Electromagnetic Theory
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Module - 2
Electrostatics
Lecture - 22
Dielectrics

We have so far been discussing the physics of electric field in a dielectric medium, and with this lecture today, we will be completing our discussion on electrostatics. And I wish to briefly remind you about what we were doing in the last lecture.

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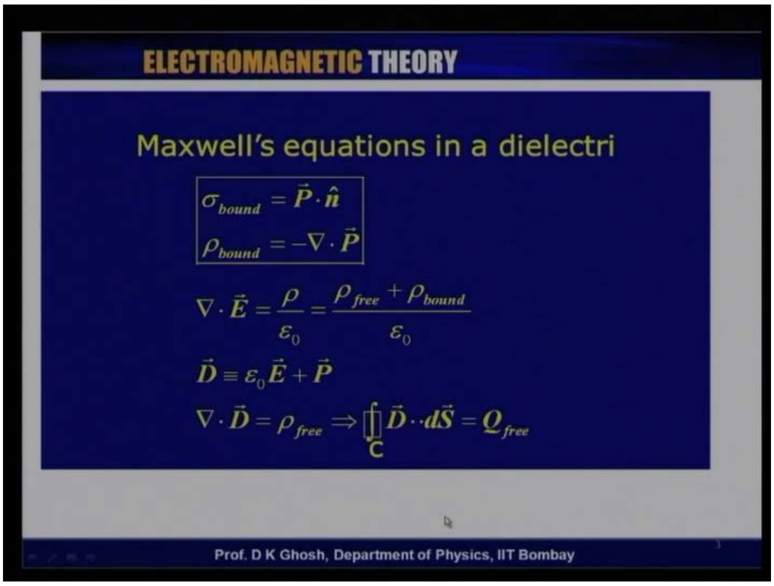

We had talked about polarization. If you recall we defined polarization as the dipole moment per unit volume of a material. Now what we had seen is that when I have a polarized medium, there are charges which we have been calling as the bound charge which are there and these are not fictitious charges, but these are charges which arise because an external electric field might separate the positive, and the negative charge centers. It might also arise, because the molecule has a permanent dipole moment so that there are positive and negative charges. And we have seen that how these could be aligned, so as to produce a surface charge or even a bound charge density within the volume.

And what we had seen is that the surface charge density of these bound charges is essentially given by the normal component of the polarization vector. And the bound charge density is given by negative divergence of the polarization vector. So when you describe a dielectric medium, you have to talk about two types of charges charge densities; one charge density on the surface and the other charge density which we call as the volume charge density. Now, let us look at what this does to our Maxwell's equation. If you recall we had seen that the divergence of the electric field is given by rho by epsilon 0 where rho is the charge density.

Now when we are talking about a dielectric medium, the charge density has two components. There is a free components which are the type of charges we had seen when we are talking about electrostatics in a vacuum and now there are these new charges which are the bound charges and since rho is the net volume charge density, what I have is that the divergence of the electric field is given by rho free plus rho bound divided by epsilon 0. Now, it is sometimes convenient to define a vector called the displacement vector or most physicist and electrical engineers prefer to call them just by vector d. The vector d as we had seen is defined as epsilon 0 times the electric field E plus P and this E is the net electric field at whatever location we are talking about. Now let us look at what this vector d represents. Now if you take the divergence of D, you notice that del dot of p then becomes epsilon 0; I am assuming epsilon 0 is a constant.

So, del dot of a constant time electric field; so epsilon 0 del dot E which is nothing but this rho by epsilon 0 plus del dot P which is minus the rho bound. Now when you add them up, the bound charge density is cancelled out and you are left with del dot of d equal to rho free. So in other words, d is the vector which arises out of if you could imagine that the real charges could be separated from the bound charges in a medium, then the electric field for which the responsible agent are the actual free charges; that gives rise to the displacement vector D. Now recall that there is a difference in the dimensions of the two; in the sense D and E they do not have the same dimension and so therefore del dot of d instead of being like electric field rho by epsilon 0, it is actually rho free and where rho free is the real free charge densities in the medium.

Now just as using the del dot of E equal to rho by epsilon 0 could be converted into the integral form of the Maxwell's equation E dot d S equal to total charge enclosed by epsilon 0. You can use the same thing here and get that the surface integral of the displacement vector D, D dot d S is just the free charges enclosed within that surface and notice that there is no epsilon 0 on the other side. So, it is just the total fee charge that is enclosed. Now, this is the integral form of the Gauss's theorem for in general case whether there are dielectric medium or not. If it is not there, then of course we know that D and E are simply related. Now the other problems which we started talking about last time but did not have time to complete, I will briefly sketch.

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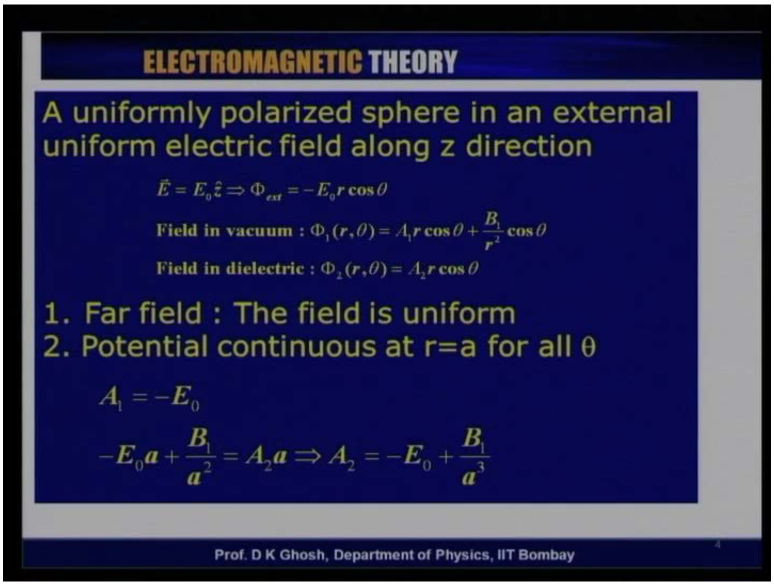

And this also I have put a uniformly polarized sphere in an external uniform electric field which is acting in the z direction. So, the electric field at far a distances is E 0 times z which obviously has arisen form an external potential v at phi external equal to minus E 0 r cos theta. We have assumed that the problem has azimuthal symmetry so that only the cosine theta is there. Now we had seen earlier that when you write down a potential, you can do an expansion in what we call as Legendre; associated Legendre polynomials which are basically expansions in powers of cosine of theta.

Now in this case what we have is that external electric field being E 0, the external field potential at large distances is minus E 0 r cos theta. So, notice that whatever general form of the Legendre polynomial I write that has to be such that, that at r as r goes to infinity namely at large distances, it must give me minus E 0 r cos theta which implies that if when I write down the potential expression in terms of the Legendre polynomial, I only need to retain powers of cosine theta and not powers of higher order cosine theta like cosine 2 theta, 3 theta, etcetera.

So, let us look at that the field in the vacuum which is phi 1 r theta, then I have got A 1 r cos theta and you remember that I had that B l by r to the power l plus 1 p l cos theta. And there again only p 1 cos theta will remain, I cannot write the others because in that case I will have cosine 2 theta, etcetera which I of course do not have and plus this is the right limit as r goes to infinity it gives me A 1 r cos theta which is the behavior at large distances. Now inside the dielectric, I obviously cannot write anything which has a power of one over r because inside the dielectric the origin is included. So as a result, the potential inside the dielectric is A 2 r cos theta and nothing else because I need up to cos theta, but I do not have one over r or any of its power because then at r equal to 0 the fields would diverge.

So, these have been the basic points which enabled us to write down the two potential expressions; having done that what we notice is that since the potential format large distance is minus E 0 r cos theta, if you compare this two expressions namely phi one with phi external, this tells me that A 1 must be equal to minus E 0. Likewise I have the following thing that since the potential is continuous on the surface of the sphere, irrespective of whatever value of cosine theta we take, these two expressions must become equal when I put r is equal to A at any theta so that theta will cancel out and I will be left with A 1 which is minus E 0. So, minus E 0 A plus B 1 by a square equal to a 2 times a; that tells me a 2 is minus E 0 plus B 1 by a cube.

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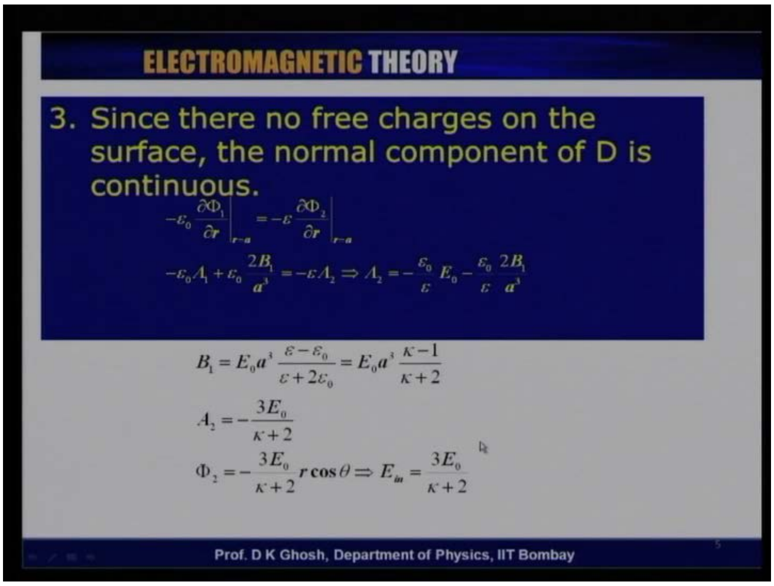

So, I have got these two expressions and let us now try to find out these remaining constants; I have already found A 1, I need to find B 1 as well as A 2. So, there are two ah conditions which we need to talk about on the surfaces we have no free charges. So, that tells me that the normal components of the D field is continuous and let us use them to find out our remaining constant.

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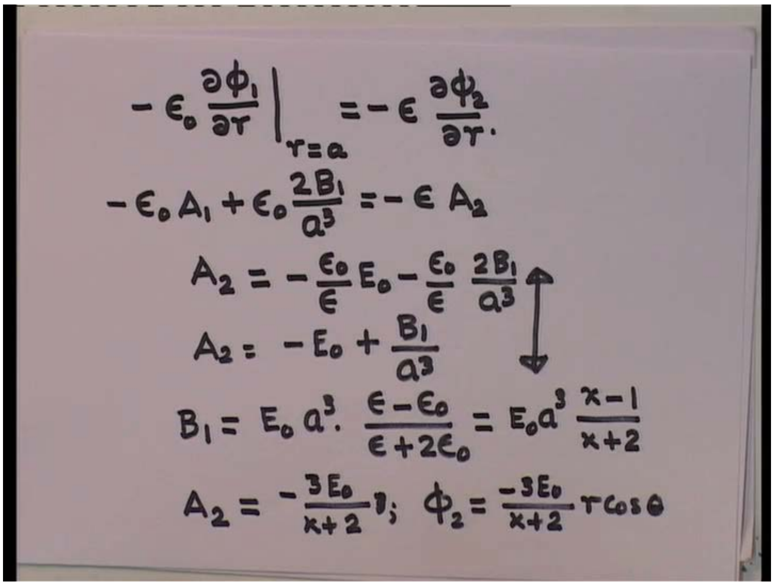

So therefore, normal component of the electric field is minus E 0; E 0 because the dielectric function outside the medium is E 0 the vacuum permittivity and this times d phi 1 by d r; that is the normal component at r is equal to a and that must be equal to from the side of the sphere. So, it should be d phi 2 divided by d r. Now recall that I already have an expression for E 1 and E 2; just to recall for you my E 1 is this expression which is the phi 1 is this expression A 1 r cos theta plus B 1 by this. So therefore if I differentiate with respect to this, I am getting minus epsilon 0 A 1 plus epsilon 0 differentiation of that B one by r square; that gives me 2 B 1 by r cube and since I am putting r is equal to a that is by a cube and that is equal to minus epsilon and d phi 2 by this, the d r which is equal to a 2 and that gives me that recall that A 1 is E 0.

So, I get a 2 is equal to minus. So, there is a minus there. So, I have got A 2 this A 2 is equal to A 1 which is minus E 0. So, minus epsilon 0 by epsilon E 0 and then I have got a minus epsilon 0 by epsilon again 2 B 1 by a cube. So that is one relation. So therefore, it tells me if I connect this with the previous expression that I had given you and that was a 2 was equal to minus E 0 plus B 1 divided by a cube. Now I need to equate these two and if I equate these two I get immediately an expression for B 1. You can do this; this will rather reveal arithmetic and what I get is B 1 is equal to epsilon 0 a cube times epsilon minus epsilon 0 divided by epsilon plus 2 epsilon 0. And basically I am equating these two terms and just doing a simplification. Remember the definition of the dielectric constant which is basically epsilon by epsilon 0. So, I can rewrite this expression in terms of the dielectric constant as well which will be E 0 a cube and dielectric constant kappa minus one divided by kappa plus two.

And if you now plug it into this expression for A 2, you will find A 2 is equal to which is minus E 0 plus B 1 by a cube; just put the B 1 into this expression, you get this is equal to minus 3 times E 0 divided by kappa plus 2 times. Well, that is it and that tells me that the function the potential phi 2 which is equal to A 2 r cos theta which is simply given by minus 3 A 0 by kappa plus 2 r cos theta. So, this is my potential and the corresponding internal electric field is simply dividing this by taking d by d z of this and which will be simply given by 3 E 0 by kappa plus 2 as is written here. So what is the effect, what does all it mean? Notice that we said there is there is an external electric field which is uniform given by E 0.

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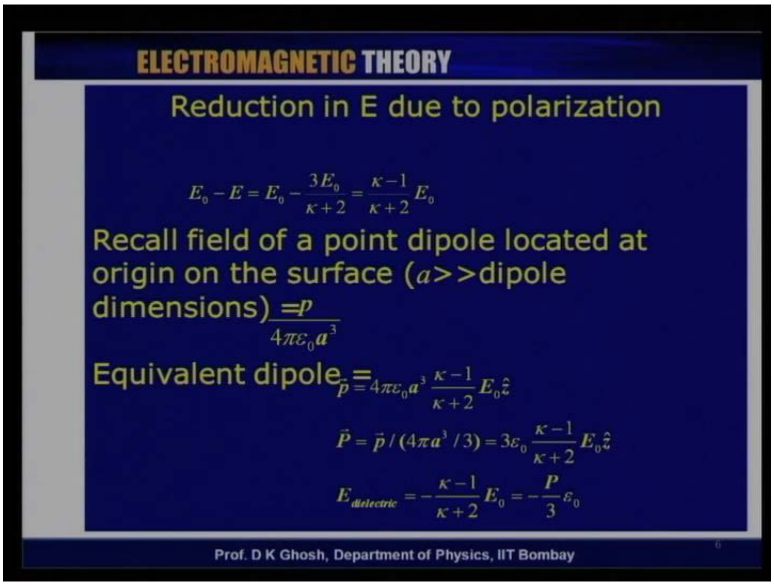

But when we calculated the electric field we found that it is given by E which was minus 3 E 0 by kappa plus two.

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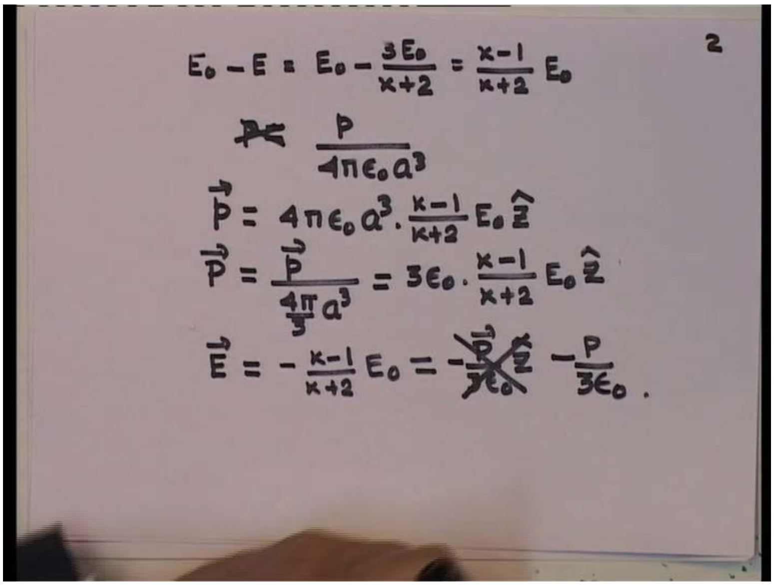

So, what it means is there is the reduction of the strength of the electric filed by an amount E 0 minus E which is E 0 minus the value of E that we just now calculated which is 3 times E 0 by kappa plus 2 which is equal to kappa minus 1 by kappa plus 2 times E 0. Kappa is the dielectric constant. Now what I want to do is to relate this to, you know how much is the effect. Now what I want to do is to relate this to a field produced by a dipole. If you recall that the field of a dipole which is located at the origin and if you want to calculate the field on the equatorial surface if you like, it is given by p by 4 pi epsilon 0 a cube.

So, what we are trying to say is this; what is this equivalent dipole? Remember we have just now calculated the reduction in the field; that is the additional field that is produced because of the fact that high field dielectric medium is given by kappa minus 1 by kappa plus 2 into E 0. So, what is the effective dipole moment? Now you have seen that the electric filed on the equatorial surface due to a dipole of strength p is given by p divided by 4 pi epsilon 0 a cube. So therefore, my effective p strength of the dipole is given by 4 pi epsilon 0 a cube times this reduction that has been produced namely kappa minus 1 by kappa plus 2 into E 0 and this time let me put a direction namely the unit vector z.

Remember again that the polarization vector p is dipole moment per unit volume. So, it is p divided by 4 pi by 3 a cube which is the volume of the sphere and therefore this is equal to 3 epsilon 0 kappa minus 1 by kappa plus 2 times E 0 times z. So therefore, the field in the dielectric is given by which we had seen is given by minus kappa minus 1 by kappa plus 2 into E 0. If you now relate this to the dipole the polarization vector that we have produced, that is simply equal to minus p by 3 epsilon 0 z should be minus p by 3 and an epsilon 0; should be p by 3 epsilon 0.

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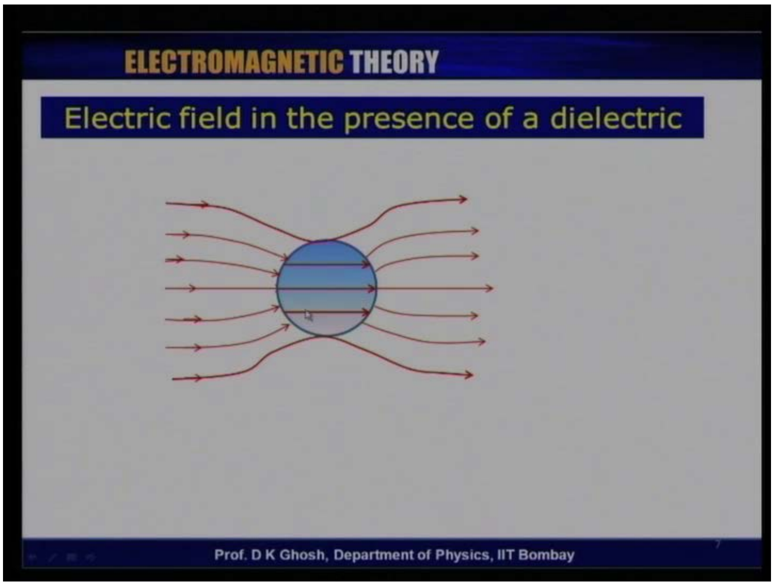

So, look at the picture of the electric field that is there. At large distances I expect the electric field to be parallel to the z direction and so notice that the field lines come and sort of approach from the side; this side. So obviously, this edge is going to become positive and the field, sort of go like this and the field inside is uniform. So, you are talking about the reduction in field due to polarization.

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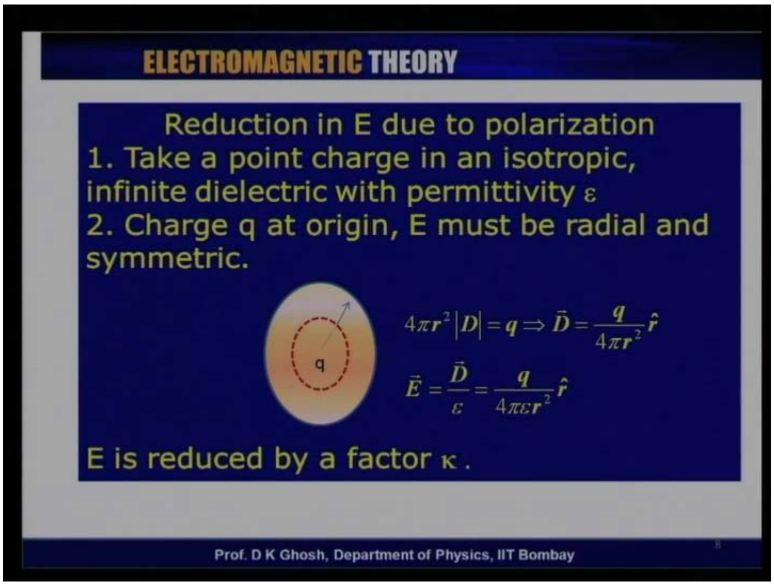

Let us look at that little more. Suppose I take a point charge in an infinite isotropic dielectric having a permittivity epsilon and let me put. So, which means I am considering a huge sphere of some radius. Let me put a charge q at the origin and clearly by symmetry, the electric field must be radial and symmetric and by Gauss's law I know that 4 pi r square times D must be equal to the charged that is enclosed and the only free charge is q which tells me that the D is q by 4 pi r square times the unit radial vector. And since the dielectric is uniform, the electric field vector E is just D by epsilon. So, which is 4 pi by epsilon r square r.

Remember that when we had empty space, it was essentially the identical expression excepting for the fact that instead of the epsilon in the denominator I had an epsilon 0 in the denominator for E. So, which means the electric filed is actually reduced because epsilon is greater than epsilon 0. It is actually reduced by a factor which is epsilon by epsilon 0 namely by the dielectric constant. So, dielectric constant is essentially a measure of the reduction the factor by which the electric field is reduced inside and I must qualify a linear dielectric medium. If it is not linear, then of course we have to worry about that what it does, but nevertheless qualitatively that is what a dielectric function of the medium; if you want to say it is not constant but it is a function. So, this gives me the effective factor by which the strength of the electric field gets reduced because of the polarization of the medium.

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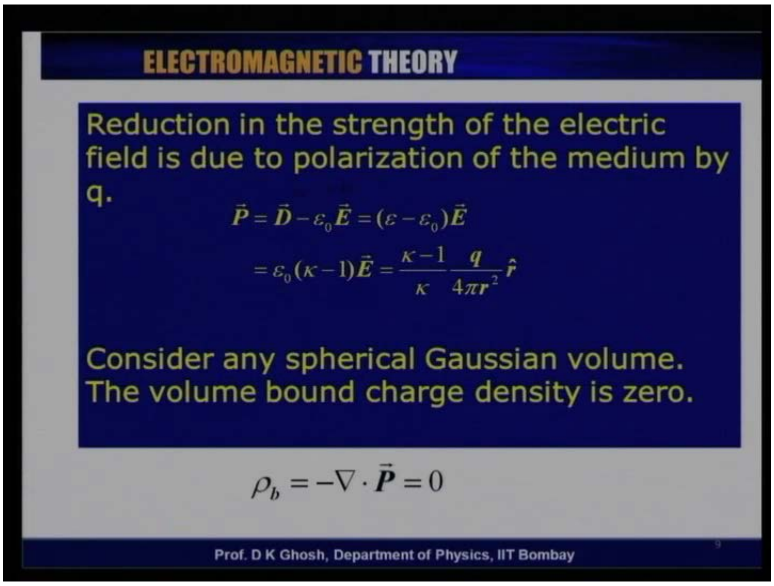

So, let us look at that what happens to what is my polarization now. So, polarization vector which is d minus epsilon 0 E which is epsilon minus epsilon 0 times E and that is simply given by this expression; that is I have simply replaced for the electric vector the q by 4 pi r square, etc. And so this gives me kappa minus 1 by kappa times q by 4 pi r square because the electric field is 1 over 4 pi epsilon r square and epsilon by epsilon 0 is my kappa. So therefore if I now look at a spherical, if you like Gaussian volume, now the polarization is given by this.

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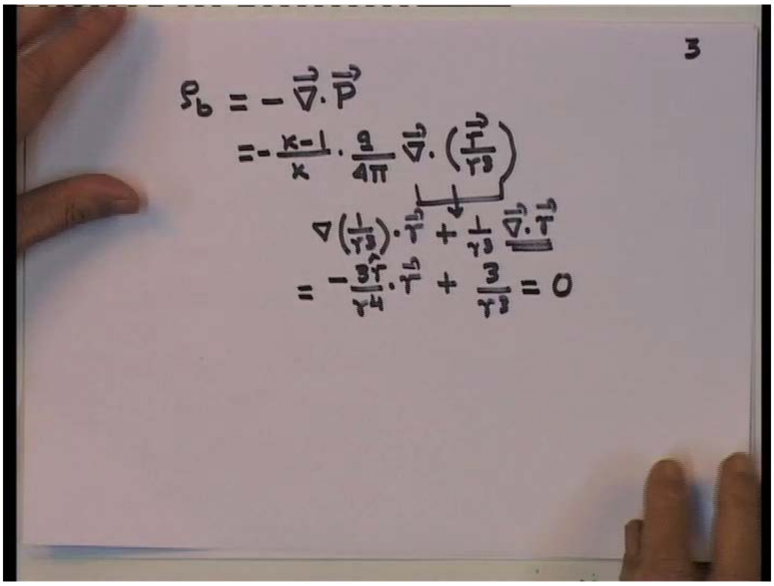

So, I can calculate how much is the bound charge in that medium because bound charge is minus the divergence of p and p we have just now seen; remember most of these are constants so they will come out. So, p is kappa minus 1 by kappa and q by 4 pi r square also I will bring it out, 4 pi I will bring it out. I am left with minus del dot of unit vector r by r square which means del dot of r by r cube. This I can easily calculate because it is divergence of a vector multiplied by a scalar. So I must have this, I am just calculating this. I must have gradient of 1 over r cube dot with r plus 1 over r cube times del dot of r.

Remember that divergence of scalar times a vector is grads of a scalar dotted with that vector plus that scalar multiplied by the divergence of that vector and this is minus 3 by r to the power 4 dot r vector and if you recall del dot of r is just three. So therefore, it is 3 by r cube and this is of course, there should have been a unit vector r here. So therefore, this is exactly equal to this; vector r dotted with r is just r r by r forth is r cube. So, this is equal to zero.

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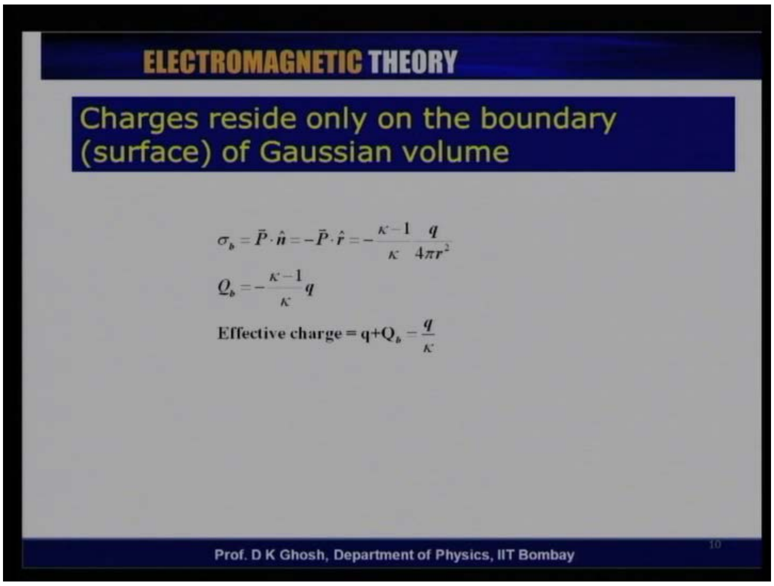

So therefore, the volume bound charge density is 0. So, what I am left with; the only thing that I can have now will be the surface charge density.

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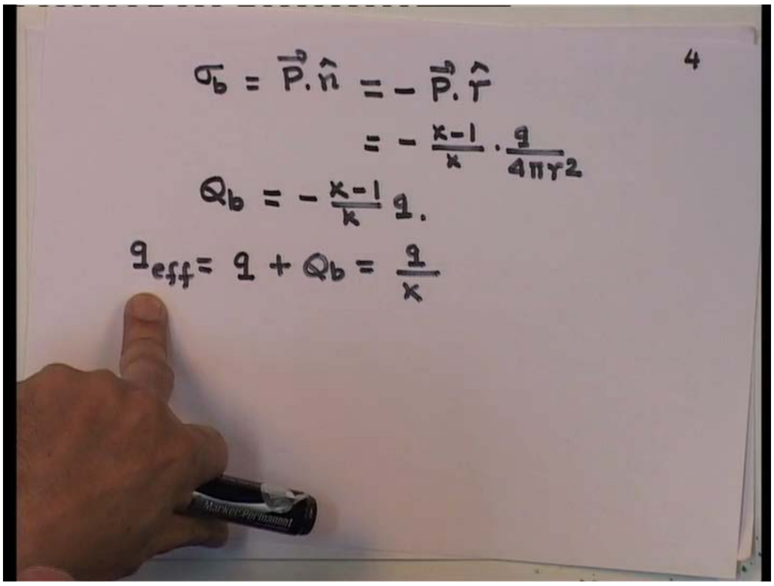

Now the surface charge density we have seen is p dot n. You remember that I have a sphere but then the dielectric medium is inside the sphere and the surface normal is not along outward radial but along inward radial. So therefore, it is minus p dot r and this is equal to minus, well, p dot unit vector r. So therefore, it is kappa minus 1 divided by kappa times q by 4 pi r square. So, this dielectric sphere that I have got with a charge inside does not have any volume charge density but has a surface charge density given by this; which means that there is a net surface charge which is included which appears on the surface of the sphere and this is simply obtained by multiplying this with the radius of the sphere namely with 4 pi r square, which give me simply minus kappa minus 1 by kappa times q.

This is a negative quantity because kappa is greater than one which tells me that the effective charge of this situation is my charge which I have put in. So, let me call it q effective. The charge that I have put in plus the bound charge which is negative which is simply equal to q by kappa. Now this again emphasizes the point that I was making that the effect of a dielectric is to provide a measure of the factor by which the electric field strength decreases. Now this tells me that you can also look at it another way by saying that this means that we though you have put a charge q but real charge q, the effective charge which a test charge will experience is as if there was a reduced amount of charge namely q by kappa. Just to continue with the same application, let us consider this is a situation which is known to you from school; that is what happens when you put a dielectric inside a parallel plate capacitor.

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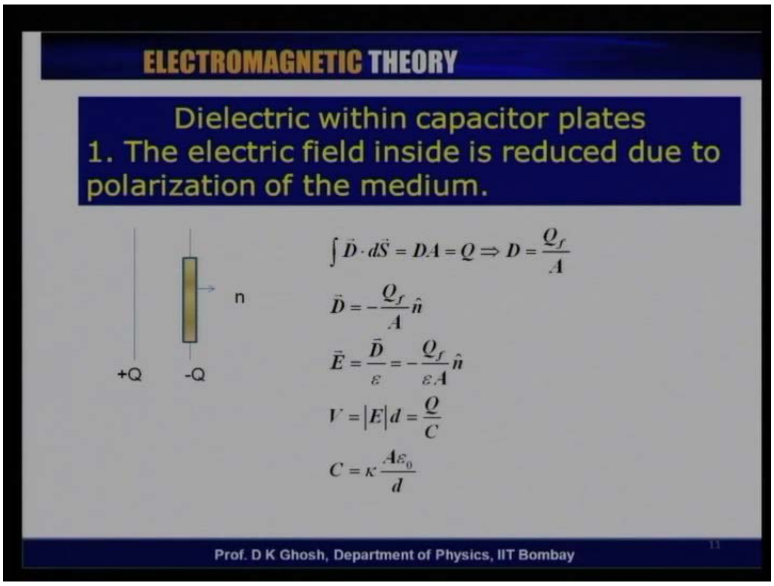

And so basically you are aware that what happens is the capacitance increases but let us look at it from our view whatever you have learnt now. So, what we say is this. We have just now agreed that if you put a dielectric inside a capacitor or in any medium, the electric field inside will be reduced due to the polarization of the medium. Now what I have done here is to have a parallel plate capacitor with a charge plus q on the left hand side and a minus q on the right hand side. Now, if I consider a Gaussian volume in the shape of a parallelepiped of certain area a let us say and sort of a negligible width l.

Now, let us apply Gauss's theorem to this namely d dot s. Now remember the d field is easy because I know that the there is a real charge on their parallel plates capacitors. In this case I have taken my Gaussian volume which is a rectangular parallely piped to be enclosed about the negative plate. So, d dot s and if the area is a, which is equal to d times a; a is the area of one of the surfaces and that amount that is enclosed is nothing but the amount of charge q on an area a of the capacitor plate. So, d times a is equal to q and so therefore, d is equal to your q and just I have emphasized q free by a. So, these are the free charges which are in the capacitor plate.

And in the next expression what I have done is to write this d as a form of a vector. Remember the magnitude of d we have calculated here is q free; q free is the same as this q that I have written down by a. I have put a minus sign because the dielectric medium is to the left of this and this is the outward normal is n. So, the n on the inside will be minus of this and so therefore, the electric displacement vector is given by Q f by A n and the electric field is D by epsilon which is then given by minus Q f by epsilon A n. Now I can calculate how much is the potential difference from the electric filed which is nothing but multiplying the magnitude of the electric field with the distance between the plates and this gives me that remember this was my definition of q by c. So if you now do that multiply this expression with d, you find that the capacitance expression is given by kappa times a epsilon 0 divided by d.

So, it tells me the effect of a dielectric is to increase the capacitance of parallel plate capacitor. Now let me slightly shift to another important point. Now let us suppose that I am looking at a dielectric medium and consider them as a collection of molecules and let us suppose that there is a field in which all these are in which I am going to call as the microscopic field. Now if you considered a collection of gas, the molecules of the gas are well separated. So, I assume that when a collection of molecules is put in a electric field, the each molecule experiences an electric field at its sight which is equal to the average microscopic field that is there in the medium. Now you see if you are considering a gas where the molecules are well separated; this is a very good description because the average microscopic field which is felt is can be consider as E and how much is that.

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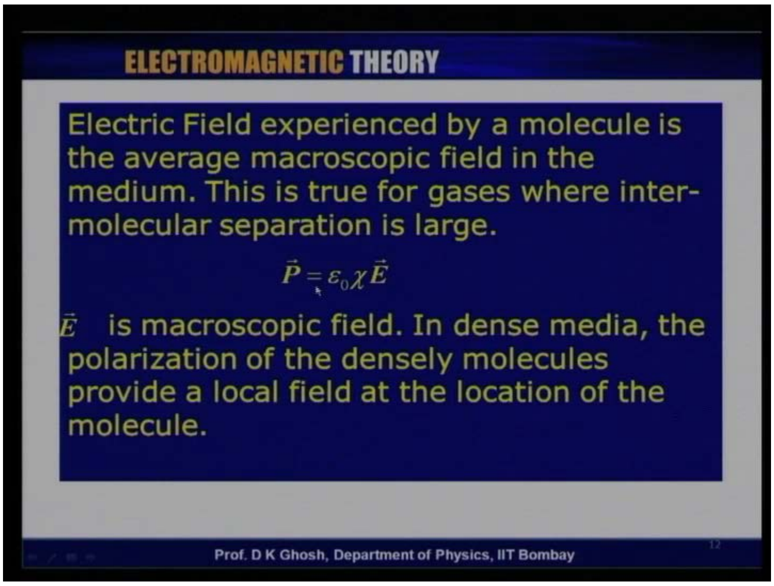

Remember that the polarization can be written as epsilon 0 chi E which is the susceptibility times the electric field and E is the microscopic filed. Now if you consider however, a dense medium however the molecules are packed close. Then if you consider a particular molecule, then the electrons in the vicinity of that molecule, they will be polarized of course and they are responsible for producing what we can call as a local field at the location of the molecule which we are considering.

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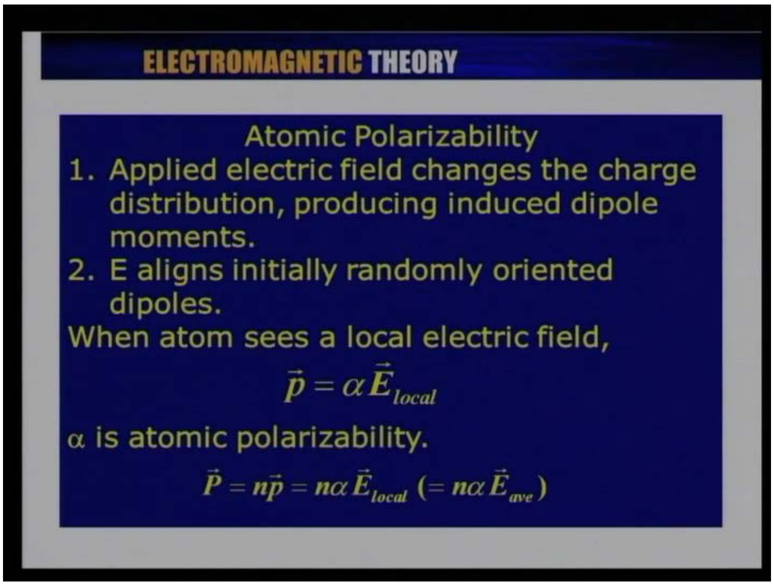

So, as a result what will happen is that; 1. The applied electric field changes the charge distribution. So, this will mean that this will polarize the molecules that is there and now when I am considering a particular atom, it will see now a local field and this local field that it sees will be written as the dipole moment is equal to alpha times the local field. We assume that the dipole the amount of polarization is linear in the field and this alpha is known as the atomic polarizability. And I know that the dipole moment per unit volume is my polarization p. So, I have multiplied this with the density. So therefore, the polarization p is given by n alpha times E local, but there is small problem.

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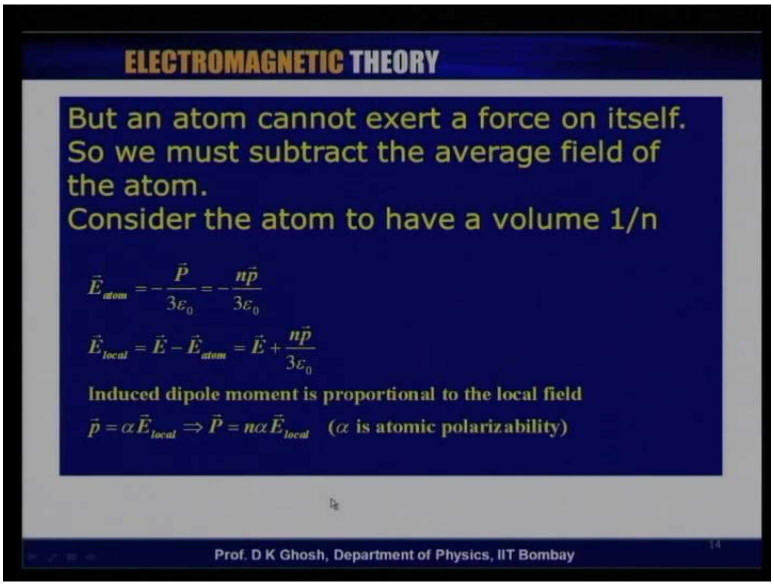

So, I have considered what will the neighboring atoms do but I know that an atom cannot exert a force on itself. So therefore, I must subtract; I must subtract from average field the field due to the atom. So, if I consider my atom to have a typical volume 1 over n, then the electric field due to the atom is minus p by 3 epsilon 0 which is minus n p by 3 epsilon 0. So therefore, the local field is not E, but E minus the E atom. So, which is given by E plus n p by 3 epsilon 0, and I know that induced dipole moment is proportional to the local field. So, therefore I can write this p as this.

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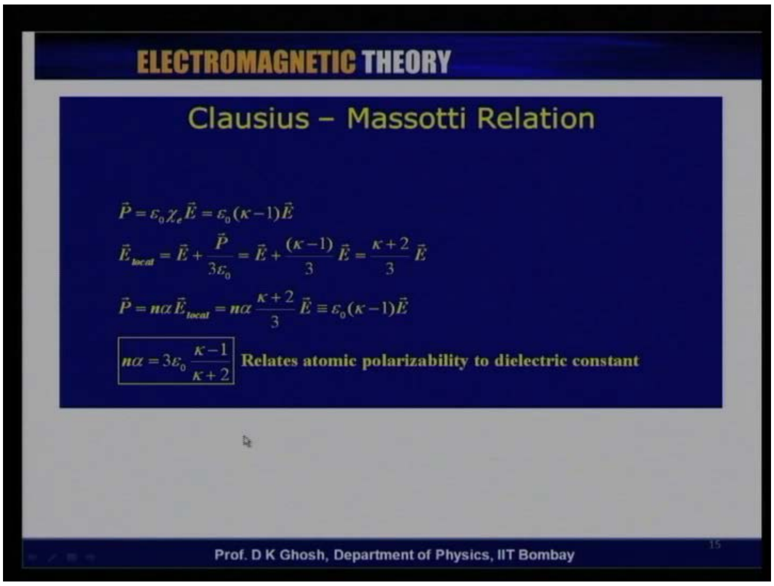

Now, I can do a bit of an algebra and the algebra is this that p if you recall is epsilon 0 times susceptibility times E which is epsilon 0 kappa minus 1 times E. So, local field is E plus p by 3 epsilon 0 which is E plus kappa minus 1 by 3 E which is this expression substituted. Add it up, it becomes kappa plus 2 by 3 E and how much is p; p is n alpha times E local. So, just write down this n alpha times k plus 2 by 3 times E and that is equal to this expression because there are just two different ways of writing. And so if you do that, that gives you an expression for the atomic polarizability which is n alpha equal to 3 epsilon 0 kappa minus 1 by kappa plus 2. What is this relation? Notice this is relating the atomic polarizability with the dielectric constant of the medium.

The dielectric constant of the medium is more an average thing because we have said it is an average effect, but what does it actually do to an atomic polarizability and this relationship is known as Clausius-Mossotti relation. We will bring this discussion of electrostatics to a close with a discussion on what happens to the energy of the charge distribution. If you recall the energy of the charge distribution, when we work it out for the case of free space that is are the collection of a charges, what we did is to assume that initially my charges were at infinity and I brought charges; first I brought one charge put it somewhere, set no work. Next time I bring in a charge, I have to do some work because the charge which has already been there has established a potential and I have to bring this additional charge in the field of that potential. So as a result, I assembled the charge distribution bit by bit and I will not go through the same argument again.

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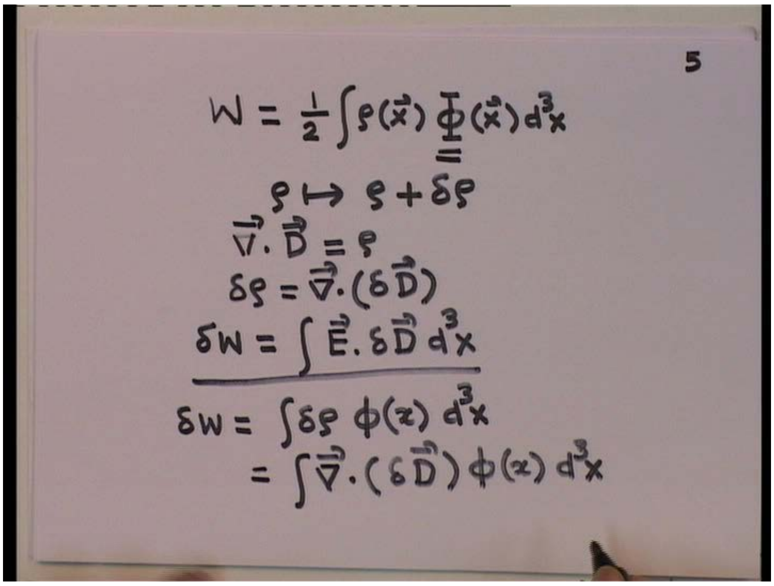

But if you recall we had shown that the work done which is stored as the energy of the electrostatic field in case of vacuum was given by 1 over 2 integral of rho x which is the density at the point x times the potential at the point x d cube x. Now it is not very clear that you can use this expression when you have a polarized medium a dielectric medium because in case of a dielectric, supposing I am assembling the charges bit by bit, the work that needs to done in addition to putting the charge wherever it should be; that is locating the charge, bringing them from infinity and putting them in their place. I also need to do or take account of some amount of work to be done in polarizing the medium because I have to produce certain amount of a state of polarization of the medium.

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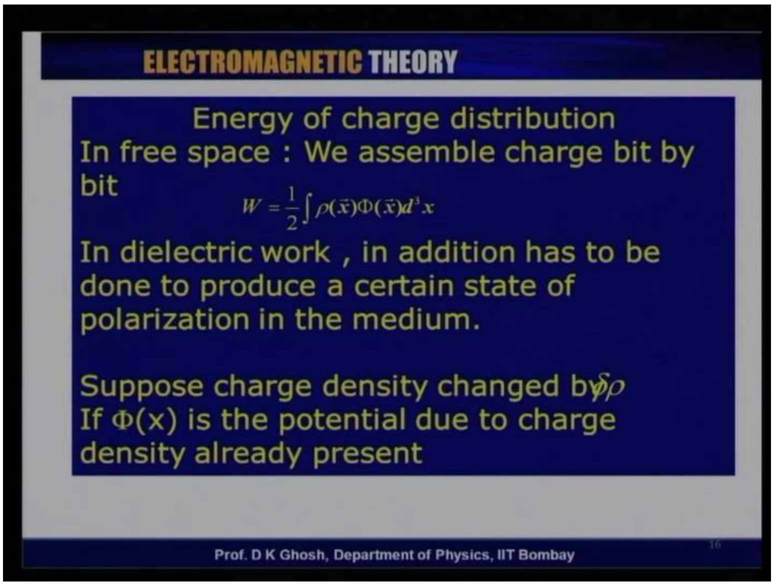

Now what do I do? Now let me give you a general expression. Suppose I have a charge density distribution already established. I have a dielectric medium and how it has been established; let us not go into at this moment. Now suppose I have a charge density rho. Now let us say that my charge density rho changes by some amount delta rho. So, rho goes to rho plus delta rho. Now the amount of charge density changes slightly; phi of x is the potential due to charge density which has already been established. Remember that as I am bringing in a bit of charge, I do not assume that the potential has already adjusted itself now. So therefore, the phi of x is the potential due to already existing charge then what I have is this.

Let us look at it; I know that del dot of d is equal to rho. So therefore, my delta rho is del dot of delta d. So, which means that the work that I am doing, I have an additional work is necessary now and that is given by, how much is the additional work let us look at that. This will be integral; now you have to be careful. Let me first write down the expression; E dot del dot of delta d d cube x. How do I get this expression? Let me come back to this. Firstly, you notice that I am now doing an additional work in changing the charge density from rho to delta rho.

So therefore, my delta rho delta w is integral delta rho phi of x d cube x. Why is there factor of half missing here? The reason is if you recall what is the origin of this factor of half? The factor of half was introduced, because so that I do not do a double counting between charge number one and charge number two. In this case, I am simply bringing in a additional charge delta rho so therefore how much work is done. Now this is nothing but this delta rho we have seen is del dot delta d times phi of x d cube x. Now I can simplify this by doing integration by parts.

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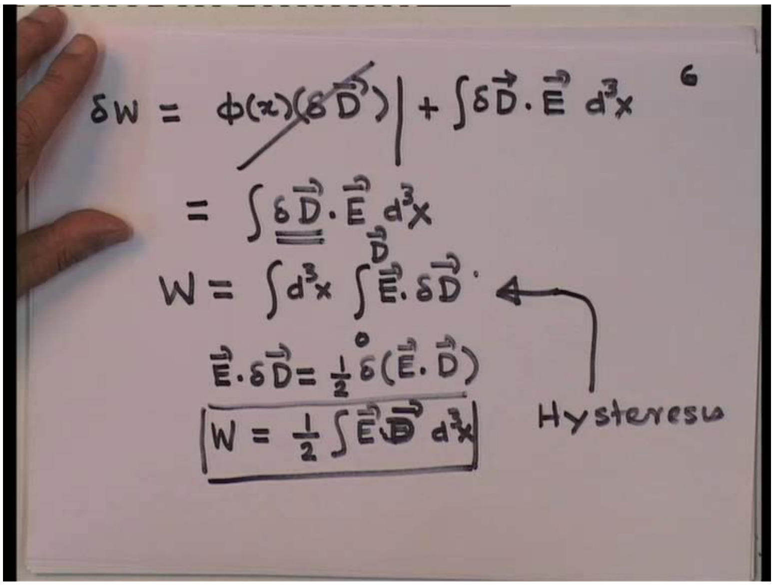

So delta w is equal to, remember I have got here divergence of delta d phi x d cube x. So, what is my integration? My integration will be let me show both of them together; my integration will be integral of this part which is nothing but. So, I will do this phi of x times the integral of this part. So therefore, I will write this as phi of x delta d in some limits; the limits will be you are bringing things from large distances. So therefore, I know that the potential has to become zero at large distances. Now minus integral of integral of the first part which is delta d times the gradient of phi but minus this minus which is there and the gradient of phi will give me an electric field e. So, this minus will become plus and I will be left with d cube x. So, this is the expression that I get.

This term will vanish; this term will vanish because my fields at large distances they are zero. So, this is my delta w which is integral delta d dot E d cube x. So, what is my total energy? So, my total energy E would be now remember that this is a small displacement vector that I have produced and in principle what I require when is to bring from zero, that is when I did not have any charges to the state which actually exists. In other words, I need d cube x integral and this delta d should go from zero to its full value which means o to vector d and I have got it is like E dot delta d. Now this is actually the correct expression for calculating the energy of the electrostatic field. It is not a very easy expression to calculate; however, if you take a linear dielectric, then I can write remember linear dielectric means E and d are linear. I can then write E dot delta d as equal to half of delta of E dot d because E and d are parallel to each other. So, d is some epsilon e.

So, which is epsilon times E square and so differentiation of E square gives me the vector of two that is why this vector of two is there. So this then would give me, then this integration is easier because it is no longer E dot delta d, but its delta E dot d and you can now do that integration and get w as equal to half of integral of E dot E d cube x, E dot d cube x. Now notice this expression w is equal to E dot d cube x which appears with a slight mistake on this. It is the expression from which you can get back the original form of integral rho x phi x d cube x, half of that. But in getting that, you need to assume that my dielectric is linear. If the dielectric is not linear, then the right expression is this and the reason is that as you are bringing in charges, as the medium it getting polarized, there is some history which is being built up and this effect is known as a hysteresis effect. This is included here in this expression; certain amount of hysterics is included.

So, this expression is valid for a linear dielectric. This is of course always valid and starting with this expression, I can go back to the other expression that we have talked about. So all these days, we have been talking about the physics behind electrostatics that is primarily a discussion of the electric field and its effects when we consider static charges. The only thing that is important to realize is that static charges can give many many effects. Last few lectures we have been talking about the medium how it is affected; you know the medium gets polarized and so far we have been talking about static effect of charges. In the second part of our talk which will begin from the next module, we will be talking about what happens when these charges are allowed to move, and that will open up another part of electromagnetic phenomenon that we have so far not touched.