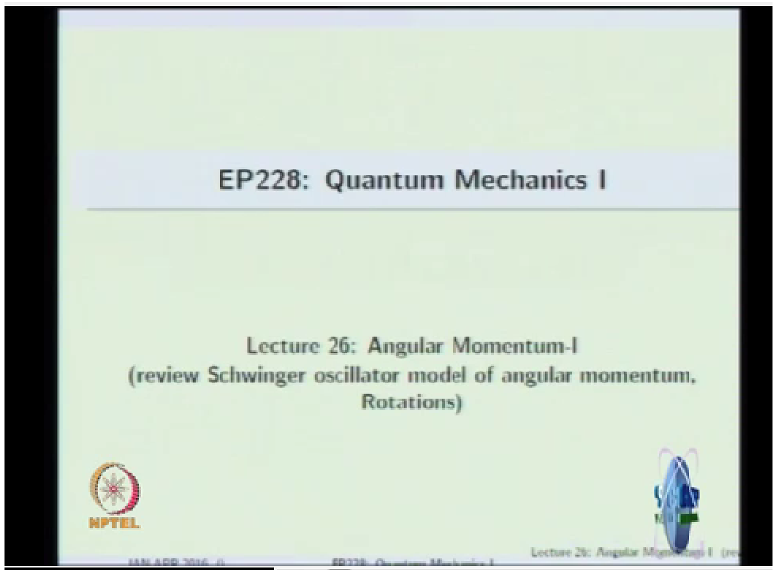
Quantum Mechanics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology - Bombay

Lecture – 55
Angular Momentum - I

So today I am slowly taking you on to diagonal momentum, okay.

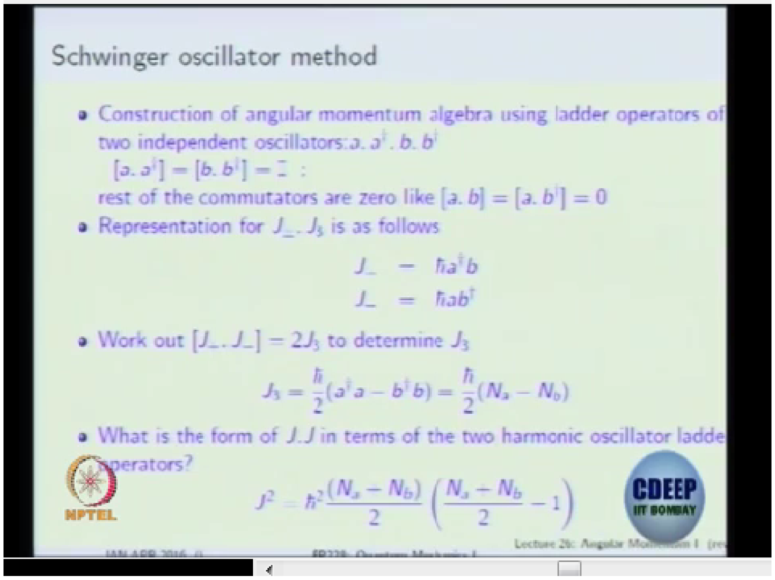
(Refer Slide Time: 00:32)



Some bit we have already seen from Stern-Gerlach experiment and also from hydrogen atom but the formalism you should know what exactly goes in the angular momentum algebra? How the states are defined? How we find under the operation of, the ladder operators similar to the raising and lowering operators in the harmonic oscillator?

What happens in this case? So this is what we started in the last lecture, last week. So let me continue with reviewing this Schwinger method oscillator model of angular momentum and then we will get on to how to see what is the rotation operation, is the one which corresponds to the angular momentum, okay. So this is the theme for today. So recall what I was trying to do in the last lecture.

(Refer Slide Time: 01:26)



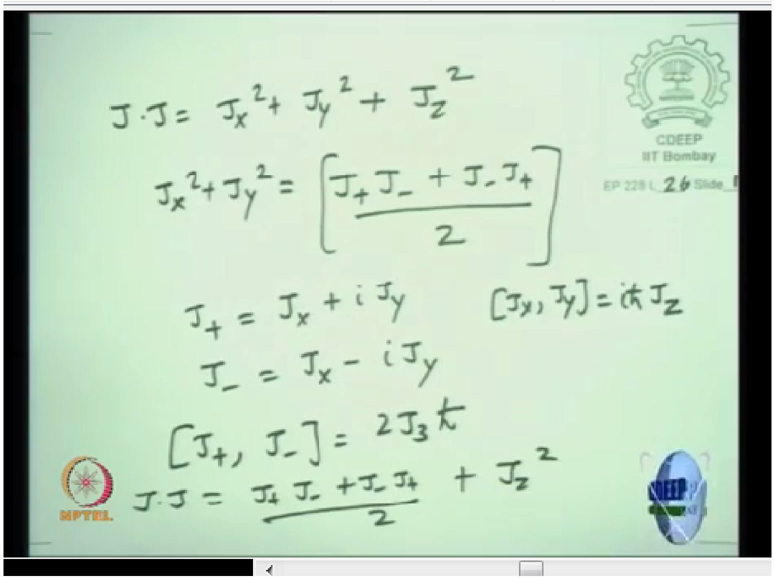
So I was trying to say that let us take 2 independent oscillators with 2 a and a dagger as the raising and lowering operators, lowering and raising operators for the one oscillator and b and b dagger for the other oscillator, but they are independent. What that means is that the commutator of any of the a operators with the b are 0. So we wanted to construct the angular momentum operators using these 4 operators, okay.

So that is what we were trying to do. If we take J+ to be this, the order really does not matter because they commute, could be a dagger also, does not matter, right. So if you take J+, it is not Hermitian. If you take the Hermitian of that or the dagger of that conjugate, you get J- and J- is this. Is that right? Then I said that the angular momentum algebra, please go back and use the angular momentum algebra to determine what is J3 constructed out of these 2 operators.

So this is the representation for the J+ and J- in terms of 2 harmonic oscillator operators and using this representation, we wanted to be satisfying in angular momentum algebra which will give you what should be the representation for J3. So J3 if you work it out, it will be the difference between the number operator for the harmonic oscillator given by a which I call it as N subscript a.

Similarly, that is the number operator, -the number operator corresponding to the b harmonic oscillator given by the operators b and b dagger. Please verify this. Once I know J+, J- and J3, can I write J dot J in terms of J+, J- and J3 sum one? What will that be?

(Refer Slide Time: 03:50)



Right. So can I write this Jx squared+Jy squared as J+J-, is this correct? I need to do this. Why? Because J+ and J- do not commute. So what is J+ and J-? Jx+iJy. J-=Jx-iJy, is that right? So if you take the product of these 2, it is Jx squared+Jy squared but you also have a JyJx JxJy which do not commute. So you need to take the other combination so that that cross term cancels. Classically you could have just written it as a product.

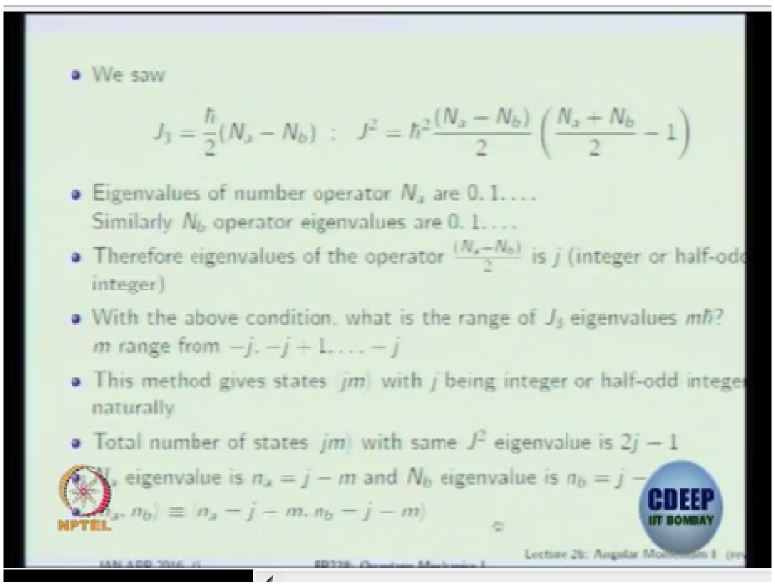
Quantum mechanically you have to symmetrise it so that it takes care of that J+ and J- do not commute. So you have seen that J+ with J- is 2J3h cross. May be I missed the h cross there, I should put the h cross. This can be derived from using the fact that Jx Jy=ih cross Jz. Use this fact and you can derive this. So Jx square+Jy square is this. And Jz square you can just write it, add to it and that will give you the J dot J.

J dot J will be J+J-+J-J+/2+Jz square. So the J+ is given to you and J- is given to you. J+ is given to you in terms of these harmonic oscillator ladder operators to commute in harmonic oscillators to independent harmonic oscillators. And J3, you have derived but if you have not done, please go back and check this. And using this point what is J dot J, please work it out. If you work it out, this is also an exercise for you.

You will get the result for J squared. So each one has a h cross. So when you take a product, that will be h cross squared. Substitute in terms of the representations involving a a dagger and b b dagger. Finally can be neatly written as the sum of the number operators, 1/2 the sum of the number operators*1/2 the sum of the number operators+1. Please verify this. But please check this out and you need to use those properties which I gave you.

Do not write blindly Jx squared+Jy squared as J+J-, you will not get anywhere. You have to write the symmetry combination/2. Is that clear? So what next? J3 operating on eigen state and J squared operating on eigen state, they are simultaneous eigen states. Because J squared commute with J3.

(Refer Slide Time: 07:28)



So what are the eigen values of the number operators of a harmonic oscillator? It starts from 0. The number operator eigen value starts from 0 and goes on, there is no upper bound. Similarly, the number operator for b operator is also similar. They are independent. It is not required that the number operator eigen value of Na if you find it to be in the first excited state, number operator Nb can be in the second excited state or a some other excited state or ground state.

They are independent oscillators. So the eigen values of the operator Na+Nb/2, suppose you take the sum of the number of operators, 1/2 the sum of the number operators, what will be the eigen values looking like? 1/2 integers, right. It could be integers or 1/2 odd integers. So let us call that as J to make contact with the angular momentum. J you know that it could be integer including 0 and 1/2 odd integers, very beautiful.

Naturally comes the angular momentum algebra when we try to do, this Na+Nb/2, the 2 factor is very interesting. It allows for J which can be integers or 1/2 odd integers. We will call this as J. We will see the meaning very soon, why we call it as J? What is the range of the J3 eigen value? You would have blindly written J3 eigen value as mh cross. And you would have said mh cross should go from -J to +J, that is what you would have said from the hydrogen atom problem.

Does it fit in here? It is Na-Nb. For a given J, if suppose Na is 0, Nb has to be J which means the J3 eigen value will be a -Jh cross. If Nb is 0, it is +Jh cross. For any arbitrary values which is non-0, you are restricted it to be J. So therefore, the J3 eigen values of m should go from -J to +J. It comes natural. Is not it interesting? I do not know. I constructed an algebra using 2 harmonic oscillators and I insisted that this is an angular momentum algebra.

I am just trying to explore what are the eigen values. What is the eigen value of J squared? h cross squared J*J+1 which is exactly what you have said. And the range of m comes natural. It goes from -J to +J. So the m explicitly is difference between the 2 number operators and similarly, the J in the J squared eigen value is the sum, 1/2 the sum of the number operators. This is what is the identification.

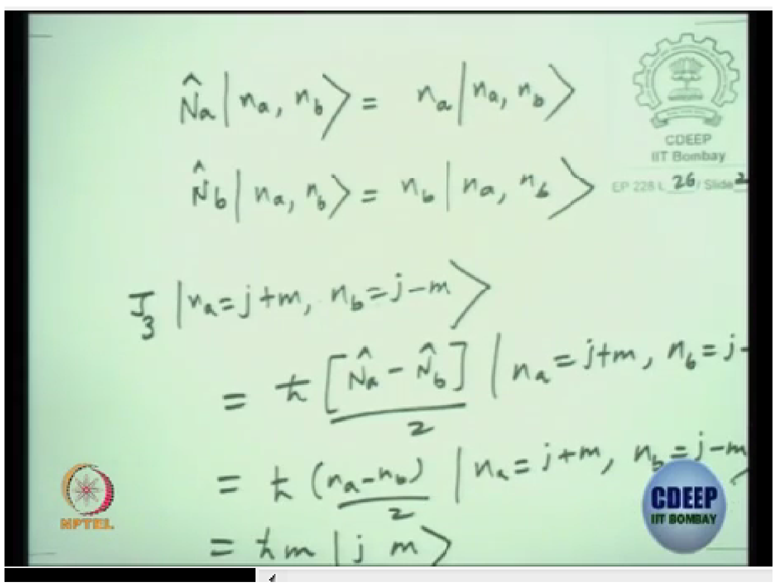
You identify that it will take 2 harmonic oscillators, 1/2 the sum of the number operators of both the harmonic oscillator, should be identified as the angular momentum J and the magnetic quantum number m will be 1/2 the difference between the number operators. Once I take that, then my range of them is automatically fixed to be -J to +J. Any questions on this? So range of m goes from -j, -j+1, the highest value can j take is +2, okay.

So this method gives states jm with j being integer or 1/2 odd integers naturally and the total number of states jm with the same j squared eigen value is, m goes from -j to +j. How many are there? m is going from -j to +j. So the number of states with the same j squared eigen value will be 2j+1. You can also rewrite the eigen value of na in terms of j and m. We will see why we want to do this, right.

Given Na+Nb/2 is J, Na-Nb/2 is m. You can try to write na as eigen value of the number operator with j+m and similarly, the eigen value of the number operator b is j-m. Will this always be integer? Yes or no? j-m? Has to be integer. j is 1/2 odd, then m also is 1/2 odd, so the difference between two 1/2 odd will be integers or the sum will be integers. Very beautiful. Conversely also it is right.

It is consistent. If you start getting the number operator eigen values to be 1/2 odd integers, we are stuck. We cannot use this, right. Now what do we want to do? We can work either with these 2 independent harmonic oscillator with Na and Nb eigen values, rewrite the states of the J3 operator in Na, Nb eigen values or you can even rewrite it in terms of the small j and small m, that is what we would like to do, okay. So let us do that.

(Refer Slide Time: 13:34)

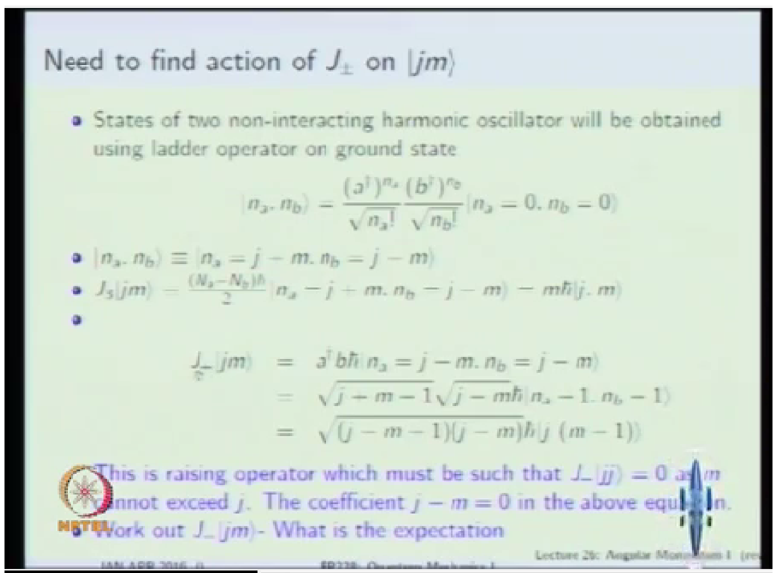


So what did we see? We saw that Na operator, it could operate on a state which is na and nb. It could give me na. I can write the state as a simultaneous eigen states of both the oscillators because they commute. Similarly, Nb. Is that right? We can do this. I can also write na as j+m and nb as j-m. And look at what this J3. J3 is nothing but h cross*Na operator-Nb operator/2, right. I can do this?

What will you get? h is h cross na-nb/2 and na-nb is nothing but 2m/2 will be m. You will get h cross m. You can rewrite, instead of using na nb, I can also write it as jm, there is no problem. It is synonymous or equivalent. We have a representation or the angular momentum operators in terms of the ladder operators of the harmonic oscillator, I can play back and forth. No harm in doing this.

Is that right? I can do this. So formally, I would say that I can either use na, nb which is equivalent to, na is j+m and nb is j-m which I can formally write it as j,m. Either the state can be denoted by na, nb or I can denote the state by j,m, there is no problem. Where the relation between na and nb to j and m is exactly this, okay.

(Refer Slide Time: 16:05)

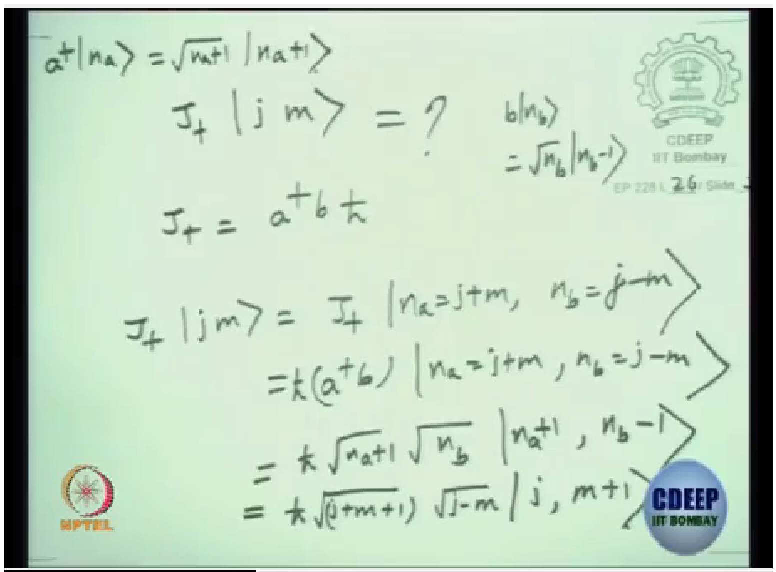


What is the next thing you need to do? J3 it was an eigen state. We need to do the, like the way we did in harmonic oscillator, how the a operates on the number operate eigen state. It was not an eigen state of a or a dagger. We try to determine what does a do and what does a dagger do? Right? We want to do a similar thing here, right. How do we do this? Very simple now.

You can write any arbitrary number state with an arbitrary na and nb in the harmonic oscillator context, 2 independent harmonic oscillator, as if you have this raising operator na*, on the ground state or vacuum state with this normalization. And similarly, the b dagger, nb*, with this normalization on the ground state. So this is an operator representation or operator acting on vacuum which will give you any arbitrary state of 2 independent harmonic oscillator.

Also saying verbally what this mathematical equation means. Now we are going to use the fact that na nb is nothing but connections with the angular momentum, magnetic quantum number and the j quantum number is this. So do this J3 on jm, is mh cross jm. This is what I derived for you. Let us work out. J+ on jm is nothing but a dagger bh cross, that is the representation in terms of the harmonic oscillator operators, the J+, that was my first slide in the Schwinger method. But you know how the b operates, how the a operates? So let us do this, this for.

(Refer Slide Time: 18:23)

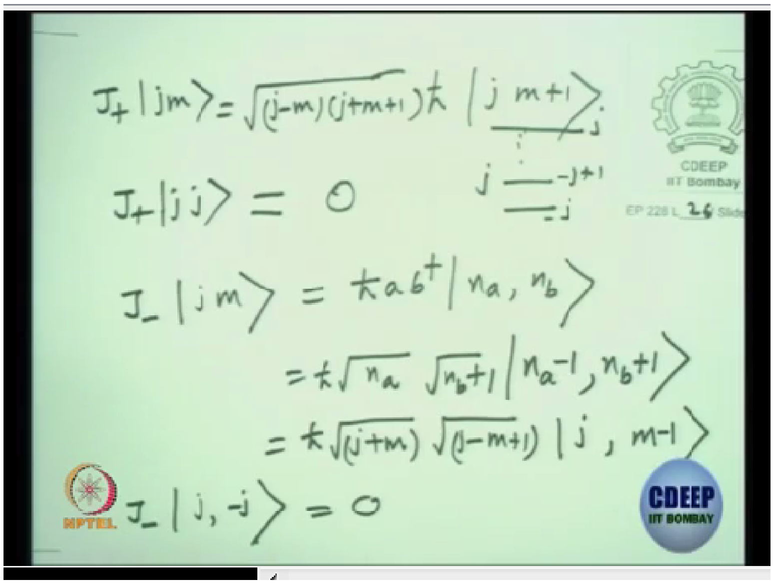


So I want to know what is J+ on jm? J+ has a representation which is a dagger bh cross. So you want to operate J+ on jm as equivalent to J+ on na=j+m and nb=j-m, the same, the state is the same. And here I can operate a dagger b, h cross is anyway a number, na=j+m, nb=j-m. What will b do? b will operate on the first state or the second state? b will operate on the number state nb.

What will it do? Recall for me what is b/nb? What is b/nb? Root nb*nb-1. What about a dagger on na? Root na+1*na+1. Let us use that fact. What will this give us? It will be root of na+1 root of nb on na+1 nb-1. Na is j+m, nb is j-m. What happens to the J3 eigen value and the J eigen value? J eigen value is same which is what you expect. J3 is increased by 1, okay. So we can rewrite this as h cross, instead of n, I will put it as j+m+1.

Instead of nb, I will put it as j-m, is that correct. nb is j-m, na is j+m and what will I write here? I can write this as j, j does not change. m changes to m+1. There is one more thing you can check. If m is j, this also you know. You cannot go to j, j+1, right. m range is from -J to +J. What is J+ on j, j?

(Refer Slide Time: 21:20)



So we find here J+ on jm to be square root of j-m j+m+1 h cross on jm+1, this is what we find. What is J+ on jj? 0 or equivalently this is also making sure that your m do not go above j. Unlike your harmonic oscillator, we have an upper cut-off also. In harmonic oscillator, there is a lower cut-off that you cannot go below 0 in the number eigen values. But upper eigen values can go anywhere.

But here you see that there is an upper bound also. A redo the same thing for J-. Can you do the same thing for J-, all of you? What is the operator for that? ab dagger on na, nb. We will substitute later. What happens? This will be, a will give you na, b dagger will give you nb+1 and state na will shift to na-1, b will shift to nb+1. What happens to the J eigen value? What happens to the magnetic quantum number?

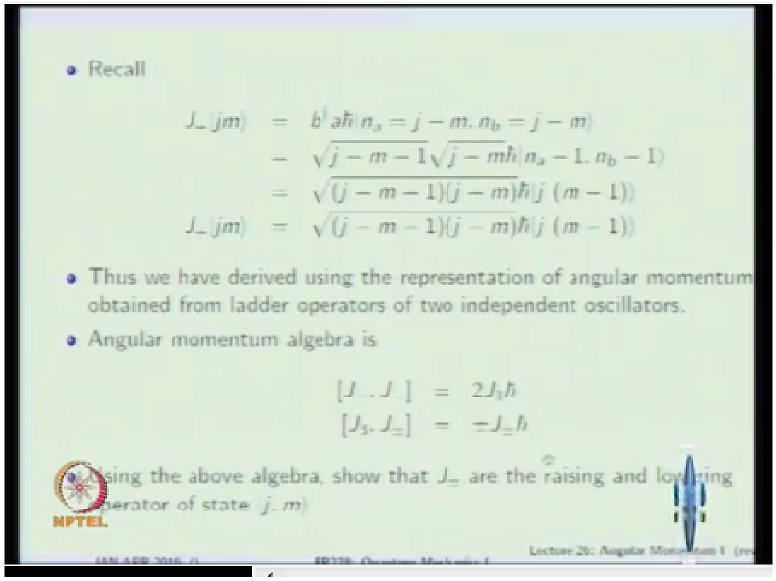
Rewrite. j is same but m shifted by -1. So let us rewrite na as j+m, j-m+1, right. I am substituting na as j+m, sorry j+m and nb as j-m, this is what I am doing. And this I can write it as m-1. Is that right? So this is like a raising operator which takes the magnetic quantum number by 1 unit up with this coefficient and this coefficient we have tried by writing a representation for J+ in terms of ladder operators of 2 independent harmonic oscillator and we have figured this out.

We could do this independently from the algebra but right now, I have done it by the Schwinger method. And we see that there is an upper bound that the raising operator cannot go to infinity if it operates on this j where the magnetic quantum number is same as the j squared quantum number, is going to be 0. What happens for J- -j? m=-j if I operate. Look at this eigen value. It will become; so what do we get?

We get stage for a specific j, it goes from -j+1 dot dot dot and there is an upper bound which is +j for the magnetic quantum number. For a given J eigen value, this m, magnetic quantum number will take values from -j to +j. So this multiplied with a given j or all states with a given j will be 2j+1. So just to rego over it on the slide. So this is what we did and you can try to write J3 eigen value as mh cross and then J+ if you try to do this, you will get na to be raised to 1 and nb to be decreased by 1 which you can write it as j.

And magnetic quantum number increased by 1, j does not change and you have rewritten all these na eigen values and nb eigen values in terms of j+m and j-m. And this is a rising operator which has this property that there is an upper cut-off as m cannot exceed j. And you see this naturally and you do this, you do see that when m=j, this term becomes 0. So similarly J- is a lowering operator and we have done this also now. And the lowering operator has a lower cut-off. It cannot go beyond m=-j. So you can check this out.

(Refer Slide Time: 26:29)



Please reverify things? You can rewrite it as a lowering operator, okay. So just to summarize, we have derived using representation of angular momentum obtained using 2 independent oscillators and we want to, main aim was to keep in mind the angular momentum, algebra generated out of them, that is they have to satisfy this, okay. J+J- is 2J3h cross and the J3, the 2 ladder operators, raising operators in the angular momentum context which has an upper cut-off, cannot go indefinitely and - is the lowering operator.

This has to have + or -, J+ or -*h cross, okay. So this is what we call it as a angular momentum algebra. What you could do is just like we did for the harmonic oscillator algebra, this is for you to check. We will discuss this on Friday. Independently take it to be simultaneous eigen state of J dot J and J3 which is j,m. Use this algebra to figure out that J+ and J- will reduce the m by + or -1 or change the m by + or -1 and we need to fix the coefficient.

Is there a systematic way in which you can fix the coefficients, just like the way we did for the harmonic oscillator and we fixed root na and root na+1, I want you to do it just for say looking at this algebra without using Schwinger method of 2 independent oscillator. That was a representation or a dress for this algebra and we derived this top 2 but can we derive these 2 by just looking only at this algebra, without giving a representation or dress in this fashion.

By dress or representation I mean we wrote J- in terms of the ladder operators of the harmonic oscillator. Now we do not give anything but we just look at this algebra alone and can you fix that you get J- as this and J+ raises it by 1 unit with this explicit coefficient. I will leave it you to try it out yourself and we will discuss on it.