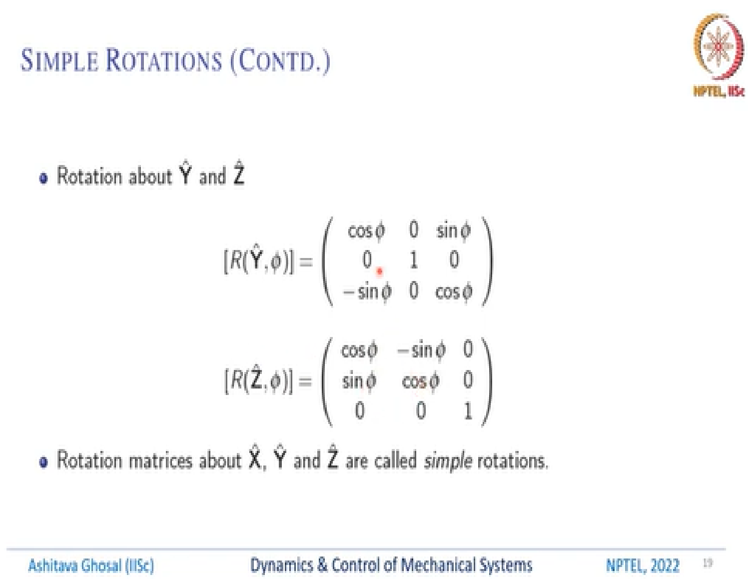

How about rotation about Y axis and Z axis again we can either go and see these pictures and see or we can go back and use the formula in terms of k_x, k_y, k_z and that angle. So, in that case k will be 0 1 0 and this is ϕ angle. So, we will get cos ϕ 0 sin ϕ, cos ϕ 0 - sin ϕ and so on. Similarly, for the Z axis we will get cos ϕ - sin ϕ 0, sin ϕ cos ϕ 0 and 0 0 1. So, I am sure this you have seen in very many basic mechanics problems in undergraduate.

So, if I rotate a rigid object about the Z axis the X and Y components are cos ϕ - sin ϕ sin ϕ and cos ϕ. So, something like this rotation about X, Y and Z are called simple rotations.

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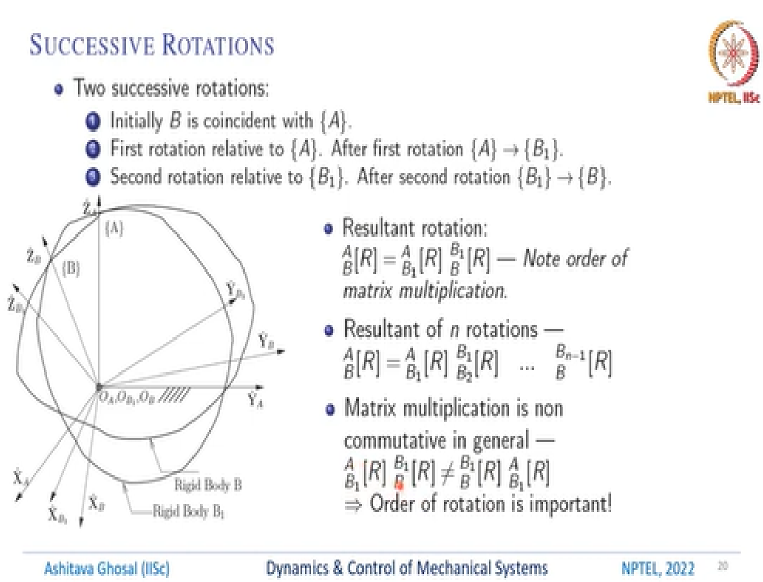

Next let us look at another very important concept which is two successive rotations. So, what I want to mean by two successive rotations is that initially the rigid body B is coincident with A. So, let us this picture so initially $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ is the reference coordinate system, the B rigid body is aligned with respect to this $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$. So, $\hat{X}_B$, $\hat{Y}_B$, $\hat{Z}_B$ at this initial instant is aligned with $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$.

The first rotation is relative to A coordinate system after the first rotation A will go to B_1 . So, the rigid body is now described by a coordinate system $\hat{X}_B$ 1, $\hat{Y}_B$ 1 and $\hat{Z}_B$ 1. So, basically what is happening? You can think of it that this A coordinate system which was the reference coordinate system the rigid body was in this reference coordinate system it has gone to or it has been oriented and gone to another coordinate system which is B_1 .

The second rotation is relative to B_1 it is very important. The second rotation is not relative to A it is related to B_1 , the moved coordinate system. So, after the second rotation $\hat{Z}_B$ 1 will go to $\hat{Z}_B$, $\hat{X}_B$ 1 will go to $\hat{X}_B$ and $\hat{Y}_B$ 1 will go to $\hat{Y}_B$. So, the coordinate system B_1 goes to B. So, there are two successive rotations which are happening first A to B_1 and then B_1 to B. So, the question is what is the resultant rotation matrix?

And it turns out that the resultant rotation matrix which is the rigid body B with respect to A so A B_1 , B_1 B so B with respect to A is nothing but the product of the rotation matrices $B_1A[R]$ and then $BB_1[R]$. So, it is important to notice the order of the matrix multiplication. So, we went from A to B_1 and then B_1 to B. So, from A to B_1 there is a rotation matrix $B_1A[R]$. So, basically B_1 with respect to a then second one is we went from B_1 to B.

So, we have B with respect to B_1 so this and again if you follow my notation used here sort of you can think of B_1 and B_1 cancelling and we are left with A and B. So, the resultant rotation matrix is nothing but the product of the rotation matrices in the sequence it occurred. We went from A to B_1 , B_1 to B so it is not in the opposite order. And if you think about it a little bit you can see that if I have n such successive rotations.

So, I want I go from A to B_1 , B_1 to B_2 all the way from $B_{\{n-1\}}$ to B, I make n's successive rotations. Then the resultant is nothing but the product of all these rotation matrices in the order of the rotations and matrix multiplication is non-commutative in general. So, we cannot switch the order. So, we cannot say that it is A B_1 , B_1 B is not the same as B_1 B, A B_1 . So, if you multiply this matrix before this and you know so if you do $BB_1[R]$ into $B_1A[R]$ it is not the correct one.

It is not the same as $BA[R]$ which is $B_1A[R]$ into $BB_1[R]$. And again, in the notation I have used you can sort of see that here B_1 and B_1 is cancelling and we are left with the superscript A and A subscript B which is nothing but B with respect to A. But in this case, nothing like that is happening. So, this is cancelling out and we are left with some A and B not in the right way.
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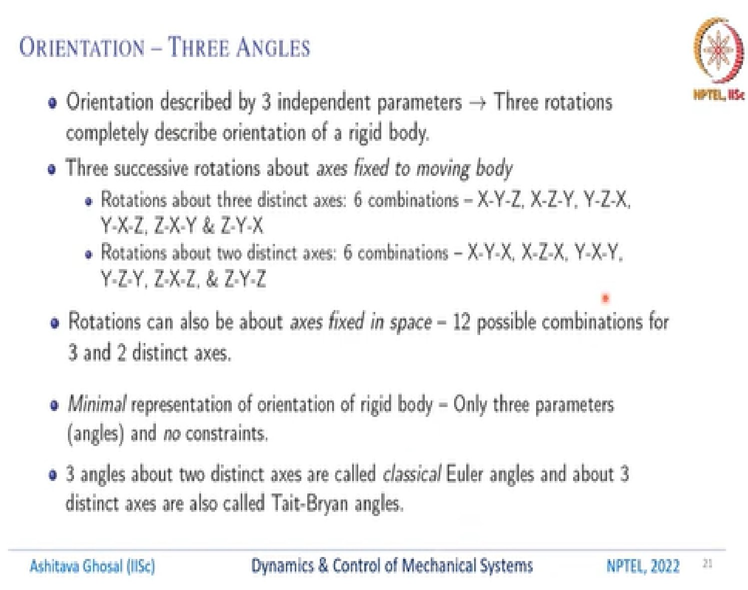

So, important observation is this orientation can be described by three important independent parameters. So, remember in the rotation matrix each column vector has unit magnitude and each of these column vectors are perpendicular to each other say hence out of the nine parameters in the rotation matrix only three were independent. So, we can at least think of representing a rotation matrix completely by three parameters and this is indeed possible.

So, we can do three successive rotations about fixed access or access fixed to the moving body. So, there are two ways of doing it. One is we can have rotations about three distinct taxes. So, we have six combinations we can rotate about X then Y and then Z. But you can change the order we can have X Z Y Likewise Y Z X and so on. You can also obtain by rotation about two distinct axes.

So, a distinct here means combinations like this so you can have X Y X, X Z X and so on. So, there are 12 possible ways of finding rotation of a rigid body by rotating about three axes by three angles in some sense. We can also have rotations about axis fixed in space. So, remember these are access rotated about the moved coordinate system. So, remember we went from A to B_1 and then from B_1 to B.

The second rotation which was with respect to B_1 or the rotated coordinate system. But we can also rotate about the first with respect to the fixed A then with respect to fixed Y and third with respect to fixed Z not with respect to the axis which are attached to the moving body or rotating body. And then there are 12 possible combinations of rotations about access fixed in space and these are about three and two distinct axes.

So, we will see later most of the time we use this axis which are fixed to the moving body and rotations about axis which are fixed to the moving body. This is a minimal representation of orientation of a rigid body only three parameters which are basically angles and no constraint. So, I can have one angle about X, one angle about Y and one angle about Z. So, the product of these three rotation matrices are rotations about X, Y and Z will give me a rotation matrix which represents the orientation of B with respect to A.

So, if you have three angles about two distinct axes so, these ones X Y Z, Z Y Z. These are called classical Euler angles and about three distinct axes which is X Y Z, X Z Y these are also called Tait Bryan angles. In some books all of them are called Euler angles.

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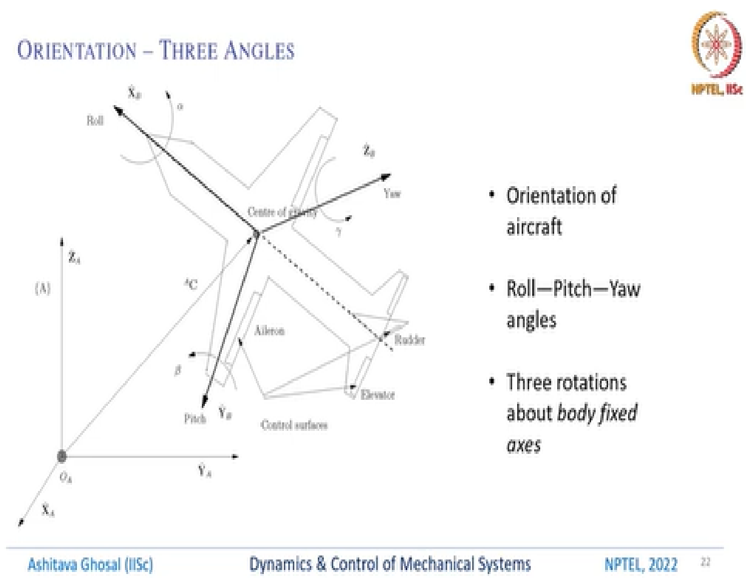

Let us look at orientation of a rigid body using three angles. So, here is a typical example this is a sketch of an aircraft. So, in this figure we have a reference coordinate system which is $\hat{X}_A$, $\hat{Y}_A$, $\hat{Z}_A$ with origin O_A and this aircraft is located by its centre of mass or centre of gravity which is this vector AC. And at the centre of gravity, we have these three axes which is $\hat{X}_B$, $\hat{Y}_B$ and $\hat{Z}_B$. So, $\hat{X}_B$ is along the body or the fuselage of the aircraft, $\hat{Y}_B$ is perpendicular to the fuselage, and $\hat{Z}_B$ is perpendicular in this direction.

So, these are also sometimes called as the role which is rotation about the $\hat{X}_B$ by an angle α is the role. Then rotation about $\hat{Y}_B$ which can be denoted by β as the pitch and rotation about $\hat{Z}_B$ by an angle γ this is called the yaw. So, pitch means the nose of the plane goes up and down, yaw means it is rotating about the Z axis and roll means it is rotating about the X axis and although we are not going to go into the details.

This role, pitch and yaw could be done using what are called as control surfaces. So, we have elevators we have radar and early run. So, the I want to describe the orientation of this aircraft. So, we can describe the orientation of this aircraft by these three angles by this roll angle α , β and pitch angle β and the yaw angle γ . So, these are what are called as three rotations about body fixed axis.

So, remember the axis are fixed to the aircraft and these rotations are about axis which are fixed to the aircraft.

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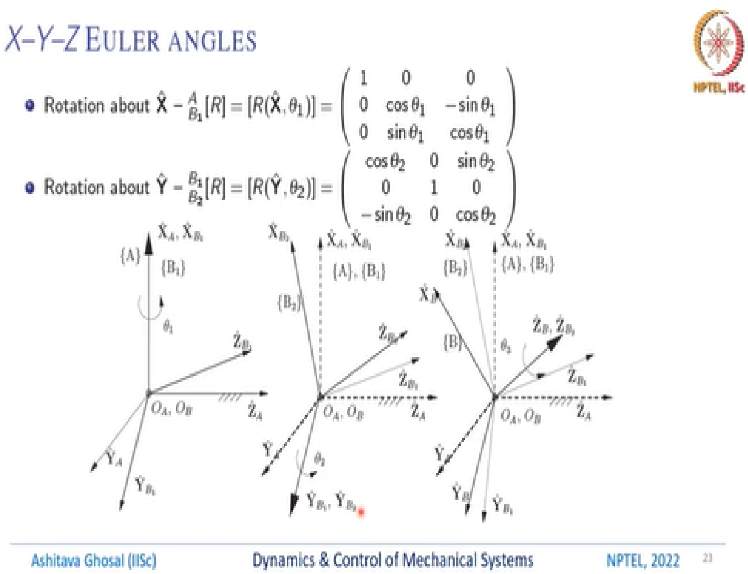

So, if you go back and remember we were going from one coordinate system A to B_1 then B_1 to another coordinate system and then B let us call that B_2 and then from B_2 we go to another coordinate system which let us call that the B coordinate system. So, there are three rotations which are happening. So, the first rotation in this X, Y, Z Euler angles is that we go from A to B_1 . So, this is I have introduced this notation earlier.

It is a rotation matrix about X axis by an angle θ_1 . So, what will be the elements of this rotation matrix? So, you can see this will be 1 0 0, the X axis is same Y axis is given by 0 cos θ_1 sin θ_1 , Z axis is given by 0 - sin θ_1 cos θ_1 . So, this we can obtain using the general formula of r_{ij} in terms of k_x, k_y, k_z and angle ϕ. So, here ϕ is θ_1 which is the rotation about the X axis.

Likewise, we can find the rotation about Y axis which is taking B_1 to B_2 and then we find the elements of the rotation matrix as cos θ_2 0 sin θ_2 , 0 1 0 and so on. So, pictorially what is happening is first rotation is about the X axis. So, A and B_1 are like this $\hat{X}_A$ and $\hat{X}_B$ 1 are at the same place $\hat{Y}_A$ is going to $\hat{Y}_B$ 1, $\hat{Z}_A$ is going to $\hat{Z}_B$ 1. The second rotation is about B_1 it is the moved axis.

So, as you can see $\hat{Y}_B$ 1 and $\hat{Y}_B$ 2 are at the same place and then the third rotation is about the move Z axis. So, now $\hat{Z}_B$ and $\hat{Z}_B$ 2 are at the same place so this is the third rotation θ_3 . So, I have given you the rotation matrix corresponding to rotation about X which is the rotation about Y which is this and pictorially this is what is happening. So, it is important to notice that the first rotation $\hat{X}_A$ and $\hat{X}_B$ 1 are the same place.

In the second rotation $\hat{Y}_B$ 1 and $\hat{Y}_B$ 2 are at the same place and there is a third rotation $\hat{Z}_B$ and $\hat{Z}_B$ 2 are at the same place. So, you can see there are all these different lines depending on which is the fixed axis or about which axis you are rotating and how the other axis is changing.

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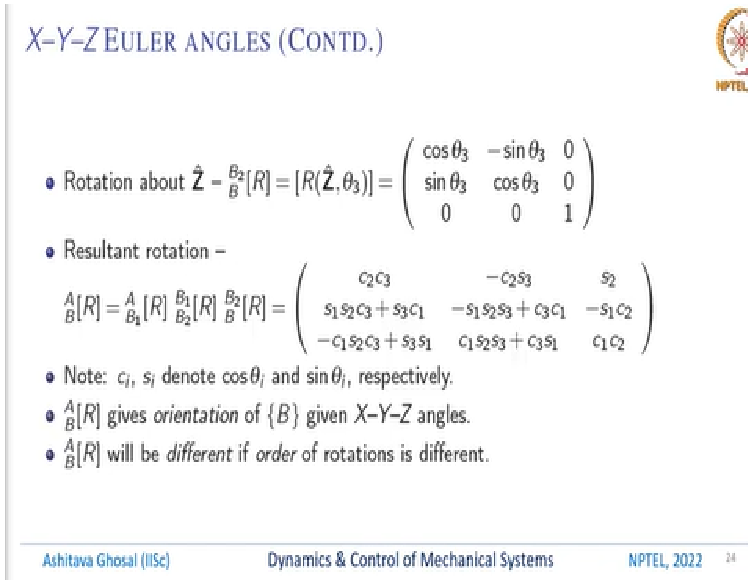

So, the rotation about the Z is given by $R(\hat{Z}, \theta_3)$which is given by this rotation matrix cos θ_3 - sin θ_3 0 sin θ_3 cos θ_3 0 and 0 0 1. So, the resultant rotation as I have mentioned is in the order of the rotations. So, we went from A to B_1 which is rotation about X axis B_1 to B_2 which is rotation about Y axis then B_2 to B which is rotation about Z axis. So, we multiply the rotation matrices in the order in which it happened.

So, then we can see that the resultant rotation matrix is this. So, little bit of math little bit of algebra that you multiply three rotation matrices and then you will get back this as the resultant rotation matrix where c_i and s_i denote the cosine θ and sin θ respectively. So, what is the English meaning of this? This tells you that the rotation of B resultant rotation of the object and B with respect to A is given by cos and sin of this various angles θ_1, θ_2, θ_3.

So, c_2 here denotes cos θ_2, s_1 denotes sin θ_1. So, as mentioned here c_i, s_i denote cosθ_i and sin θ_i respectively. So, it is again I want to stress it again and again that we need to multiply the matrices in the order of the rotations that we did. So, if you were to switch the order you will get completely different terms here and they are not the rotation of this airplane or orientation of airplane with respect to A coordinate system.

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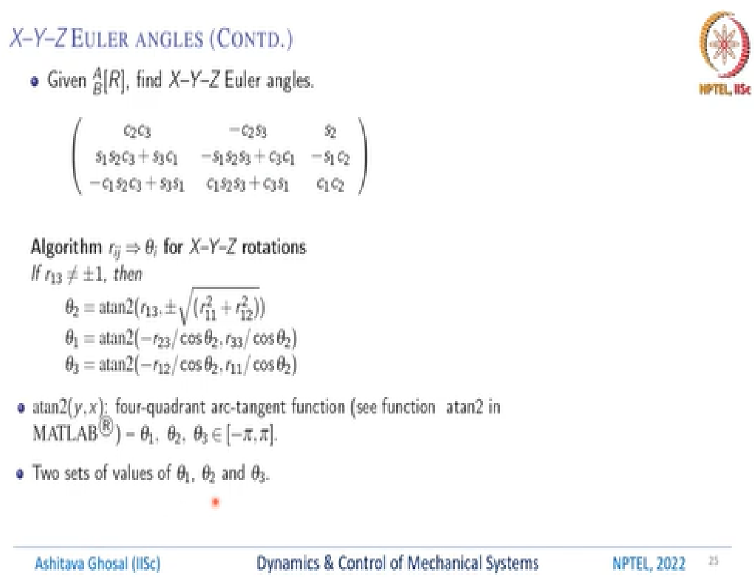

In the last slide we had obtained the rotation matrix from X Y Z Euler angles. So, when we had θ_1 rotation about the X axis θ_2 rotation about the Y axis θ_3 rotation about the Z axis, we obtain the Euler $BA[R]$ which is given by these elements. So, for example the r_{11} is c_2 c_3. Recall c_2 means cosine of θ_2 c_3 means cosine of θ_3. So, for example here stands for sin of θ_1. So, we had obtained this 3 by 3 rotation matrix.

A natural question is if you are given a rotation matrix and if you are told that these are the X Y Z Euler angles can we find out θ_1 θ_2 θ_3? So, some numbers are given. All these 9 numbers for which populate this rotation matrix we want to find out θ_1 θ_2 and θ_3. So, we can look at these elements of the rotation matrix and derive an algorithm. So, let us look at this element of the rotation Matrix.

So, if r_{13} which is sin θ_2 is not equal to ± 1. So, in that sense θ_2 is not equal to plus minus 90. So, then we can find that r_{13} $\pm\sqrt{(r_{11}^2 + r_{12}^2)}$. So, $r_{11}^2 + r_{12}^2$ will be left with cos θ_2. And when you take the square root, you can have 2 plus minus signs. And we can find θ_2 because sin θ_2 is known r_{13} is known and also this term under the square root is known because r_{11} and r_{12} is known.

And we can use this atan2 which is that basically a tangent inverse of Y by X but it gives you in the right quadrant. So, I can find out θ_2 from this expression. Once I find out θ_2 then I can take these 2 terms which is r_{23} and r_{33} I can divide by cos θ_2. So, I will be left with sin θ_1 and cos θ_1. So, θ_1 can again be found as using atan2. Of these 2 terms r_{23} divided by cos θ_2 and r_{33} divided by cos θ_2.

So, again we go back to this thing that if θ_2 is $\pm 90°$ ° or r_{13} is $r_{33} \pm 1$ then cos θ_2 will be 0. So, we could not divide by cos θ_2. So, this is a special case which is why we are in the algorithm. We have a special case of if r_{13} not equal to ± 1 we can find out θ_2 we can find out θ_1 and we can also find out θ_3 because again we can divide r_{12} by cos θ_2 we will be left with sin θ_3.

And we can divide r_{11} with cos θ_2 and we will be left with cos θ_3. Again, we can use atan2 formula and obtain θ_3. So, atan2 is this MATLAB supplied routine which gives you the angle in the right quadrant. It is basically tan inverse of Y by X but it looks at the sign of Y and X. So, tan inverse of minus 1, minus 1 will be in the third quadrant. So, as I said atan2 y, X is a 4 quadrant R tangent function. It is available in MATLAB.

And it will give you the angles in this - π to + π. So, what is the algorithm telling you? That if I give you these 9 numbers, I can get 2 sets of values as θ_1 θ_2 and θ_3.

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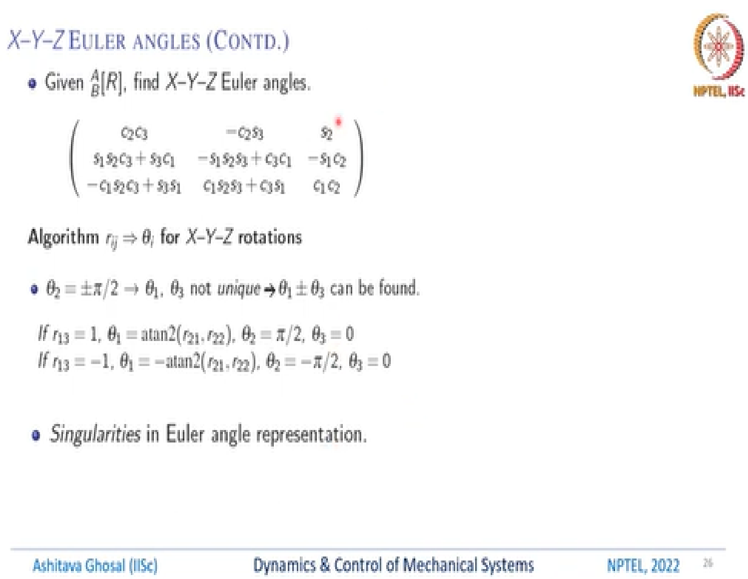

So, let us continue. So, we are basically deriving an algorithm where if you are given the rotation matrix $BA[R]$, we can find X Y Z Euler angles. In the last slide I had showed you that starting from this rotation matrix where sin θ_2 is not ± 1. Then we obtain θ_1 θ_2 and θ_3 . So, let us see what happens when you have θ_2 is $\pm \frac{\pi}{2}$

. So, if θ_2 is $\pm \frac{\pi}{2}$ so this will become 1 these 2 terms will become 0. So, cos $\frac{\pi}{2}$

is 0.

Here also these 2 terms will be 0. And now we are left with these 4 terms so, this is r_{21} r_{31} r_{22} and r_{32}. So, if you substitute θ_2 as $\pm \frac{\pi}{2}$ in this expression so you will have c_3 + s 3 c_1. So, if you go back to your trigonometry this is nothing but sin of θ_1 + θ_3 . So, similarly if you substitute θ_2 as $\frac{\pi}{2}$ here with - c_1 c_3 + s 3 c_1 which is nothing but cos of θ_1 + θ_3 .

So, what you can see is when θ_2 is $\pm \frac{\pi}{2}$ I cannot find out both θ_1 and θ_3 because this term which is the only left term because these are all zeroes everything else has become 0. You will see that it is sin of θ_1 $\pm$ θ_3 . So, we cannot find both θ_1 and θ_3 uniquely. So, we cannot give up. So, there is a convention when you want to find Euler angles given a rotation matrix and in this special case of θ_2 is $\pm \frac{\pi}{2}$

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We make a convention which says the following. If r_{13} is 1 so sin θ_2 is 1 which means θ_2 is + $\frac{\pi}{2}$. Then we can say θ_1 is atan2(r_{21}, r_{22}) it will be atan2 this r_{21} which is sin of $\theta_1 + \theta_3$ and this is cosine of $\theta_1 + \theta_3$. So, we can find out atan2 and then since we know we cannot find θ_1 and θ_3 uniquely. We say θ_1 is this and θ_3 is 0. So, in summary if r_{13} is θ_1 is obtained from this atan2 formula θ_2 is of course $\frac{\pi}{2}$ And θ_3 is chosen as 0. If r_{13} is - 1 then θ_1 will be – atan2 of r_{21}, r_{22} again, we can check these terms, one of them is sin of a + b another one is cos of a + b. And then we say θ_2 is - $\frac{\pi}{2}$ and θ_3 is 0. So, this condition θ_2 is $\pm \frac{\pi}{2}$ is known as a singularity condition. So, this happens in all Euler angle representations. So, there are special angles when we cannot find the other 2 angles uniquely.

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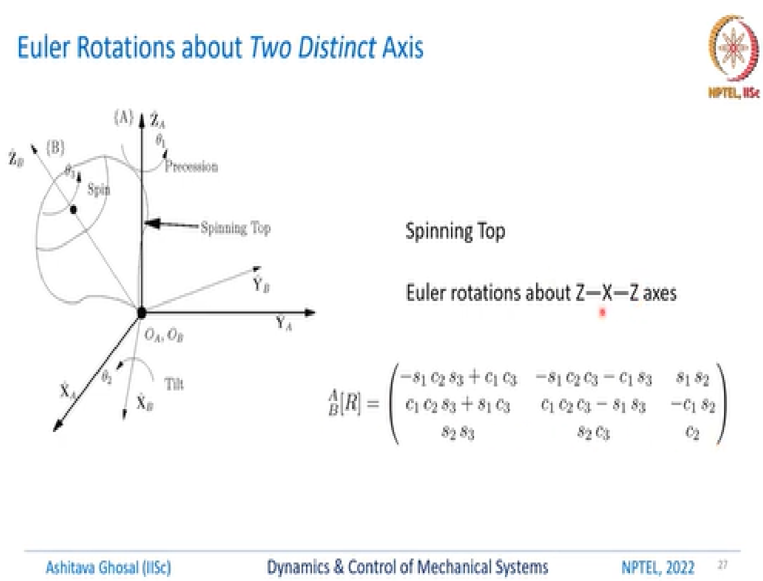

So, is there a practical example for Euler rotations about 2 axes? And the answer is yes. So, one of the well-known problem in dynamics is this problem of a spinning top. So, this is a sketch of a top which is spinning about this point which is fixed. So, O_A and O_B is the point about which this top is spinning. So, typically in a top there are 3 possible angles. One is called precession which is rotation about the Z axis.

Then there is this tilt or this top which can tilt from the vertical that is about the X axis this is θ_2 and then the top itself is spinning about its Z axis which is θ_3 . It is called the spin axis. So, in this case we have Euler rotations about Z X Z. So, you can see Z θ_1 X θ_2 and Z θ_3 . And again, we can find individual rotation matrices rotations about Z, rotations about X, rotations about Z and pre multiply all of them in the order in which it is happening.

And we will get a resultant rotation matrix which looks like this. So, again we can see that 3 3 element is cos θ_2 and so on.

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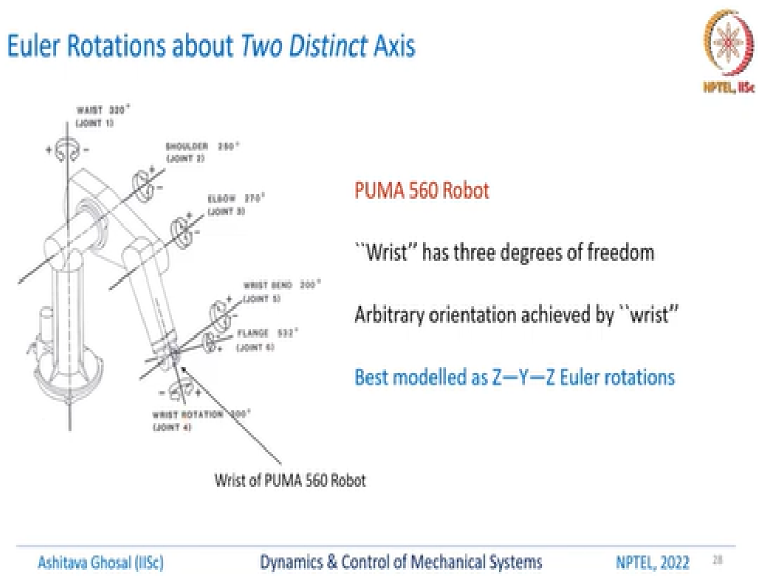

How about rotations about 2 distinct axes? There are many such examples in actual robotics and various other places. One such example is this robot. So, this is the sixth degree of freedom robot. It is a very well-known robot called the Puma 560 robot. It consists of 6 motors. So, one rotation is about this vertical axis, waist then shoulder then elbow. More importantly as far as we are concerned here there are 3 rotations which are happening at the wrist.

So, wrist bend flange and wrist rotation. So, in this case for this Puma robot the wrist has 3 ° of freedom. You can achieve arbitrary orientation by the wrist, and it can be modelled as Z Y Z Euler rotations. So, we do not want to go into too much detail. But in a robot the Z axis is typically the rotation axis. So, in one case you have one Z axis then you have another one which is perpendicular to the Z which is the Y axis and again there is a third rotation which is happening about the Z axis.

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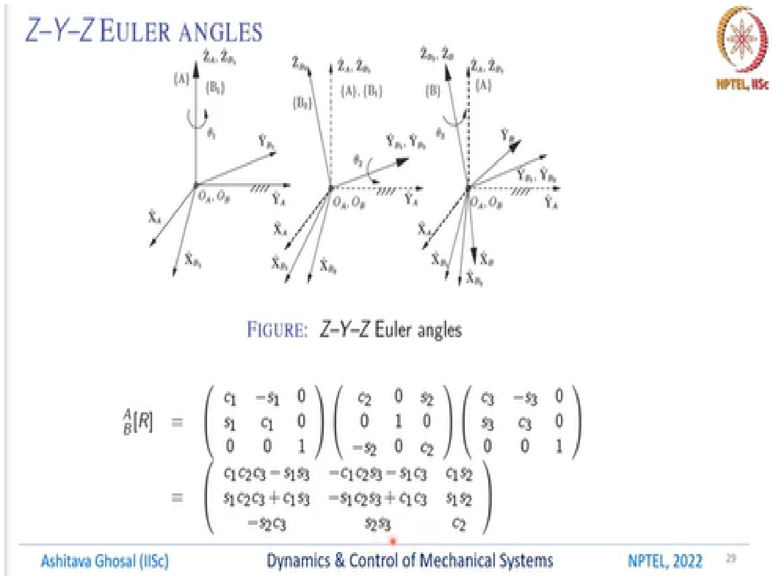

Next, we have an example of Z Y Z Euler angles. Basically, there are 3 consecutive rotations about the Z axis, then the Y axis and then again, the moved Z axis. Pictorially what is happening is what you can see is that the first rotation $\hat{Z}_A$ and $\hat{Z}_B$ 1 are at the same place so, this is θ_1 . The second rotation is about the Y axis. So, $\hat{Y}_B$ 1 and $\hat{Y}_B$ 2 are at the same place and rotation axis is Y axis it is about θ_2 . And the third rotation is about the moved Z axis.

So, $\hat{Z}_B$ 2 and $\hat{Z}_B$ are at the same place so, this is θ_3 . So, the locations or the orientation of the different access are shown in all these pictures. So, how do I find what is the resultant rotation matrix? So, basically, we multiply 3 rotation matrices first one is about Z axis. So, here c_1 means cos θ_1 - sin θ_1 and so on. And you can see that the Z axis is the fixed axis so 0 0 1 that does not change. Then the second rotation is about the Y axis moved Y axis.

So, cos θ_2 , 0, sin θ_2 and again the Y axis is not changing, and the third rotation is back to Z but it is the moved Z. so, in this picture it is about the moved Z axis. So, again we have this rotation matrix. And we can multiply all these 3 out and then you can see that you get a resultant rotation matrix which looks like this. So, the important thing here is that the 3 3 element is cos θ_2 . This element is sin θ_2 sin θ_3 .

And the three one element is - sin θ_2 cos θ_3 and these 1 3 elements is cos θ_1 sin θ_2 and this one is sin θ_1 sin θ_2 . So, you can notice that the rotation matrix here is very different from the previous rotation matrix.

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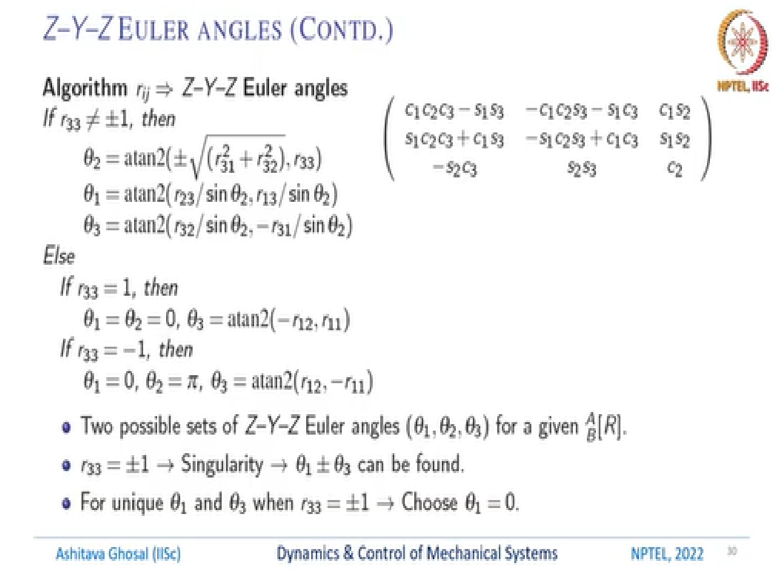

And again, we can find the algorithm which given r_{ij} are given the rotation matrix. And I know it is a Z Y Z Euler angle I can find out all the 3 angles θ_1 , θ_2 and θ_3 . So, for example θ_1 is given in this form θ_2 is given in this form θ_3 is given in this form. So, again here you can see that there is some problem in sin θ_2 is 0. So, when r_{33} is $\pm$ 1 you have a singularity, and we cannot find $\theta_1 \pm \theta_3$.

So, this is a typical thing which happens in all Euler angles. So, there are certain angles which you cannot find uniquely for some cases. And again, we can find out θ_1 is this θ_2 is given by this in terms of r_{ij} . So, first we find θ_2 then we find θ_1 then we find θ_3 . So, that is similar to whatever we have done before. And then there are these special cases if r_{33} is 1 then we find θ_1 to 0 and θ_3 is this.

If r_{33} is - 1 then θ_1 is 0 θ_2 is π and θ_3 is this. So, again for a given rotation matrix the 3, Z Y Z Euler angles θ_1, 2 and 3 there are 2 possible sets.

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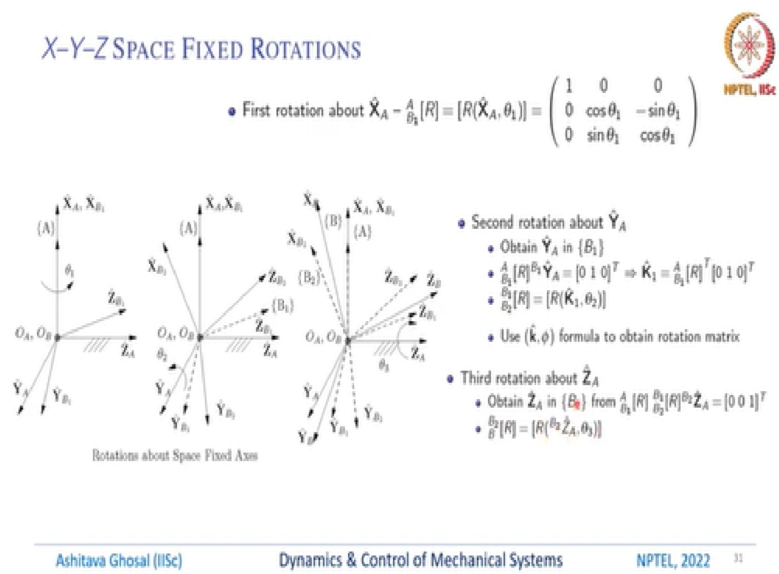

We can also have rotations about axis which are fixed in space. So, these are called space fixed rotation. So, here is an example. So, first rotation is about the X axis. So, you can see $\hat{X}_A$ and $\hat{X}_B$ 1 is at the same place and this is rotation about θ_1. The second rotation is about the Y axis and not the $\hat{Y}_B$ 1 axis not the moved B axis but the original Y axis. So, it is about $\hat{Y}_A$ by θ_2 and finally the third rotation is about the Z axis.

And again, it is not this moved Z axis in which case would have been $\hat{Z}_B$ 2 but it is about $\hat{Z}_A$. So, what we are doing is 3 successive rotations but not about the body fixed access. It is about the original reference or X Y and Z in the space fixed axes. So, as I said the first rotation is about X axis. So, I go from A to B_1 and we can find this rotation matrix very easily. So, $\hat{X}_A$ and $\hat{X}_B$ 1 is at the same place.

So, we have 1 0 0 X axis is at the same place Y axis you know cos θ_1 and sin θ_1 and so on. The second rotation is about $\hat{Y}_A$ I know the formula if I rotate about the moved Y axis. But I do not know where the moved Y axis is. So, what we want to do is we want to find out $\hat{Y}_A$ in the moved coordinate system in the rotated coordinate system B_1. And we can find this? Yes. So, we know that this rotation matrix $B_1 A[R]$ into B_1 $\hat{Y}_A$ is 0 1 0.

Why? Because this is I am rotating about the A axis. So, A B_1 into B_1 this will give me the Y axis in its original coordinate system in its own reference coordinate system so, it is 0 1 0. So, solving this equation I can show that $\hat{K}_1$ is A B_1 transpose R transpose into 0 1 0. So, inverse of this is same as transpose. So, I want to find out what this Y axis is in the B coordinate system. Let us call it $\hat{K}_1$ and this can be found out by this simple transpose of the rotation matrix into 0 1 0.

Likewise, now I can know what is B_1 to B_2. So, I can find out where is B_2 with respect to B_1 by rotating about K in by θ_2. And can I find this? Yes, because I have the general formula for a rotation matrix given an axis and an angle long time back, I had shown you r_{11} in terms of k_x square into 1 - cos ϕ and so on. So, various r_{ij}'s in terms of k_x, k_y, k_z and ϕ. So, I know what is $\hat{K}$ I know; what is ϕ in this case it is θ_2.

So, hence I can use the general formula to find B_2 with respect to B_1 rotation matrix. The third rotation is about the $\hat{Z}_A$. Again, I want to find out why this $\hat{Z}_A$ axis is with respect to the moved coordinate system. So, I want to find out what is $\hat{Z}_A$ with respect to the B_2 coordinate system. So, can I find that out? Yes. So, again we use this simple relationship that $B_1 A[R]$ into

$B_2B_1[R]$ into B_2 $\hat{Z}_A$ should be equal to 0 0 1 because you can see B_1 B_1 cancels B_2 B_2 cancels.

And you have A $\hat{Z}_A$ which is 0 0 1. So, do I know what is $B_1A[R]$? Yes, this is $B_1A[R]$. Do I know what is $B_2B_1[R]$? Yes, I know this from this formula once I expand it. So, then I can find out B_2 $\hat{Z}_A$ $\hat{Z}_A$ in the B_2 coordinate system. And then I can find the rotation meter is going from B_2 to B and this is B_2 $\hat{Z}_A$ by θ_3 . So, this is that K ϕ formula I can easily find this out.

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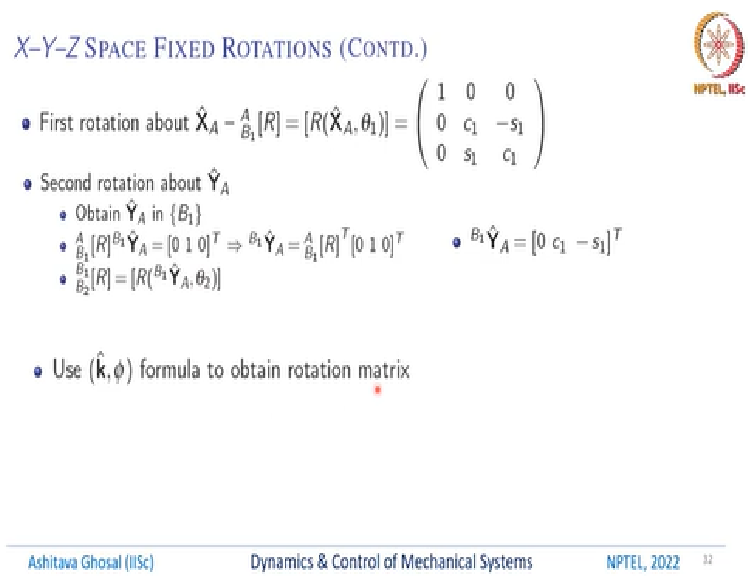

So, as mentioned in the last slide the X Y Z space fixed rotation the first rotation is about $\hat{X}_A$ axis. Remember A is the fixed coordinate system. So, we have a rotation matrix $B_1A[R]$ which is nothing but rotation by angle θ_1 about the $\hat{X}_A$ axis. We can obtain the rotation matrix it is 1 0 0 first row 1 0 0 first column. This is c_1 - c_1 so, is sin θ_1 c_1 is cos θ_1 . The second rotation is about $\hat{Y}_A$.

So, we need to find out $\hat{Y}_A$ in the moved coordinate system in the B_1 coordinate system. And that we can obtain by using this transformation which is $B_1 A[R]$ into B_1 $\hat{Y}_A$ is nothing but 0 1 0. The $\hat{Y}_A$ in its own coordinate system is 0 1 0. And again, we can see that B_1 and D 1 will cancel out. So, we will get $\hat{Y}_A$ in the A coordinate system which is 0 1 0. So, from this formula we can find out what is B_1 $\hat{Y}_A$ which is nothing but you take the inverse of this rotation matrix multiply on the left and right.

The inverse is same as the transpose. So, you will get B_1 $\hat{Y}_A$ is $B_1 A[R]$ transpose 0 1 0. So, hence we can find out what is B_1 $\hat{Y}_A$. And if we want to go from B_1 to B_2 coordinate system so the rotation is θ_2 about the $\hat{Y}_A$ in the B_1 coordinate system. Again B_1 $\hat{Y}_A$ after doing this transformation we can find out that it is 0 c_1 - s_1 column vector 0 c_1 - s_1. And we can find out what is this B_1 B_2 R by applying this and we again use the k ϕ formula to obtain the rotation matrix.

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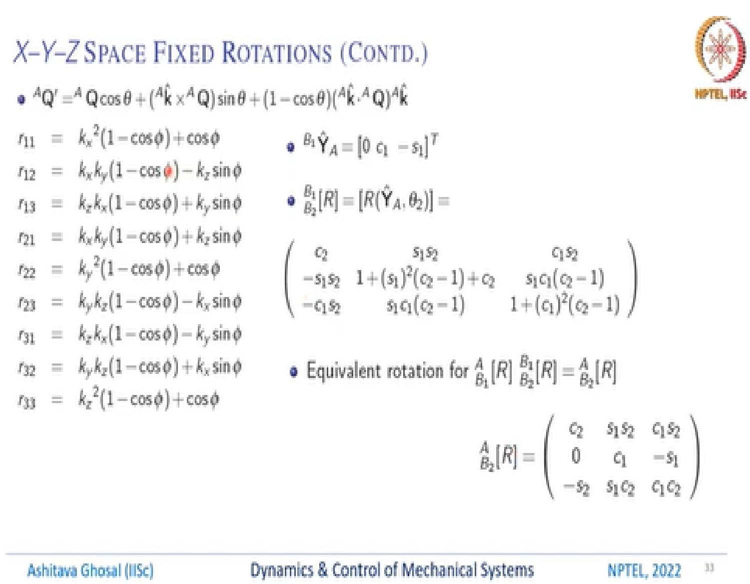

To recapitulate we obtain the k ϕ formula using this rotation of an axis of a vector AQ by an angle θ about the k axis. And this was the famous Rodriguez formula which is AQ' whereas

AQ cos θ + $\hat{Ak}$ cross AQ into sin θ + 1 - cos θ $\hat{Ak}$ dot AQ along $\hat{Ak}$ vector. And this formula was derived earlier. So, we obtained r_{11} which is nothing but k_x^2 (1 - cos ϕ) + cos ϕ and so on. So, for example r_{33} is k_z^2 (1 - cos ϕ) + cos ϕ.

So, we had seen these formulas before. So, we use the Rodriguez formula and then we obtain the rotation matrix elements r_{ij} in terms of k_x, k_y, k_z and the rotation angle cos ϕ. So, now we have the rotation x axis is this $\hat{Y}_A$ in the B coordinate system. So, the k here is 0 c_1 - s_1 and then $B_2B_1[R]$ it is the rotation about this $\hat{Y}_A$ by θ_2. So, if you substitute k as 0 c_1 - s_1 and then you have θ_2 which is the rotation about this k axis. And then you use these formulas for r_{ij} you will get the rotation matrix going from B_1 to B_2. So, it is a little bit complicated. Some simplification is required. So, you can see that the r_{11} is cos θ_2 r_{12} is sin θ_1 sin θ_2 r_{13} is c_1 s_2, r_{21} is - s_2 r_{31} is - c_1 s_2 the r_{22} term is much more complicated. It is 1 + $s_1^2(c_2$ - 1)+ c_2. This term is c_1 into c_2 - 1 and so on. So, it is a rotation matrix.

Because what have we done? We have used this k ϕ formula. Now k is not one of the X Y or Z axis. The k is 0 c_1 - s_1 and the ϕ here corresponding to θ_2. So, if you just substitute these X you know in these expressions you will get this rotation matrix. So, now we have obtained from A to B_1 and then B_1 to B_2. So, if you multiply A B_1 into B_1 B_2, we will get $B_2A[R]$. And then if we have seen earlier what was A B_1 it was nothing but a rotation about the X axis.

And then $B_2B_1[R]$ is this complicated rotation matrix. And then if you multiply these 2 you will get a nice simple rotation matrix which tells you what is B_2 with respect to A. So, you will get c_2 s_1s_2 c_1s_2 0 c_1 - s_1 then - sin θ_1 sin $\theta_1 cos$ θ_2 c_1 c_2. So, all these squares and product of square into (c_2 - 1) and so on will be simplified and you will get this simple rotation matrix.

And this rotation matrix is a result of 2 rotations about X and Y. Both the X and the Y are the space fixed rotation matrices about space fixed axis.

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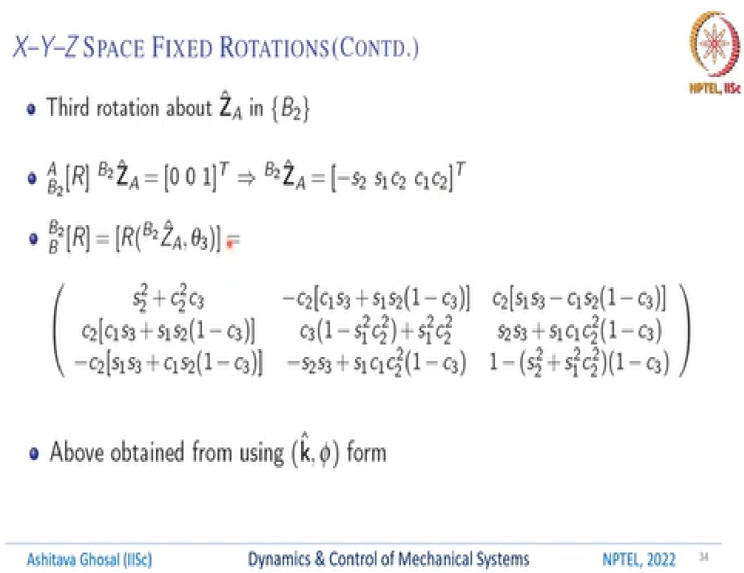

So, in the X Y Z space fixed rotations the third rotation is about the Z axis and this is the original Z axis the $\hat{Z}_A$ axis. So, the fixed axis which is $\hat{X}_A$ $\hat{Y}_A$ $\hat{Z}_A$. And how do we find out what is this $\hat{Z}_A$ axis in the moved B_2 coordinate system. So, again we can use this formula which is $B_2A[R]$ into B_2 $\hat{Z}_A$ will be 0 0 1. Again, you can see that these B_2 and B_2 will cancel out sort of and then we are left with these 0 0 1 the Z axis in its own coordinate system.

And now again we can pre multiply by $B_2A[R]^T$. So, then B_2 $\hat{Z}_A$ will be that $B_2A[R]^T$ into 0 0 1. And we have obtained what is $B_2A[R]$ in the previous slide. So, hence $B_2\hat{Z}_A$ will be given by this column vector - s_2 c_2 c_1 c_2. And again, we can use the k ϕ formula. Now in this case k is this axis which is not a 0 0 1 or 1 0 0 one of those simple rotations. But it is a rotation about an axis which is - s_2 s_1c_2 c_1 c_2 and the angle of rotation about this axis is θ_3.

So, if you go back and substitute k is this and θ_3 in that k ϕ formula, we will get some really complicated rotation matrix. And what is this? This is B_2 and it is finally going to B. So, we will get this expression which is $s_2^2 + c_2^2$ into c_3 and so on. So, as you can see now the rotation matrix is much more complicated. So, for example this r_{33} term is $1 - (s_2^2 + s_1^2 c_2^2$ into (1 - c_3) r_{11} is $s_2^2 + c_2^2$ into c_3 .

So, you will get this very complicated terms. And this can be again done if you are careful, or you can use this one of this computer algebra tools to obtain this rotation matrix. And this as I said this rotation matrix was obtained from again using k ϕ where now the k axis is this axis - s_2 $s_1 c_2$ $c_1 c_2$ and ϕ here corresponds to θ_3 .

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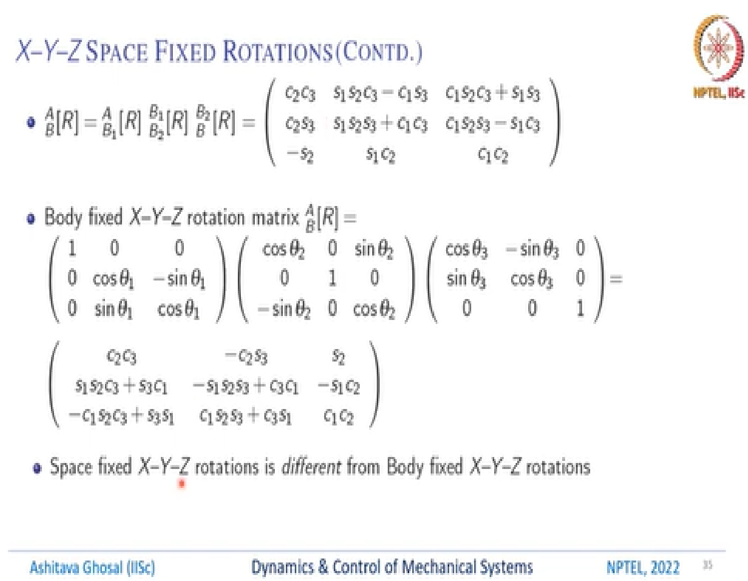

But finally, if I multiply these 3 in the order that it has happened remember we went from A to B_1 , B_1 to B_2 , B_2 to B, A to B_1 was rotation about X. This is about that K_1 this is about K_2 and then if you multiply all these things, you will get a nice much simpler rotation matrix example. So, the x axis is $c_2\, c_3$ $c_2 s_3$ - s_2. So, the Z axis is $c_1\, s_2\, c_3 + s_1 s_3$ and so on. This looks sort of little bit more familiar but clearly it is not the same as X Y Z about body fixed access.

So, if you were to do about body fixed X Y X rotations then it is much simpler. This is the body fixed X rotation this is the body fixed Y rotation this is the body fix Z rotation. And you multiply it again in that order it happened, and you will get this rotation matrix. So, what you can see is these 2 are not anywhere similar. So, for example this is $c_2 c_3$ so this one look okay $c_1 c_2$ it looks okay.

But here it is s_2 whereas here it is this complicated term which is $c_1 s_2 c_3 + s_1 s_3$ whereas here it will be just s_2. So, the body fixed X Y Z rotation and the space fix X Y Z rotations are not the same. So, space fixed X Y Z rotation is different from body fixed X Y Z rotations.

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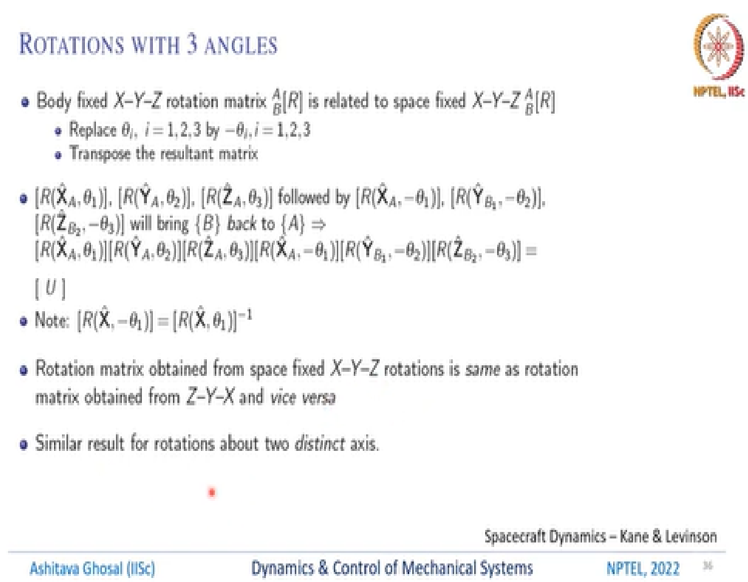

However, they are related. This is a very interesting observation. So, if you were to take the body fixed X Y Z rotation matrix replace all θ_i, i=1, 2, 3 by - θ_i, and then transpose the resultant matrix you will get the space fixed rotation matrix and this can be proved. So, if you think about it if I rotate about the space fixed X axis by θ_1 then Y axis by θ_2 then Z axis by θ_3 followed by X axis by - θ_1 followed by body fixed Y axis by - θ_2 and body fixed Z axis by - θ_3 .

It will bring B back to A. You can draw it yourself and see, so hence this order of matrix multiplication R($\hat{X}_A$, θ_1) ($\hat{Y}_A$, θ_2) ($\hat{Z}_A$, θ_3) followed by ($\hat{X}_A$,- θ_1) ($\hat{Y}_{B1}$, - θ_2) ($\hat{Z}_{B2}$, - θ_3) will give you identity. And what is R (X ,- θ_1) ? It is nothing but the inverse of the matrix. So, more details of this you can see in this book by Kane and Levinson. So, hence rotation matrix obtained from space fixed X Y Z rotation is same as rotation matrix obtained from Z Y Z and vice versa.

So, this is in many books that you say that if you do X Y Z then if you invert sometimes the invert the order, they will say this is same as Z Y X and this is what is exactly happening. So, body fixed X Y Z is same as space fixed Z Y Z and vice versa. And you can get similar results for rotations about two distinct axes.

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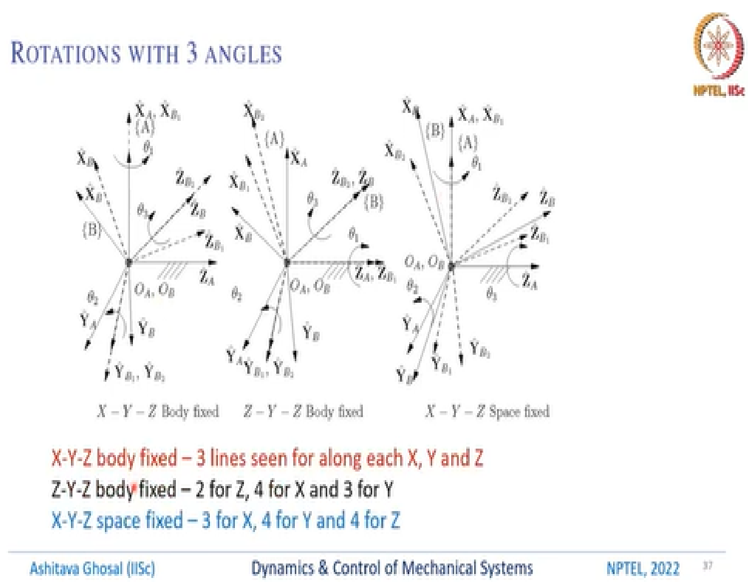

So, just let us look at pictorially what is happening with rotations about 3 angles. So, if you have X Y Z body fixed so first, I rotate about X then I rotate above the moved Y and then I rotate about the moved Z. So, this is $\hat{X}_A$ $\hat{X}_{B1}$ is same. Then I rotate about $\hat{Y}_{B1}$ then I rotate about $\hat{Z}_{B2}$. So, you can count the number of lines. So, these are the X axis Y axis and Z axis. So, what you can see is there are 3 X axes one is $\hat{X}_A$, $\hat{X}_B$ then $\hat{X}_B$ and then $\hat{X}_{B2}$ and so on.

Similarly, there are 3 Y axis lines and there are 3 Z axis lines. If you have rotations about 2 distinct axis body fixed which is Z Y Z. So, meaning what? First rotation is about Z then it is rotated about the new Y. And then it is rotated about finally again the new Z or the moved Z. So, you can see that there are 2 lines about Z and there are 3 lines about Y and there are 4 lines about

Z for the X- axis. If you see space fixed on the other hand what you can see is the first rotation is about X.

So, there is one $\hat{X}_A$ and $\hat{X}_{B1}$ is same. Then the next rotation is about $\hat{Y}_A$ and the third rotation is about $\hat{Z}_A$. So, you can see the number of lines. So, there are 4 lines in Y. There are 4 lines in Z and there are 3 lines in X. Actually, two of these are coincident here. So, the reason why this picture is drawn here is you can get a good geometric field as to how many different lines are there in each one of these different kinds of Euler angles rotations and they are different.

So, this is 3, this is some 3 2 4 and this is 4 4 3. And this is mentioned here. So, X Y Z body fix 3 line seen along each X Y and Z, Z Y Z- 2 for Z for X -3 for Y. X Y Z -3 for X -4 for Y and 4 for Z.

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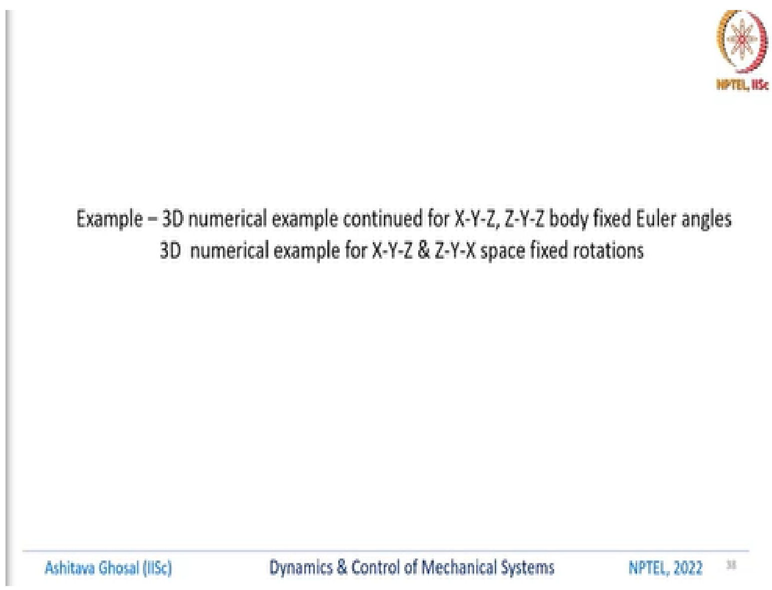

So, let us look at some examples.

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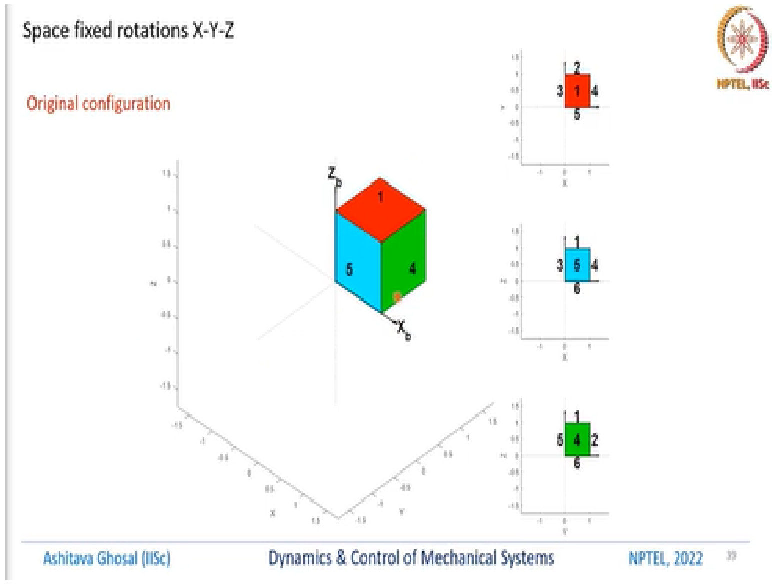

So, these examples were all created in MATLAB. So, we will quickly go through them. So, let us see this cube again that dice and the original configuration is in this form. So, the top one is 1 , this is 4 and this is 5. And then the other 3 views you can see what the other views. So, the bottom will be 6 which is shown here and this side will be 3 which is shown here and so on. So, if I rotate about x axis it will look like this.

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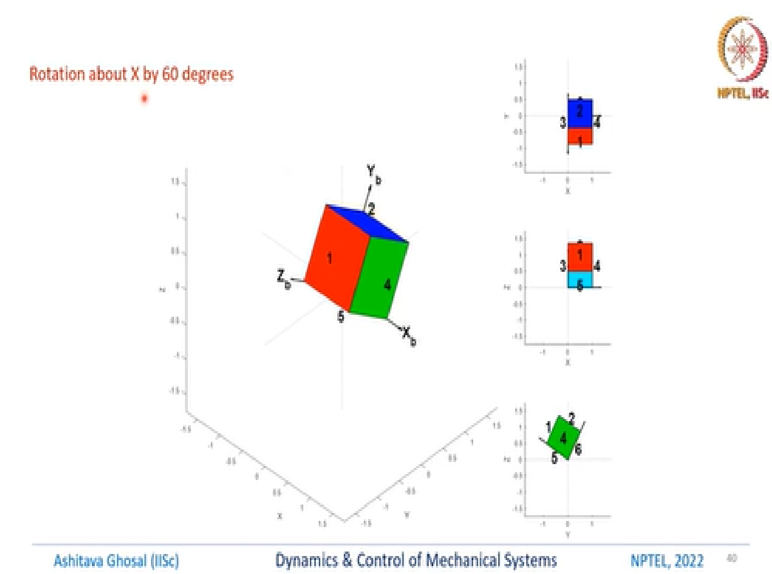

So, if I rotate by 60 ° so X b and the same new X are at the same place. The Y and Z looks like this. So, now you can see that you see different faces you see 1, 2 and 4 is same. But you see 1, 2 and something else is happening.

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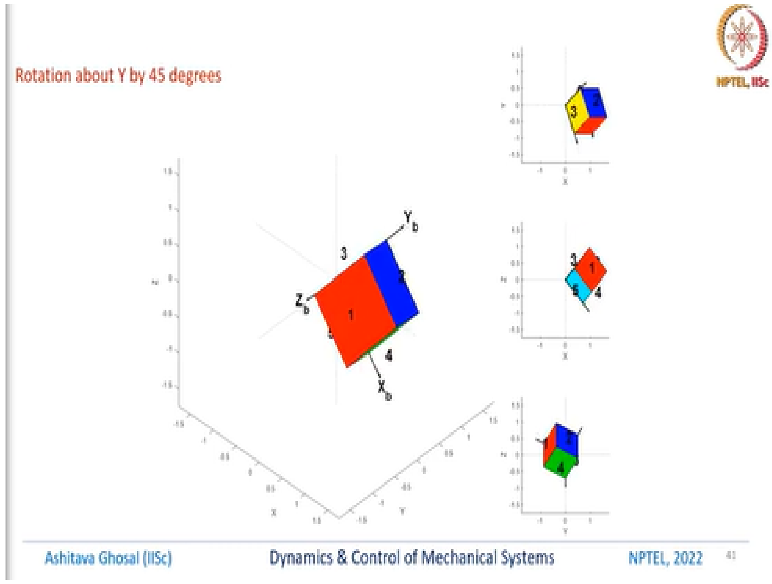

If you rotate now about Y axis by 45 ° it will look like this. So, you see 2 second phase 1 phase a little bit of you know maybe the fourth face. So, it looks different.

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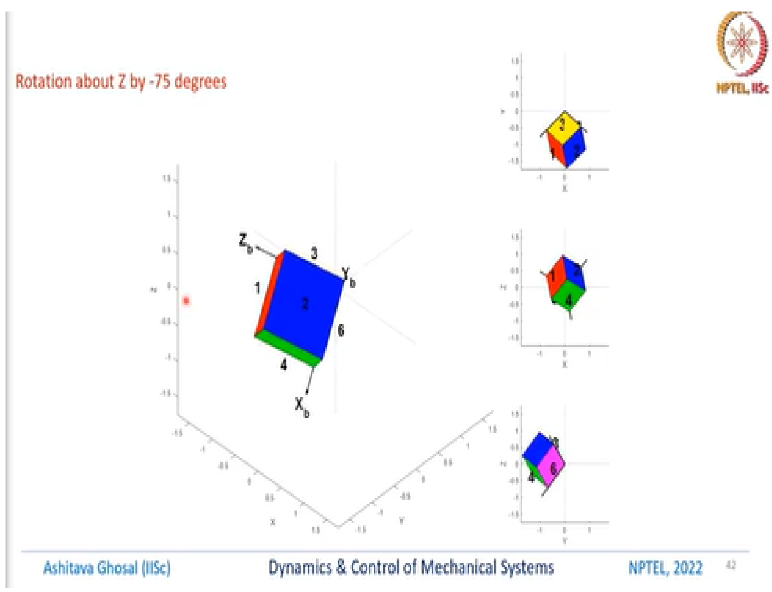

And finally, if I rotate about Z by - 75 ° it looks like this. So, these angles were chosen randomly. So, there is a program which is available which you can rotate this cube or this dice and see how it looks like. And you can see that these rotations are different depending on which way you rotate. And what is the sequence of rotations?

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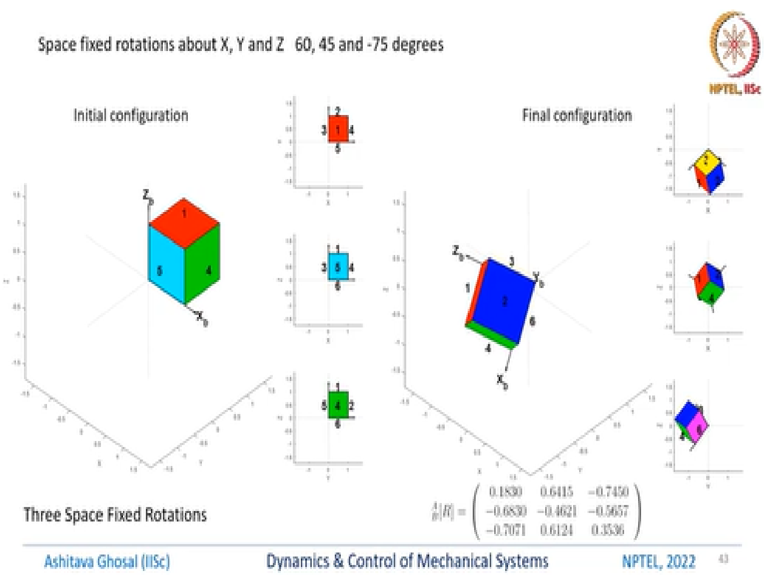

So, here is an initial configuration here is the final configuration and this is what it looks like. So, we have rotated about the original X axis Y axis by 45 ° and the original Z axis by - 75 °. And this is what it will look like the initial and the final configuration.

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And here is a video which shows how these rotations are happening. So, what you can see here is the rotation matrix as it is changing. If it is rotating about $\hat{Z}_A$ axis in this case. But you can see the third row is not changing similarly when it was rotating about the X axis the third row. So, in space fixed rotations the rows will not change.

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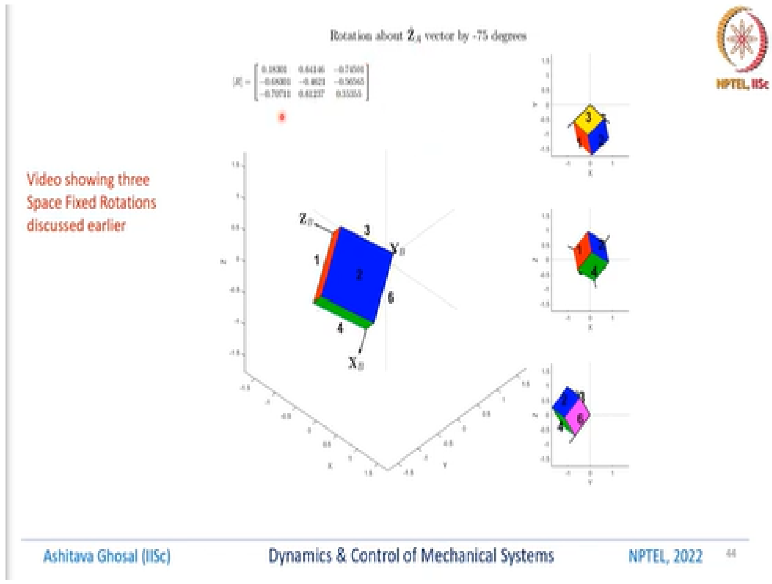

So, for Z rotation in space fix the third row was not changing, X rotation first row was not changing. Now let us look at body fixed X Y Z. This is the original configuration. This is I am rotating about X axis by 60 °. Then I am rotating about Y axis by 45 ° and then I am rotating about Z axis by- 75 °. So, we are using the same angles as in the space fixed. But in one case it is about the original X Y and Z reference $\hat{X}_A$ $\hat{Y}_A$ $\hat{Z}_A$ but now it is about $\hat{X}_B$ $\hat{Y}_B$ $\hat{Z}_B$.

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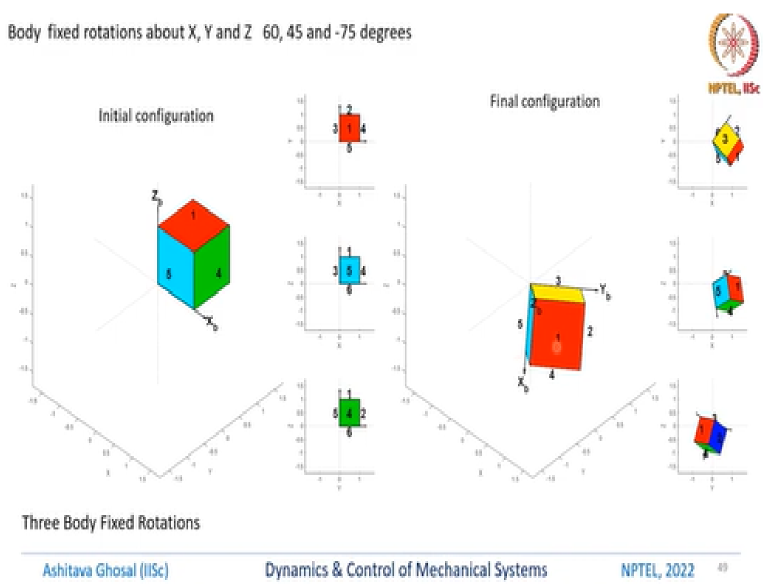

So, again you can see that the initial configuration is this. We started with the same initial configuration. And then the final configuration looks like this and the rotation matrix is this and again you can see this video.

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So, if you watch little bit carefully if you are rotating about $\hat{X}_B$ the column vector $\hat{X}_B$ is not changing it is still staying 1 0 0. And next you will see that when it is rotating about $\hat{Y}_B$ the second column is not changing. The second column remember, is for the Y axis. And third is when we are rotating about the moved z axis the third is not changing. The first 2 columns are changing. So, this is another interpretation of space fixed versus body fixed rotations.

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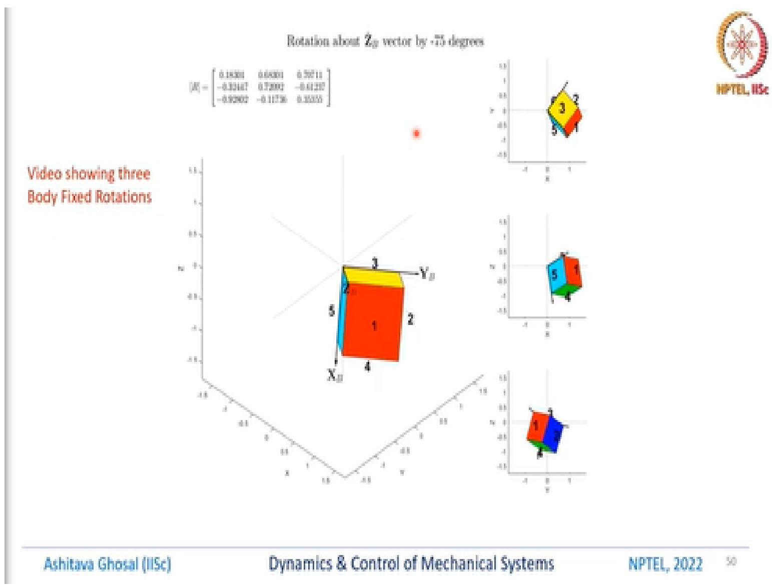

In the space fixed especially for these videos the rows were not changing depending on which axis it was rotating whereas for the body fixed rotations the columns are not changing.

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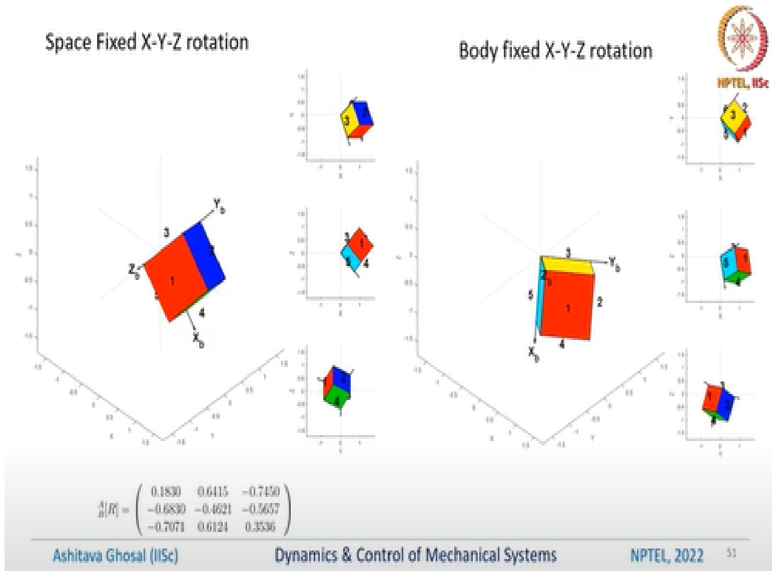

And here is a picture of the final rotation matrices. So, when you have space fixed X Y Z rotations and again those 3 angles which I mentioned earlier about X Y and Z you will get this rotation matrix and for the body fixed you will get this rotation matrix. So, as you can see both are very different. Sometimes it might look the same you know this one is looking the same as this.

This one is looking the same as this. But the other terms are very different which is what we saw in the analytical formulations also.

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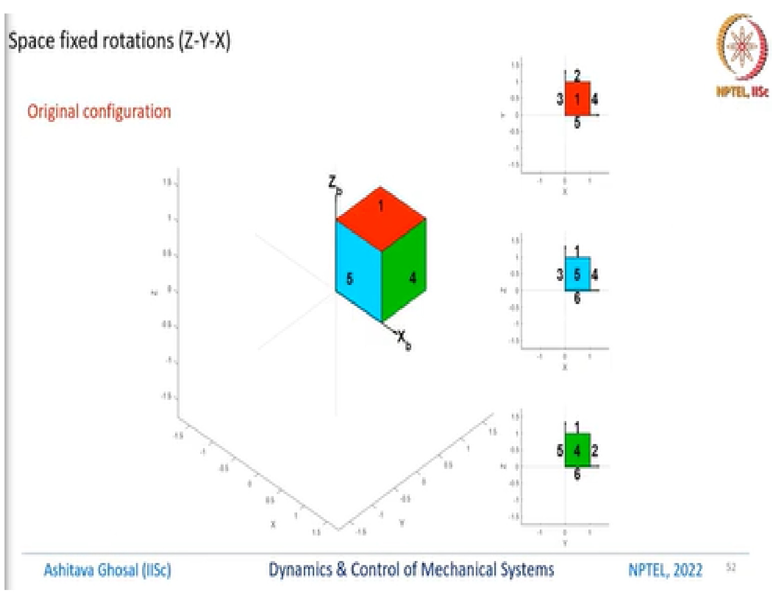

In the last slide I showed you that the body fixed X Y Z and the space fixed X Y Z rotations they give rise to very different rotation matrices. It also shows you that the picture of this cube once it is rotated by the same 3 angles looks very different. So, in this slide I want to prove to you that body fixed X Y Z is same as space fixed Z Y X rotations. So, I can show it to you mathematically but I want to show it to you numerically that after doing body fixed X Y Z.

And then on the similarly if I do space fixed Z Y X by the same angles then I will get back the same final orientation.

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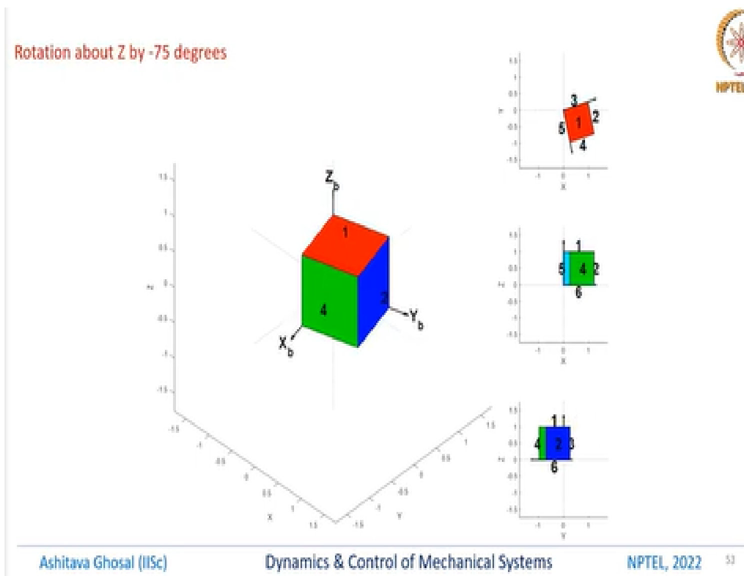

So, the first rotation is about Z axis by - 75 ° because we are keeping the angles as same.

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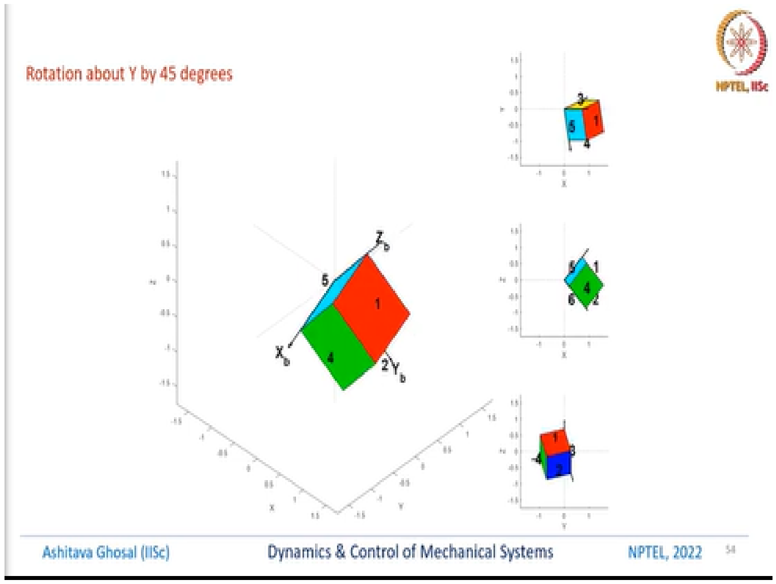

The second rotation is about Y by 45 °.

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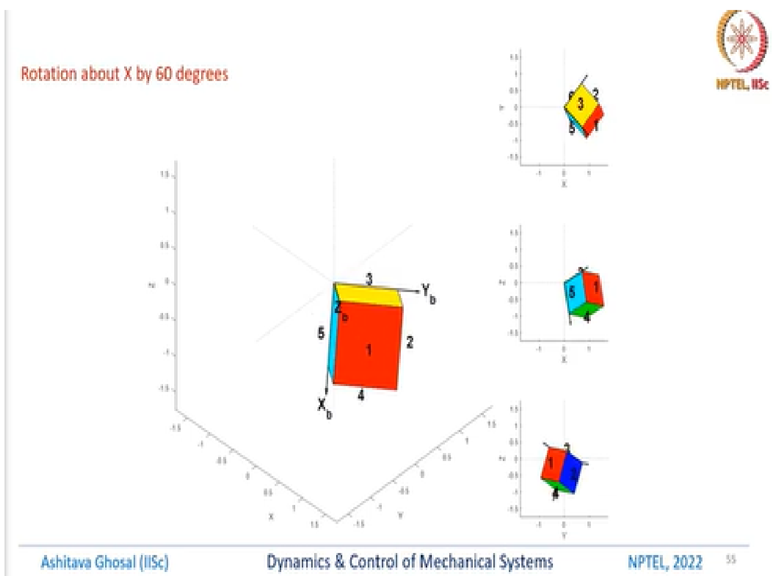

And the third rotation is about X by 60 °.

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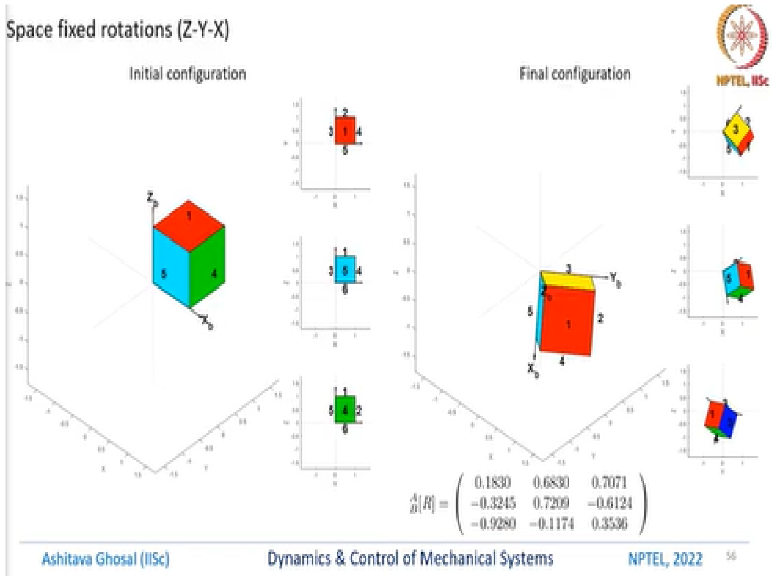

Now the initial configuration was this. Then we did this 3 Z Y X about space fixed axis. And we get this. So, this is the rotation matrix. These are the elements of the rotation matrix which is nothing new which has been seen earlier.

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And this is the video.

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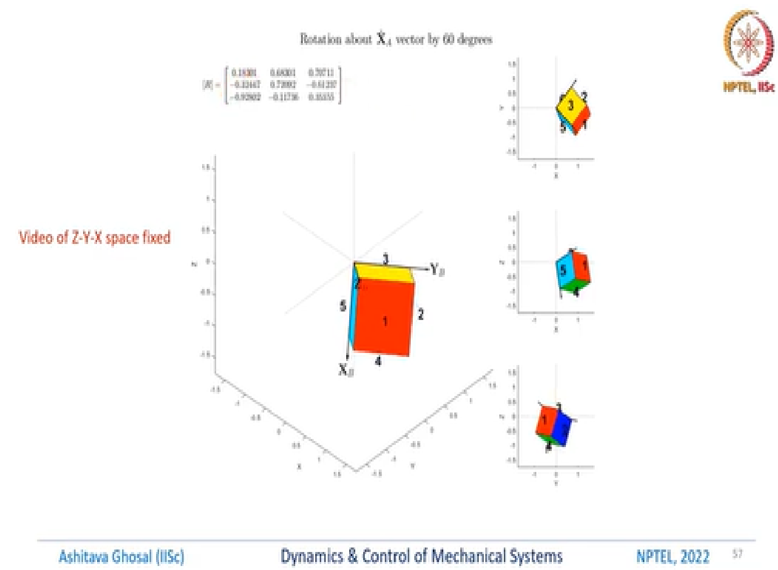

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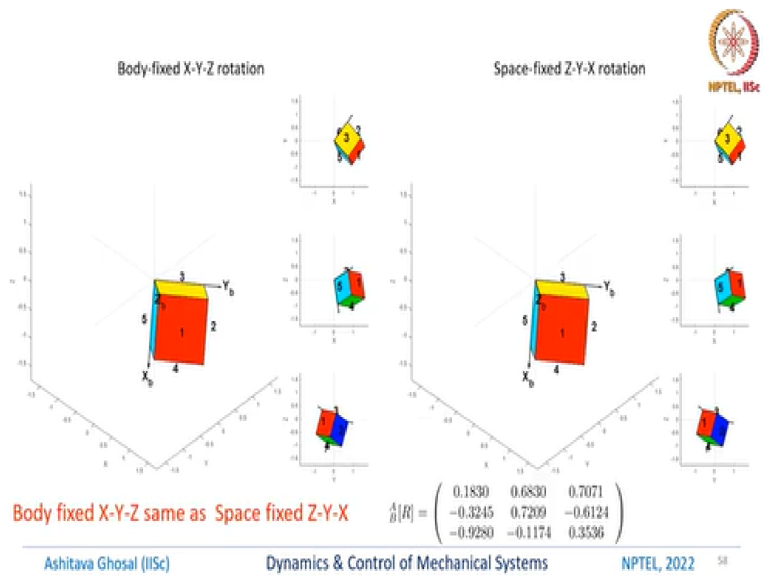

So, note again if it is space fix the first row it was not changing. So, this is the body fixed X Y Z rotation it comes to this. And the space fix Z Y X rotation looks like this. So, both looks exactly the same and the rotation matrix is this. So, at least numerically you can verify yourself of course for those 3 angles which I chose that body fixed X Y Z is same as space fixed Z Y X.

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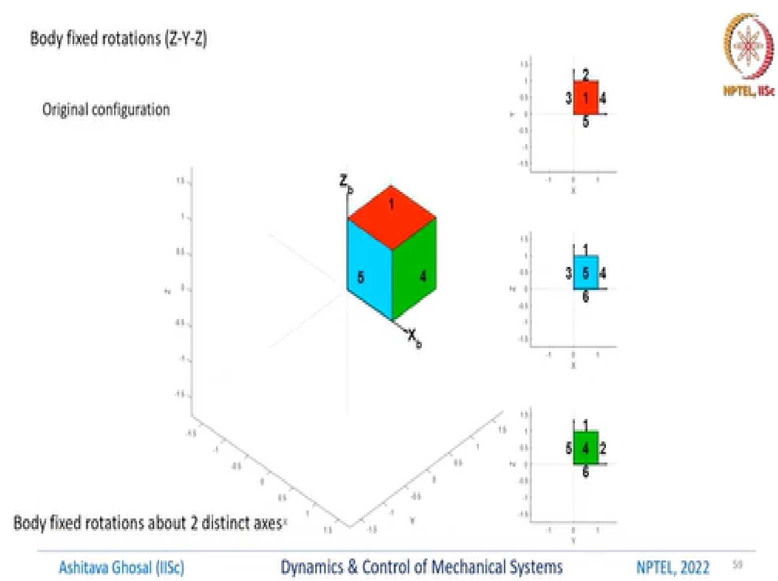

And similar results we can or numerical simulations we can do for body fixed rotations which is Z Y Z. In this case again this is the original configuration. And we are going to rotate about body fixed Z then Y and then Z. So, Y is the new Y and Z is the final again after second rotation whatever is the Z.

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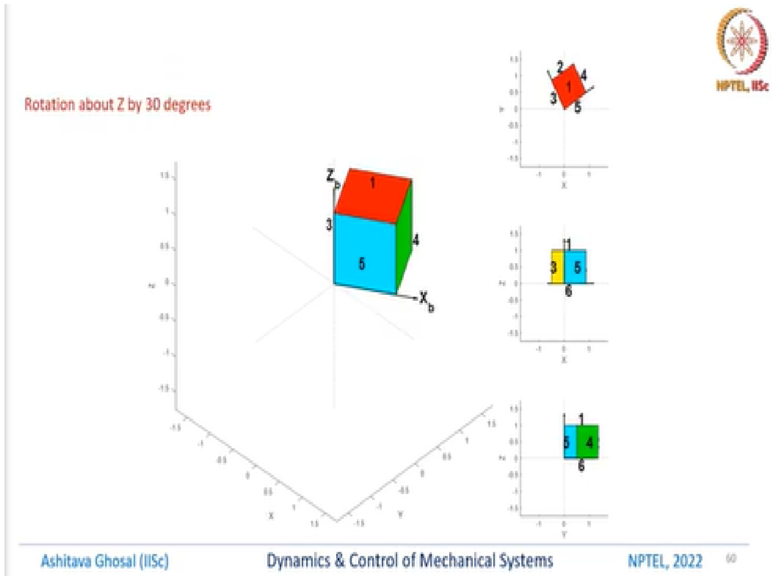

So, let us pick these angles Z by 30 °.

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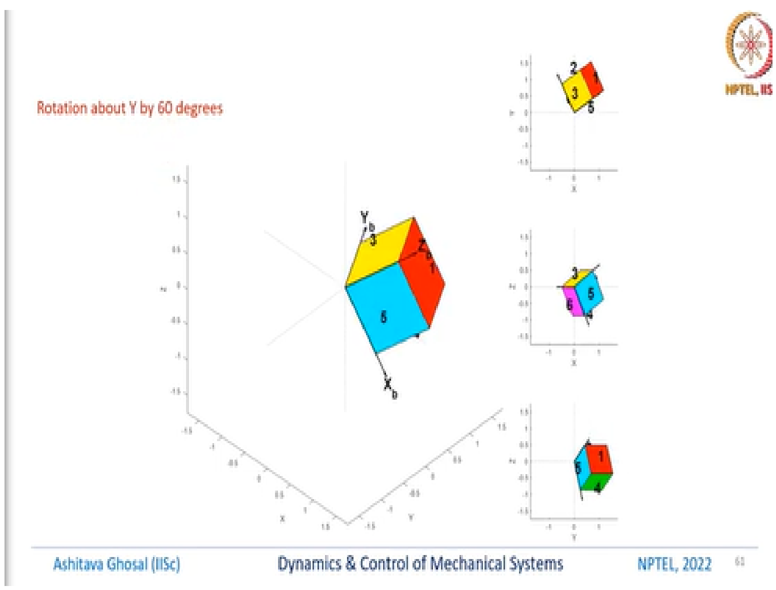

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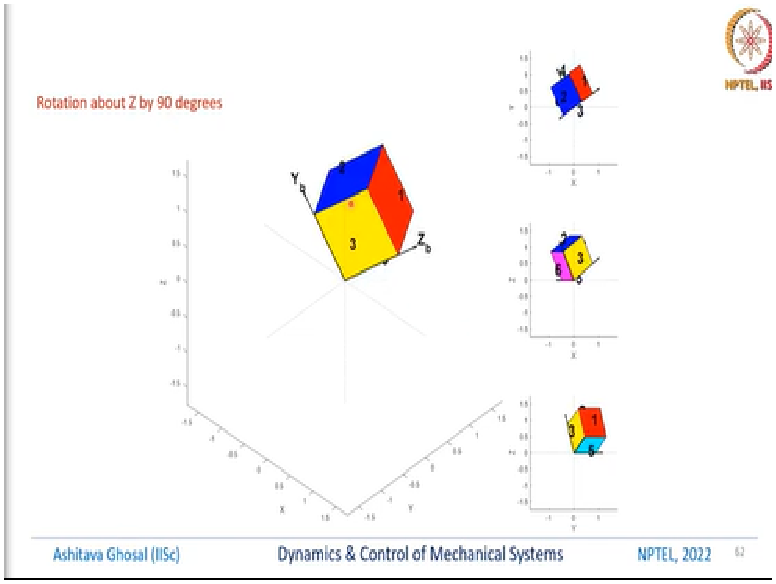

Then Y by 60 ° and Z by 90 °.

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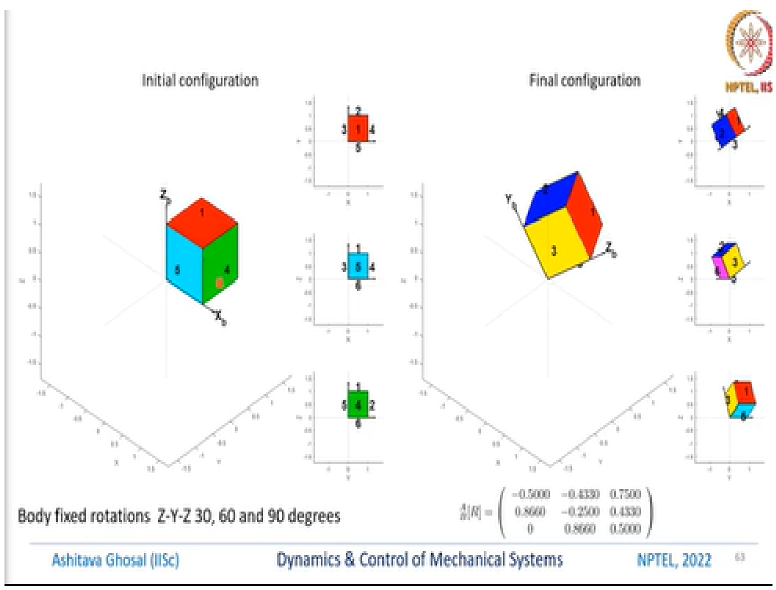

This is what it will look like. And again, this is the initial configuration 5 4 and after those 3 rotations of 30, 60 and 90 it looks like this.

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And this is a video of the body fixed Z Y Z rotations. So, remember I said Euler angles can be both about 3 distinct axes and about 2 distinct axes. So, these are examples of 2 distinct axes. And again, you can see that when it is rotating about the Z axis the last row column does not change. In space fix the row does not change numbers in the row. In body fixed the column does not change.

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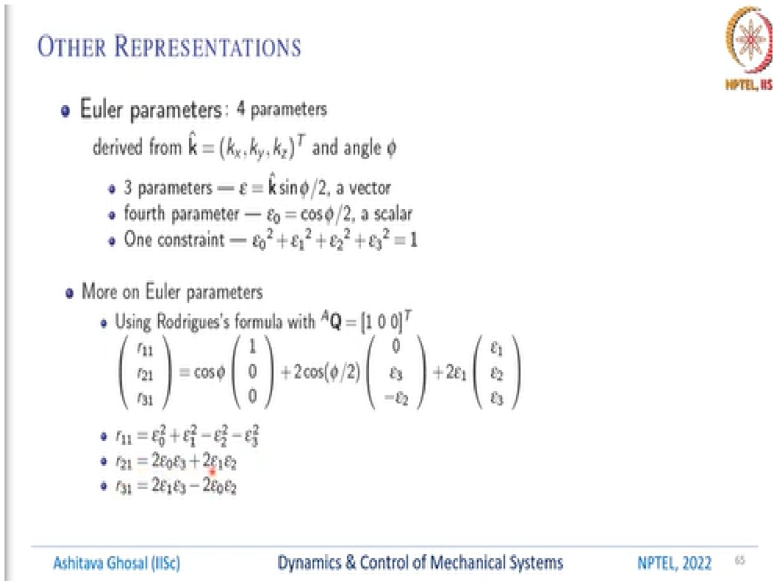

So, let us look at some other representation of orientation. So, there is a very well-known representation of orientation called Euler parameters. So, basically it contains 4 parameters and it is derived from that k and ϕ. So, in k we have k_x, k_y, k_z and then angle ϕ. This is the k ϕ representation. So, out of this k and ϕ we define 3 parameters which are vector epsilon which is k into sin ϕ by 2. So, k is a vector.

So, when you multiply by a scalar it still stays as a vector. And a fourth parameter ϵ_0 is cos ϕ by 2. This is the scalar. So, what you can see is $\epsilon_0^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2$ is 1. So, this is one constraint. In Euler parameters there are 4 parameters but there is one constraint. Remember in Euler angles there were only 3 angles and there were no constraints. So, a little bit about more about Euler parameters.

We can derive the Euler parameters and their relationship to the rotation matrix or the direction cosines r_{11}, r_{21}, r_{31} by using Rodriguez formula. So, if you keep AQ as 1 0 0 r_{11}, r_{21}, r_{31} is given by this. So, you can see that this is 0 ϵ_3 - ϵ_2 and this is 2 ϵ_1 [ϵ_1 ϵ_2 ϵ_3] . So, r_{11} is given by this expression which is $\epsilon_0^2 + \epsilon_1^2 - \epsilon_2^2 - \epsilon_3^2$, r_{21} is given by this 2 $\epsilon_0 \epsilon_3$ + 2 $\epsilon_1 \epsilon_2$ and r_{31} is given by 2 $\epsilon_1 \epsilon_3$ - 2 $\epsilon_0 \epsilon_2$.

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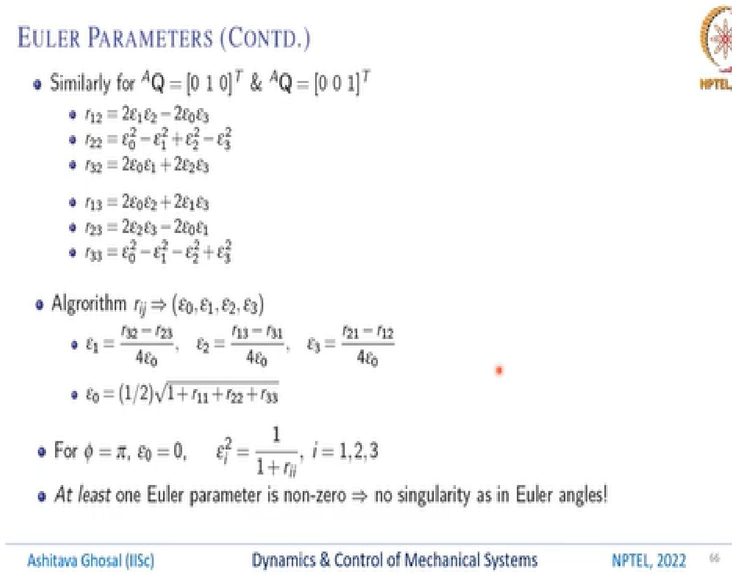

And we can obtain for all the other direction cosines by choosing AQ was 0 1 0 or AQ was 0 0 1. We can find how all the 9 direction cosines are related to the 4 Euler parameters. These are useful expressions to have. And given r_{ij} or the direction cosines I can find out also the Euler parameters. So, previously given the Euler parameters I can find out r_{ij}'s. So, we are going from so for example if all the ϵ_0 ϵ_1 ϵ_2 and ϵ_3 is given.

I can substitute on the right-hand side and get r_{12} r_{22} r_{32} and so on. We can also do the reverse. If I give you r_{11}, r_{22}, r_{23} and r_{12} and all the 9 parameters then I can find out ϵ_1 is given by this ϵ_2 is given by this expression ϵ_3 is given by r_{21} - r_{12} into 4 ϵ_0 where finally ϵ_0 is 1 / 2 square root of this. So, again as you can see, we can go from Euler parameters to direction cosines and direction cosines to Euler parameters.

Little bit more on Euler parameters. So, for ϕ = π. So, remember the rotation axis is k and the angle which it is rotating about is ϕ. So, if that were π then ϵ_0 will be 0. But ϵ_i^2 is still non-zero. So, at least one Euler parameter is non-zero. So, unlike Euler angles there is no singularity in Euler parameters. So, this is one very big advantage. So, remember in Euler angles there were always these problems of some angle being either 0 or $\frac{\pi}{2}$

. So, you could not find all the Euler angles uniquely. So, no such problem exists in Euler parameters. So, this is one of the reasons Euler parameters are used extensively in many applications. I will mention those in a few slides from now.

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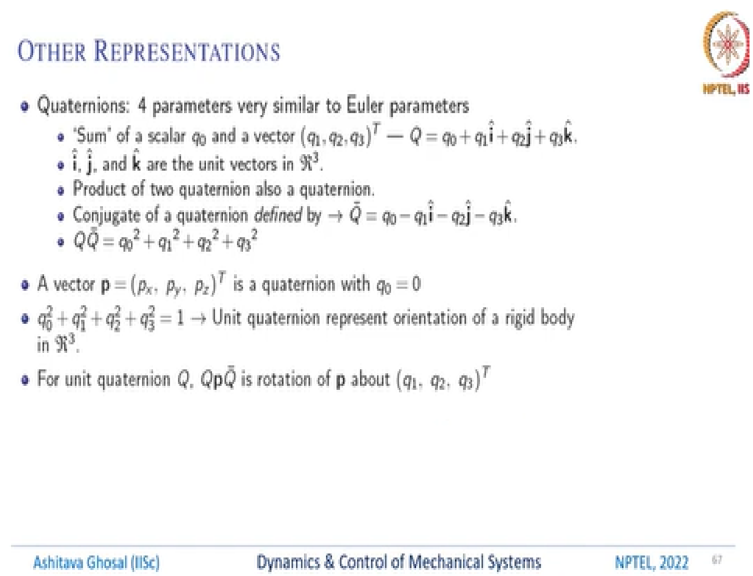

We can also have something called quaternions. Quaternion is very similar to and Euler parameter. It has again 4 parameters. However, it is set up slightly differently. So, quaternion consists of a scalar q_0 and a vector q_1, q_2 , q_3 very similar to ϵ_0 ϵ_1 ϵ_2 and ϵ_3 . But it is written in this form. It is written as q_0 + some unit vectors i, j and k. So, it is a strange beast. It is neither a scalar nor a vector. It is a combination of 2.

And so, you might think that this is a strange thing and what useful it is. It turns out quaternions are very useful. You know they have some useful ways of looking at rotations. So, if you have something 2 quaternions the product of this is also Quaternion. The conjugate of Quaternion is also defined. Conjugate is some sense like an inverse. So, $\bar{Q}$ is q_0 - q_1 - q_2
j - q_3 k and Q into $\bar{Q}$ is the square of this.

So, basically it is an approach quaternion where an approach to see whether rotations instead of using matrices can we do it like in some sense like vectors. It is not really a vector because it has a scalar and a vector part but there are some nice properties. So, a vector p which is p_x, p_y, p_z

or x, y, z is Quaternion with $q_0 = 0$. And if you have this $Q\overline{Q}$ or some of the squares of the elements of Quaternion as one this is called as a unit quaternion.

And unit quaternion represents the orientation of a rigid body in r 3. Remember a rotation matrix with 9 elements represents the orientation of a rigid body in 3D space. Whereas a Quaternion with this condition a unit quaternion represents orientation of a rigid body in 3D space. There are also some other nicer properties. For example, in a unit quaternion Q, Q p $\overline{Q}$. So, $\overline{Q}$ is the conjugate p is a vector and Q p $\overline{Q}$ is a rotation of p about q_1, q_2, q_3.

So, remember we had this k axis and then we had a ϕ angle about rotation of that and then Q went to Q Bar. So, here the same you can think of it as Q p $\overline{Q}$ where p is this vector. If you do this operation then it is rotation of this vector p about q_1, q_2, q_3.

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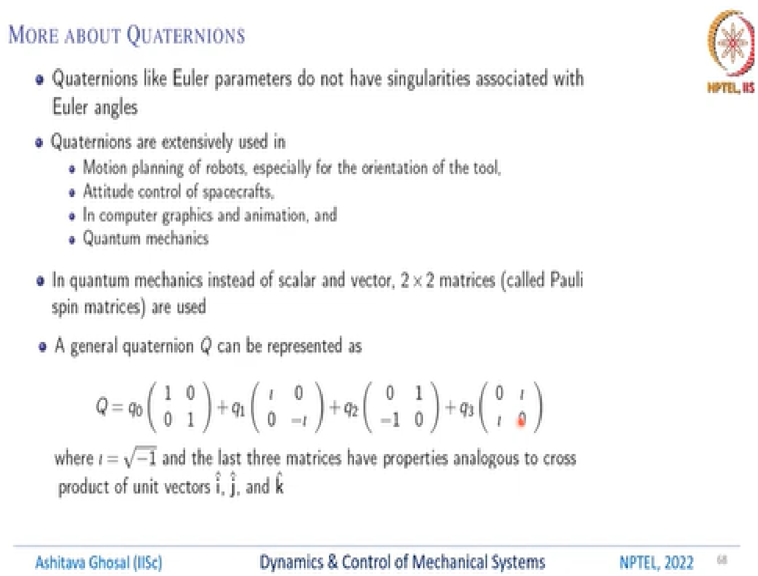

Quaternions like Euler parameters do not have singularities associated with Euler angles very big advantage. Quaternions are extensively used in motion planning of robots especially for orientation of the tool. So, I want a robot tool to follow a straight line but then the welding tool which it is carrying must be oriented in some place in some particular way. So, then we use quaternions.

It is also used in attitude controller spacecraft simply because it does not have singularities. It is also used in computer graphics and animation. And it turns out also in quantum mechanics. So, in quantum mechanics instead of scalar and vector 2 by 2 matrices called Pauli spin matrices are used. So, I do not want to represent as q_0, q_1, q_2 and q_3 where q_1, q_2 and q_3 are along i j and k and q_0 is a scalar.

But then they use 2 by 2 matrices. So, a general quaternion can also be expressed in terms of these 4 2 by 2 matrices. So, you can have q_0 into an identity matrix q_1 into here this i is imaginary number square root of -1 , q_2 is again 0 0 but 1 and - 1 and then q_3 is 0 0 i and this. So, this also represents a Quaternion. And these 3 matrices are very similar to the cross product of unit vectors in i j and k.

So, whatever you want to do with i j and k you can do sort of similar things with these 3 matrices. And these forms of quaternions are used in quantum mechanics.

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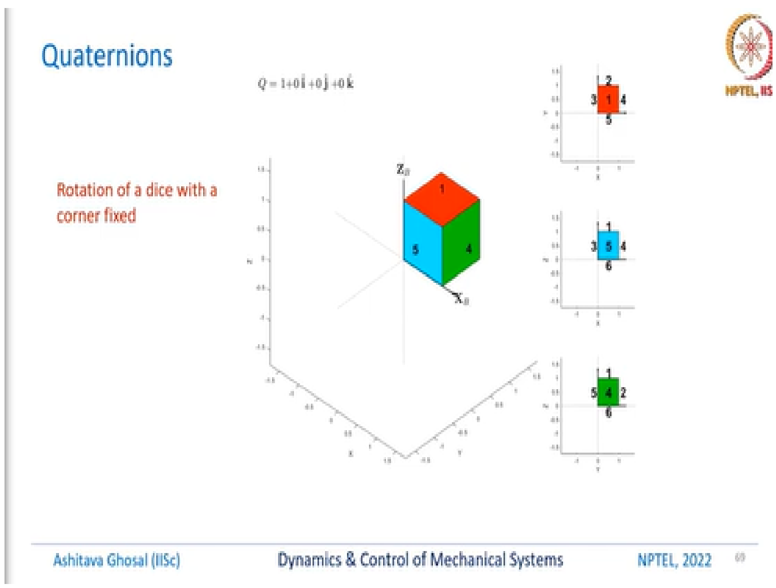

So, as I said quaternions can be used for rotations. And here is an example which shows how this dice again the same dice if I represent it using quaternions what happens.

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So, you can see here that the elements of the quaternion are changing in some particular way. And then this dice is rotating and different faces are being seen at each time. And these are the 3 views of the same dice.

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And what happens to the quaternions. So, again we can look at how Quaternion can represent orientation. And since we are rotating the orientation is changing and we can find out what the quaternion is doing.

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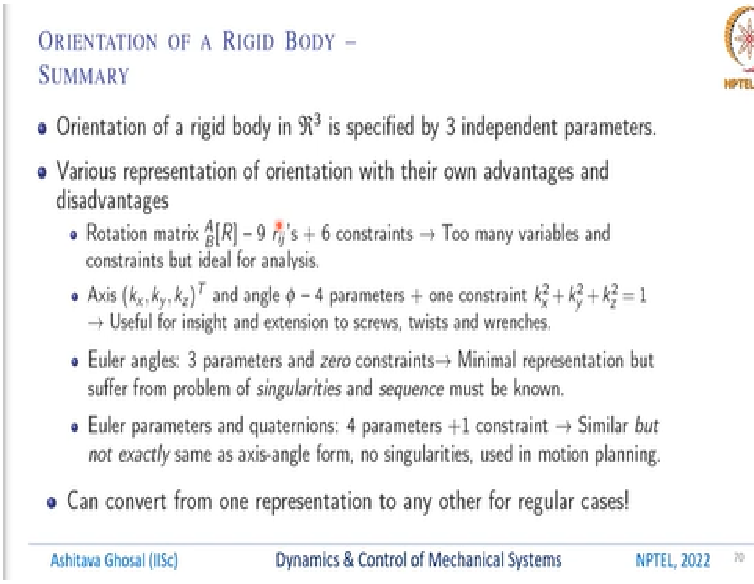

So, in summary the orientation of a rigid body in 3Ds is specified by 3 independent parameters. There are various representation of orientation with their own advantages and disadvantage. A rotation matrix has 9 directions cosines or 9 r_{ij} s + 6 constraints. There are lots of variables and there are constraints. But this is very useful or ideal for analysis. So, you can think that if you have a multiple set of rigid bodies connected to form a mechanism and you have rotation matrix.

For all these 5 different rigid bodies you will have to deal with 45 r_{ij} s and 30 constraints. So, it is a lot of effort. If you have this access and angle form k_x, k_y, k_z and angle you have 4 parameters + 1 constraint which is a unit vector. This is useful to get insight into what are called screws, twists and wrenches. These are for advanced kinematics of rigid bodies. Euler angles are very useful because they have 3 parameters and no constraints.

So, as I said for these 5 rigid bodies which made up some mechanism I have only 15 parameters to worry about. I do not need these 45 + 30 constraints. However, although it is a minimal representation it contains a singularity. And also, we should know which sequence of Euler angles we are using because X Y Z or Z Y X or X Y X depending on what sequence you are using the rotation matrix will be different.

And then you have this Euler parameters and quaternions. They all have 4 parameters + 1 constraints. They are similar but not exactly same as this angle axis form. There are no singularities, and they are very extensively used in various kinds of motion planning. More importantly I can convert from any representation to another representation. So, remember k_x, k_y, k_z and ϕ is given I can find out all these r_{ij} s all these direction cosines.

And from this rotation matrix I found out the eigenvalues and eigenvectors and I found out that k_x, k_y, k_z corresponds to the eigenvector corresponding to the real eigen value 1 and ϕ was obtained from some $e^{\pm i\phi}$ and likewise for all others.