**Classical Mechanics: From Newtonian to Lagrangian Formulation**

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**Lecture - 56**
**Small oscillation - 4**

So, we quickly recap what we have done so far for small oscillation and then we will continue.

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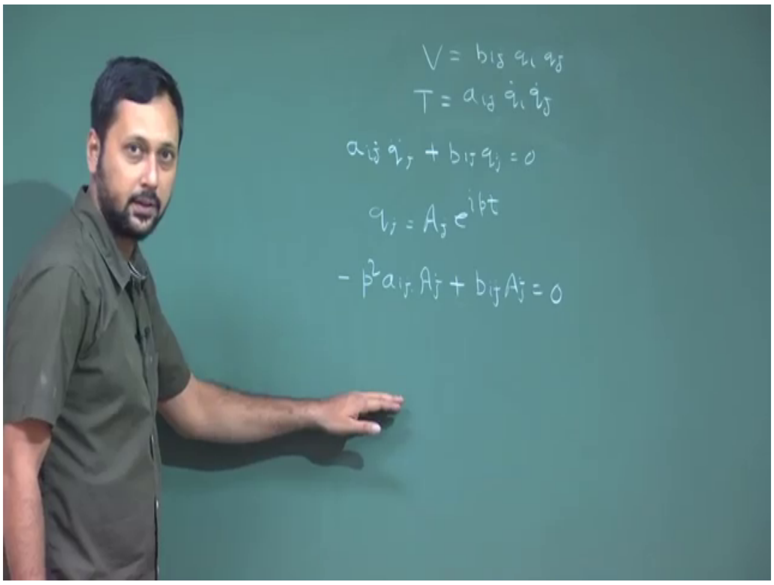


So, we started off by writing the kinetic energy potential energy V as b ij q i q j and T as a ij q i dot q j dot right so we did that. And then we have so a ij’s and b ij’s they are found to be real symmetric matrices and from there we wrote the Lagrangian, then the differential equations; so the Lagrangian and the differential equation was if I am not very wrong.

So, we took a trial solution. Essentially for the differential equation which was a ij q j double dot plus b ij q j equal to 0 and we took a trail solution of the form q j equal to A j e to the power i p t because we are assuming oscillatory motion. So, we took this trial solution when we substituted in here we got an equation of the form minus p square a ij A j plus b ij A j equal to 0 and this is a set of simultaneous linear equations. So if and only if the determinant of this coefficient term is equal to 0, so that means determinant of this equal to 0 only then this will the solution of this equation will exist.

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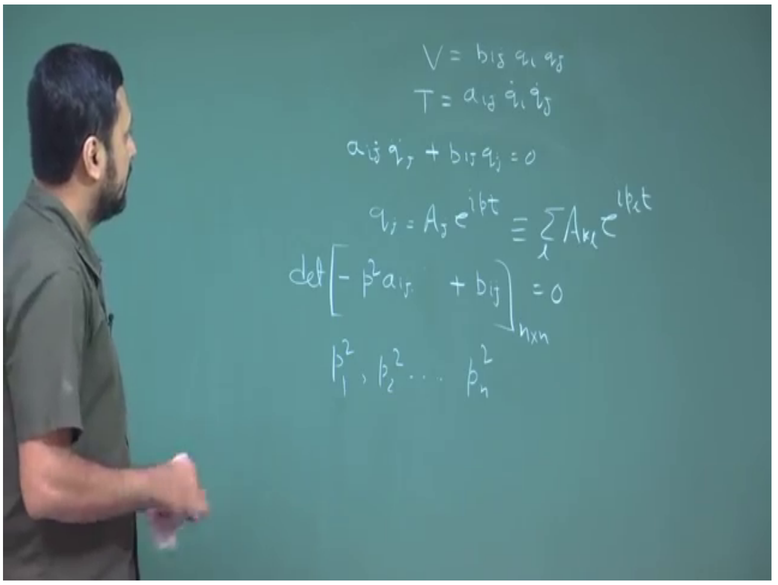


So, we took we started from here and then we found out the values of so this will be an n cross n determinant so it will have n roots in general. By solving this we would find p 1 square, p 2 square up to x square right.

Now what we do next is we have to; so the argument was each of this Eigen values will correspond to one set of Eigen vectors right. So we know it is typically it is something like an Eigen value problem this equation after substitution looked something like an Eigen value problem. And for each Eigen value problem we have seen previously we have already done that during our I mean during most probably your mathematical methods courses, you have done it already I am assuming. So what happens in is for each Eigen value there is one distinct Eigen vector. And each Eigen vector for an system of n particle each Eigen vector we will have it is a column matrix a column matrix will have n number of elements, right.

Now, for total n number of such Eigen values we have n number of such distinct Eigen vectors which will mean there will be all together n cross n that means n square number of elements. So that is what we have to that entire set is something we have to calculate. Now in order to do that we have introduced a second parameter and we have written the most general solution for this one as; sum over l A kl e to the power i p i t. And I guess you understand why it is, why we need 2 indices and why it is justified right.

Now, we will develop from here and try to see what can we make some comment on the properties of this A's and p's.

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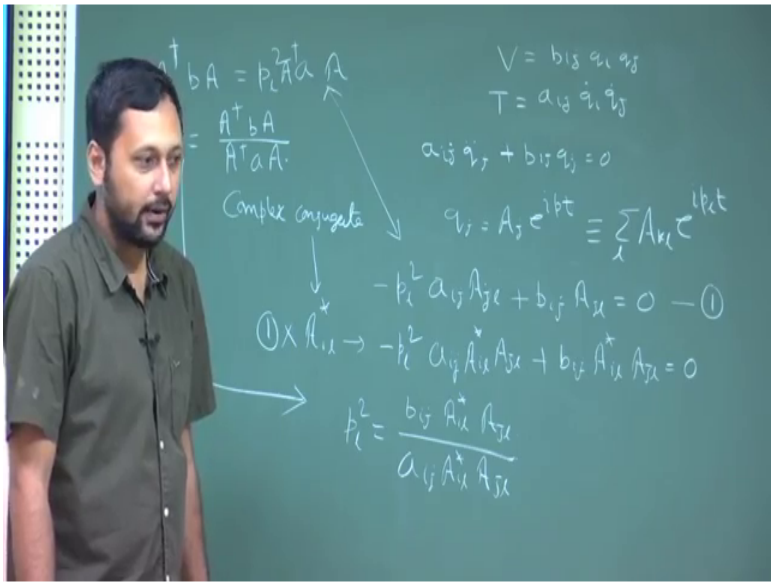


So, what do we first we substitute this into the original equation, and so if we do that we can write this as: minus p l square so basically this equation we can after this substitution we can rewrite as A kl sorry; there will be an a factor of course, it will be a ij A kl not sorry not k it will be jl right if the indices has to match plus b ij A jl equal to 0.

So, of course there will be an exponential term which will cancel out from both these terms, right. So that is why because the summation is over j right now; if the summation is over l also then we will have a problem, No, sorry it is not about summation it is just that the if term will cancel out nicely.

So, what we do now we multiply this equation; let us call this equation 1. Now let us multiply this equation by A il star from multiply this and if we do that we can write this as p l square a ij A il star A jl plus b ij A il star A jl equal to 0.

Now if I look at the matrix equivalent of these equations, this and this equation what is the matrix equivalent; look at it carefully. So the matrix equation to begin with was b A is equal to p l or rather it will be p l square no problem a A, right. So, what we did here this operation essentially mean so this is the same equation exact same equation in matrix notation so these two are equivalent; just that when I open this so matrix operation and write it by component you will get this. Now what I did here I just multiplied this from this side with A dagger. So, once we do that we have an A dagger on this side A dagger on that side. Now dagger means it is a transpose conjugate we have to take the transpose of this matrix and conjugate numbers. So if it is a row matrix or a column matrix it will be converted into a row matrix and vice versa.

So, this is to begin with it was a row matrix I sorry column matrix. So this transpose matrix is a row matrix and each element of this matrix has been converted into it is complex conjugate. So that is why there is this A star, A star means complex conjugate. I mean, I am pretty sure you are familiar with these terms, if not I think you can always check your mathematical physics textbook for this operations.

Now once we do that, now please understand this without this matrix multiplication from left this equation is a matrix equation: so b A is a matrix p square a times A is a matrix now we cannot divide 2 matrices that is not a defined operation. But once we do this multiplication it becomes a number on this side it becomes a number on that side but we can so we can always divide 1 number by the other. So, what we can do is from here we can write p l square is equal to A transpose b A divided by A transpose a A and this is a totally valid operation. But if we instead of writing this if I just write b A divided by a A this is not a valid matrix operation.

So similarly the same equation if I write in component form it will be p l square which will be equal to b ij A il star A jl denoted by a ij A il star A jl ok. So, this is one operation; this is one equation. Now each of this column matrix right now we do not know the nature of it, but all we know that if we do a general Eigen value operation on a square matrix we find out Eigen vectors and Eigen values both are complex in general.

So if we are not sure about the nature of any number a particular number or which comes out as a solution of some equation, it is safe to take this as a complex number.

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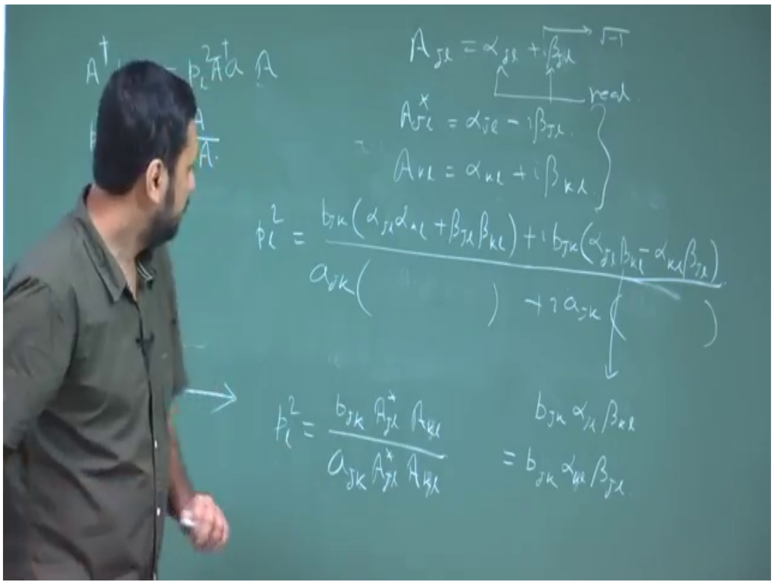


So if we do that, let us say A il is a complex number in general. So we can decompose into alpha i l plus i beta i l right. Where, alpha and beta both are real numbers. So combination of 2 real numbers and i is equal to root over of minus 1. So these are 2 real numbers in with an i it is. So, what I will first go into do is, it might be bit confusing so I will just change this index because this 2 i’s might create confusion. Anyway, these are dummy indices. So, what we can do is we can write j k, we can make it j we can make it k. Similarly this we can write j k j k, ok.

So, this is this is totally valid, because this indices are any way dummy indices we are free to change it so it will j and j. So j star will be alpha j l minus i beta j l. So now if I open the similarly we can write alpha k sorry A kl sorry so if I substitute this 2 into this equation what we will get is I will just write it here so this once again this is the complex conjugate operation. So, because these are real numbers what we have to replace i by minus i if they are complex number in general we have to also include complex numbers of the complex conjugate of this one, but right now because we have decomposed it into this format where these 2 are real we just have to replace i by minus i that is the complex conjugate operation I am sure you are familiar with this.

So, p l square is equal to open up this bracket and we will see I am just leaving it to you for thorough calculation it will be alpha j l, alpha k l plus beta j l beta k l. And so there will be a plus i b j k it will be alpha it does not really matter because any way this term will cancel out we will see soon it will be j any way let me try to get it right i j l beta k l minus alpha k l beta j l right and divided by it will be A j k same term.

So, it is identical numbers are identical terms in numerator and denominator now if you look at this term please understand that j and k indices are dummy indices. So we are free to change it what we can do is if we interchange the indices we will see that these 2 terms are exactly identical to each other. So, this term will give you 0, I mean it is not that straight forward because what we first what we have to do is what we have to break it in to 2, so it will be forget about i, it will be b j k alpha j l beta k l will be b k j.

So, we are just interchanging the indices it will be alpha j alpha k l beta j l, but now please real remember that b is a real symmetric matrix so if it is a symmetric matrix it does not matter if we write k so k j is equal to j l j k so, now if I substitute it with j k it will be exactly equivalent to the second term so not only this part, but if I do this operation totally so if I take this b j k inside and do this operation basically open up the bracket this term will give you 0, so both sides will be identical.

So, this will leave us with the following so similarly for a i k because only property we are using for that operation was real symmetric nature of b j k A j k's are also real symmetric matrices so the same thing will happen to that.

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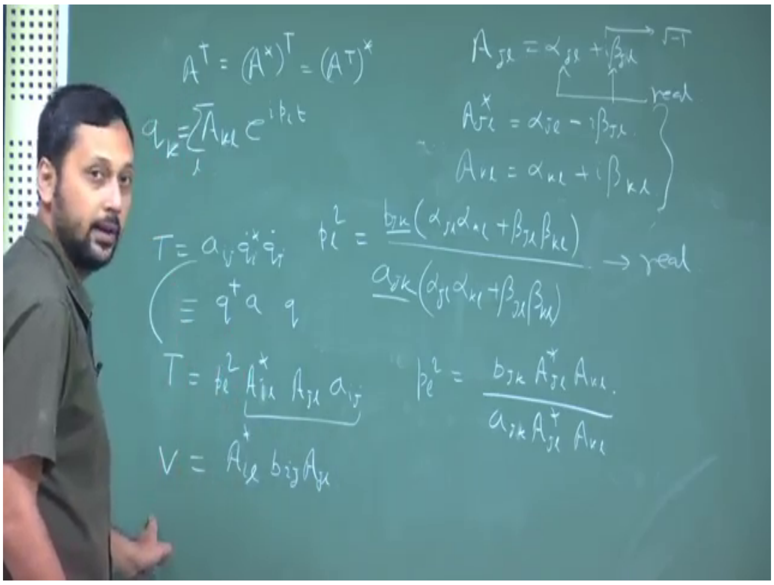

So, at the end these 2 terms imaginary terms will cancel out and we will be left with this ok. So, now let us look at this carefully first of all, all the quantities are real so the matrix element for b's and a's they are real we have already seen alpha and beta they are by choice they are real, because we have chosen it to be real right. So, this is how a total complex number can be expressed as a combination of 2 real quantities including an i, so the both numerator and denominator has real numbers, that means this quantity is bound to be real.

So one thing we immediately infer from this is the Eigen values we have obtained is real also if you look at it kinetic energy t is equal to a ij q I dot q j dot right, but once again we are now again dealing with quantities I mean q i's are combinations of a's I mean certain number of a is right amplitudes.

Now, these amplitudes are in general complex so if that is the case so we have to take we have to write this if I want to write this in matrix form I have to put a star sign here which will be equivalent to the matrix operation q dagger a ij q j right or rather i and j is not also required because these are indices anywhere so this will be the matrix operation corresponding to the kinetic energy. Now please understand that this star is necessary, because now initially when we started we have taken q's at the generalized coordinate without any knowledge whether it is a real quantity or an imaginary quantity, but now we know at this point of where we stand q's are in general imaginary.

So, that is why this operation is needed in order to write it in this particular form otherwise the simple transpose will do right if I find if we find out later on that as and q's these are all real quantity we still need to do this operation, but I mean we do not need a star here, but you have to but a transpose here right. Now we are putting a dagger; that means transpose conjugate. If you just to recall I guess if some matrix A dagger is A transpose of the conjugate matrix or A transpose conjugate so this is the dagger operation I am pretty sure you are familiar with it is used a lot in quantum mechanics it is one of the where most well known Hermitian operations any way.

Now, if you do that and we write the q as or rather q k as A l e to the power i p l t sum over l right so if I take this particular form it will be A l star just a minute it will be 2 indices first of all it will be k l so if I substitute this for q i and q j we will get some p l square A kl star A sorry i i l star A jl and that is it so T will be equal to this oh of course so there is the a ij totally forgotten and so that means, summation is implemented over i and j and l all 3 indices right so that will be the most general form of the general form of kinetic energy in the current representation.

Now if you compare this part you see this is exactly what we did right this operation the something similar operation we did to get the numerator and similarly we can write potential energy which will be something like A il b ij A jl right. So once again the summation is over A l and b. So, these are the terms we had when we wrote the original equation for p l square in the denominator and the numerator. Now we opened up the bracket just to show you that this quantity is real so this means, but originally this was written as p l square is equal to b what was it b j k A jl star A kl divided by A jl j k A jl star A k.

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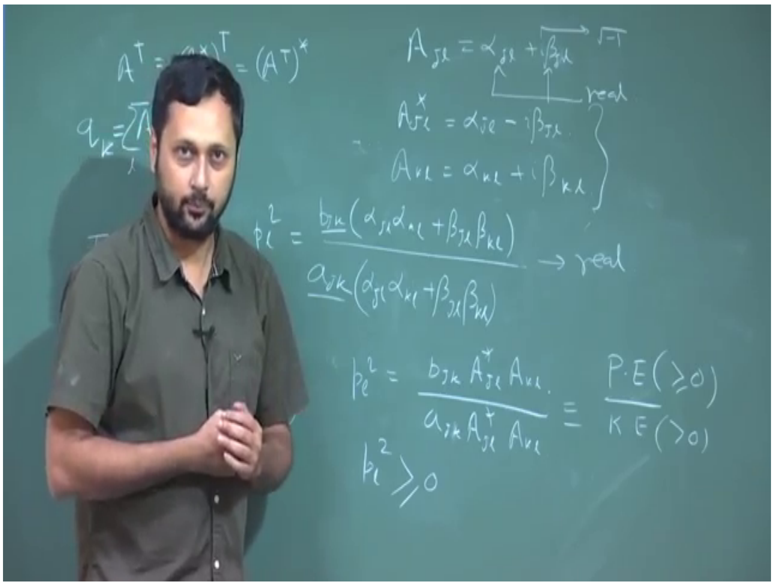


So, by virtue of these two relations we have got this is something like potential energy of the system divided by kinetic energy of the system right. Now, potential energy kinetic energy of a system which is oscillating which is in motion cannot be 0, it has to be some positive number if a in the trivial case when kinetic energy is 0. That means, there is no oscillation in the system so this is a special case of so except for the special case when there is no oscillation in the system the kinetic energy is always greater than 0.

And similarly potential energy right now please remember that for small oscillating systems we are working in the sorry my we are working in the potential minima, and any deviation from the minima if the system is even slightly deviated it is a positive number the potential energy which is which is it is gaining is a positive number. And the in case where the system stays exactly in the middle it is a 0 potential energy case.

So, the denominator of this expression cannot go to negative I mean it sorry it has to be positive greater than 0 and the potential energy that upper and the numerator is greater than equal to 0, so the limits of this is greater than or equal to 0 and has to be greater than or 0. So these 2 combined with this real nature we can we also see that it is a that the quantity t l square is greater than or equal to 0 for all the values of A’s of the system.

So we have two properties one is this is real and then this is greater than or equal to 0 combining these two we can safely state the systems the amplitude; sorry the Eigen frequencies we get by solving this equation they are all positive definite quantity or could be 0 right.

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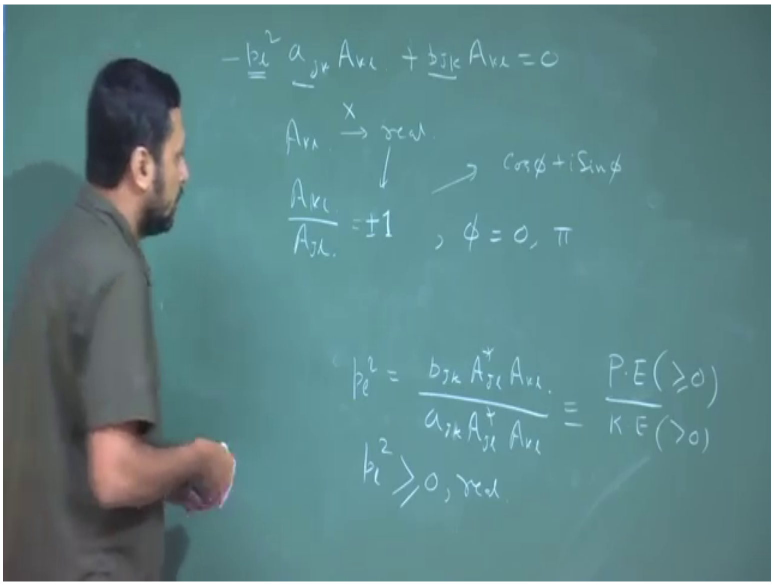


Now, once again what we do is we go back to this equation so this keep this in mind now we go back to this equation p l square a ij sorry A kl plus b j k A kl equal to 0 that was the equation we got after substitution for q q double dot and q right. Now in this equation so this is a series of equation and we will have for different values of l we will have different say I mean different equations like this, right.

So, in this equation this quantities are already real that we have seen we have I mean that is by construction of the matrices these quantities are real these quantities are positive definite or 0 I mean either I mean it is not only real, but greater than equal to 0 that we have seen. Now A k l's will be a solution of this set of equations right so if there are n such Eigen values we will have n such equations and a k l's will be come out as a solution simultaneous of this simultaneous equation.

Now if all the quantities of this participating equation is real we can safely state that A k l's are also real, but this might be little too much because sometimes what happens in at solving by when we are solving for Eigen value equations we do not find out the absolute value of the amplitude matrices, but sometime we get them in the form of ratios for a given Eigen values, so what we do is we put one particular Eigen value back right or sorry we sorry we get the differential equation for one particulars Eigen value and we have to solve for the ratios ok.

So, we can at least state if this is not totally, but we can at least say that these are real now this could be real and only if the phase factor between two you know between two amplitudes is of the form e to the power i phi where phi is equal to either 0 or pi, so we know that e to the power i phi is equal to cos phi plus i sin phi. Now if you put phi equal to 0, then we get this term vanish and we have cos phi is equal to 1. So idea is there is no imaginary part in this so it can have no imaginary part only if the phase factor between 2 amplitudes can can have either values of 0 or pi if you put pi this 1; this 1 again goes equal to 0 and this becomes minus 1. So without writing this we can also write that these are plus minus 1 ok.

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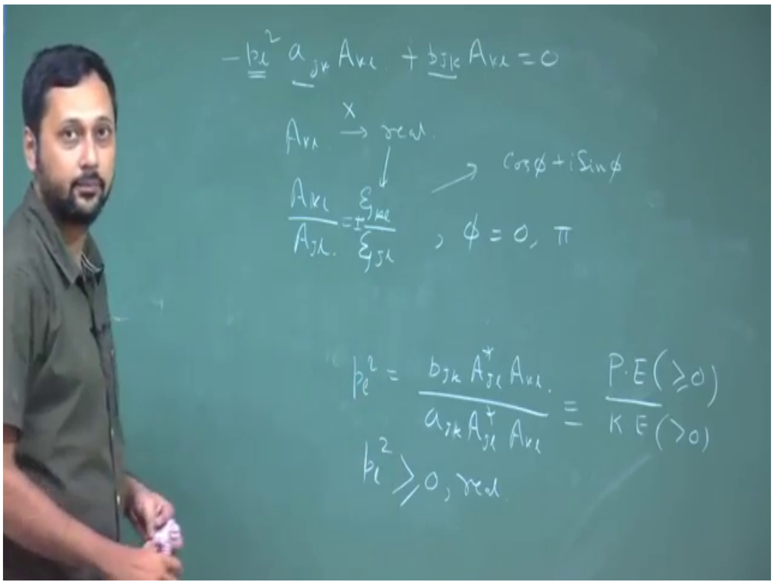


So, the phase I mean so I would so that was little mistake in my part I should write them as let us say this is the amplitude ratio the absolute amplitude ratio that is the magnitude of amplitude there will be a plus or minus sign here. So this plus or minus is coming as an outcome of e to the power i phi fine.

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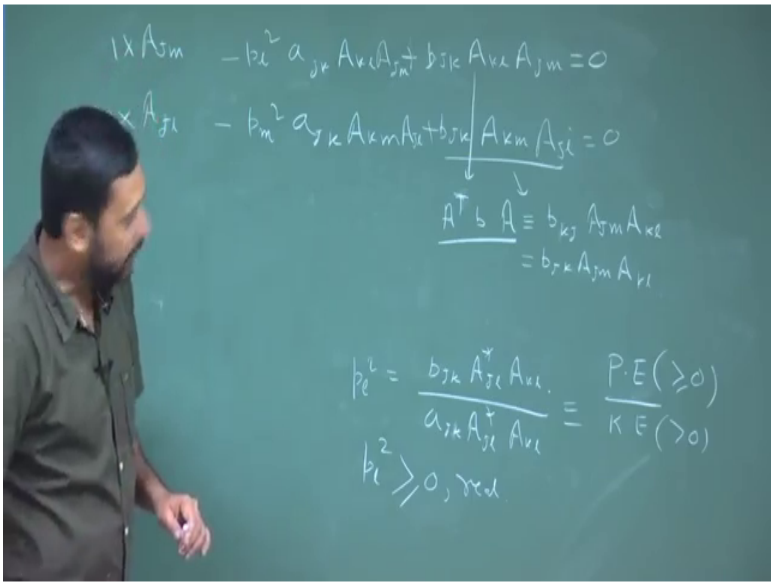


Also, let us do one thing little more and we will be almost through with this part let us do one thing this is for 1 Eigen value write let us write the same equation for another Eigen value. Now let us do one thing so it is just the same equation we wrote for another Eigen value p m let us do one thing let us multiply this one with what A j. Let us multiply this one with A jm right.

Let us do that, if I do that just give me a second; so you do that immediately by putting this number we are summing over j as well, initially we were summing over when this was not there the summation was only on k. So when we put this the summation is overall also j and so we have j m equal to 0 and so this one we just multiplied by A jm and this 1 let us call it 2 we multiply it by A kl we sorry A jl.

So, similarly if I put A jl here and A jl here, now what we get look at this operation here this operation now that we have proved that at least the ratios of this are I mean real so we do not need to take I mean complex conjugate we can only take transpose and in element form this equation can be written as A transpose b A, because once again please remember that j and k's are dummy indices we can do the same substitution without writing in elemental form.

We can also write this as b k j this one just interchange these 2 indices j k j will be A jm A kl right and as b k j and is equivalent b j k we can write this as b j k A jm A kl which is exactly equal to what we have up here. So both this terms will actually point out to this matrix operation I am just writing it in component which is a transpose b A. Now we have proved that as are most likely real we do not need a dagger, but we are good with the simple transpose operation that is the difference.

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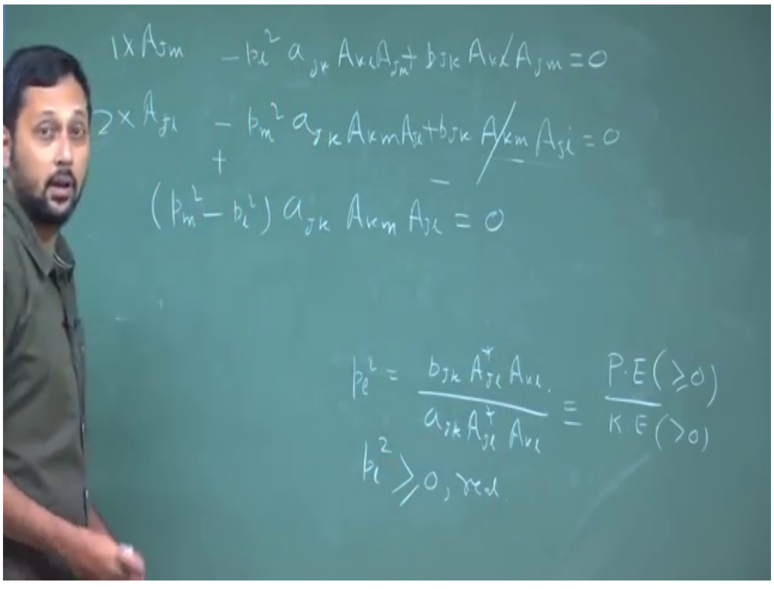

So now if I subtract one from the other of these two equations what happens is the second term, if I just make it plus make it minus this second term cancels out leaving behind p m square minus p l square A j k A k m A jl is equal to 0.

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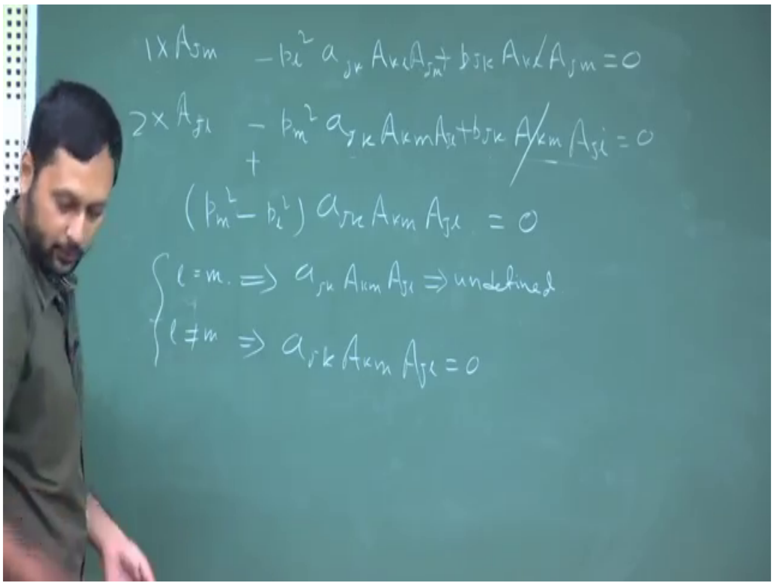


Now, in matrix notation this is nothing but oh no let us stick to the standard notation ok so these two terms are once again identical so A j k A k m A jl sorry let us stick to this now if l is equal to m. What do we see we say that p m is equal p l and this term is immediately going to 0 if this is the case then this is equal to 0 and A j k A k m A jl which is which we can you know we can set this is undefined. Now if l is not equal to n then this term is not 0.

And we see A j k A k m A jl will be equal to 0 in order to maintain this equation if this term is not 0 this has to be 0 if this term is equal to 0 this is undefined. Now these two relations together we can write as first we have to, because this is undefined this is there is no fixed value to it, so we are free to choose whichever value we are willing to put in here.

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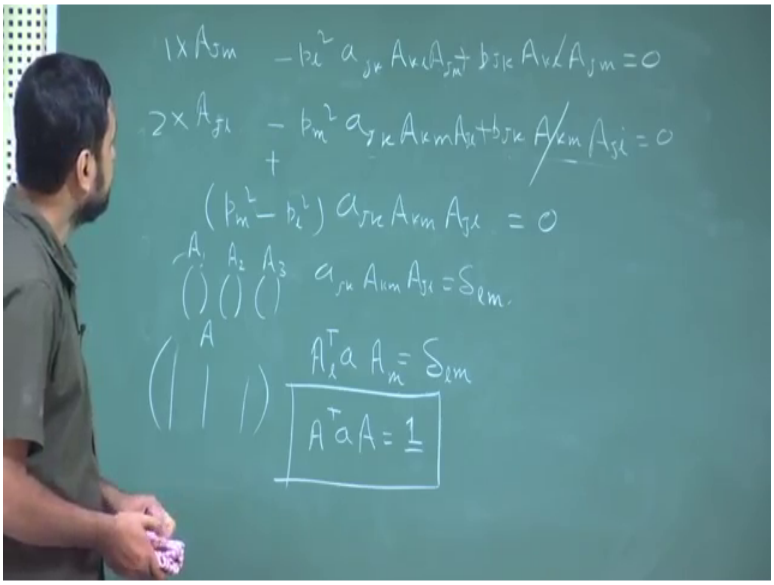

So if I put set this equal to 1, then these two relations together can be combined to write this is equal to delta l m when l is equal to m this is equal to 1, when l is not equal to m this is equal to 0 right and this again in a matrix form is just these operation A transpose a A where small a is the kinetic energy matrix big A's are the amplitude matrices amplitude column matrices is delta l ok.

So, this is one relation or you can put l here m here in more general case so what we typically do is we actually have let us say 3 individual column matrices I think we have discussed it already during the rigid body dynamics. If we have 3 individual column matrices A 1, A 2 and A 3 we club them together to form 1 matrix which we call A the first column will be A 1 second column will be A 2 and third column will be A3.

Now, if you follow the same construction here so this index will not be necessary because all capital A's and small a's are square matrices then. So we can write this as this 1 under bar is the notation for identity matrix now if we do that simultaneous there is also another very interesting thing coming out of it.

Now look at the first equation again and in it is pure form; no sorry in this particular form. Now if this is the case, right.

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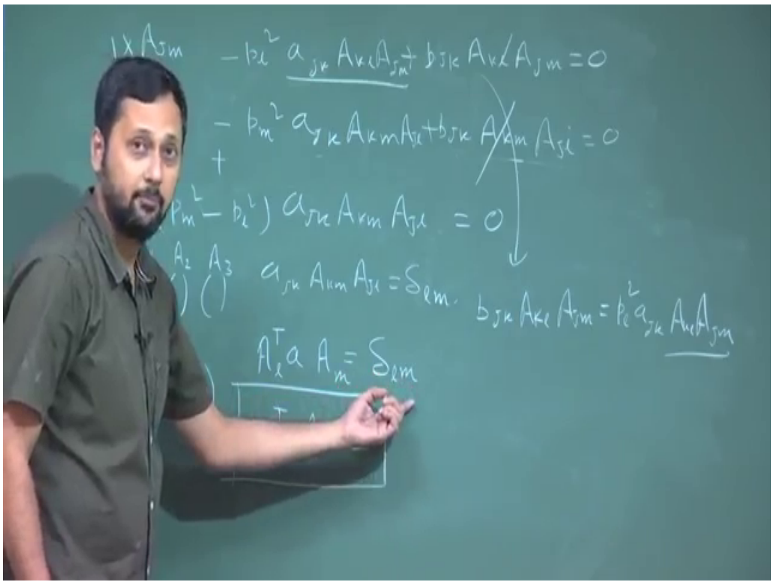


So in this first equation we have seen that this part; so this first of all this equation can be rearranged to give b j k A kl A jm is equal to p l square a j k A kl A j. Now this part we have already seen this is nothing but delta l m. So, we can write this term equal to p square delta l m right.

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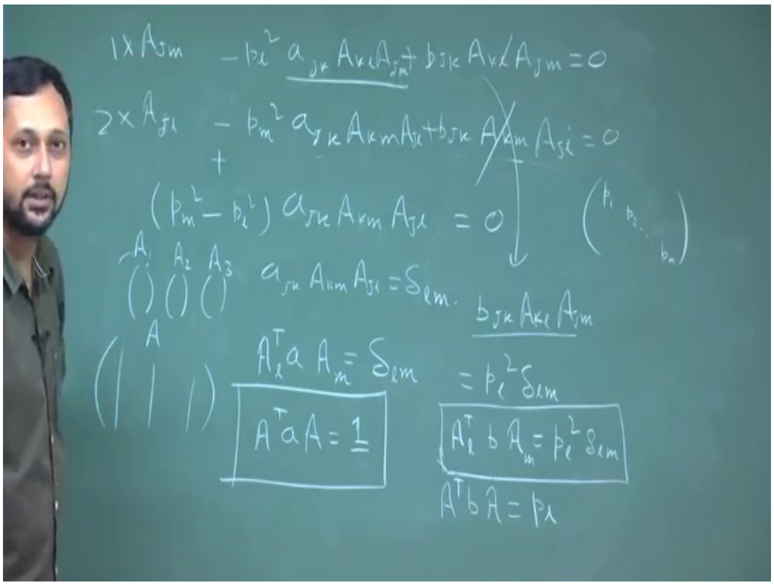


So once again this is also a matrix operation, in a similar manner if I construct it in the A matrix form we can either write it as A transpose l b A equal to p l square or delta l m. Or in more general form we can write this as if we construct this you know square matrix out of as we can write this as A b A; sorry A transpose b A equal to how do we write it- we can write this as p l.

I think this is the best representation, because this actually gives you matrix. For this I am not sure what should I write? What I mean is this operation, this square operation with square matrix will give you matrix in which we have p 1 p 2 p n. So it will be a matrix with where the Eigen values are in the dagger. So that is what I meant. But let us write it as p l square delta l m, this is also a good representation.

So, we have seen that at the end if we choose this particular normalization condition; please remember this normalization condition that this is equal to 1 is a chosen value. And we always have to do this because normalization condition there is no universal norm on how to normalize certain quantity so we are free to choose. And once we choose that we are simultaneously diagonalizing both kinetic energy and potential energy matrices in 1 go. So that is the beauty of small oscillation. Here it is the mathematics is bit complicated but we will take up examples in the next lecture and I will show you that in reality it is not that difficult, but once we solve the problem once for all we are simultaneously diagonalizing both kinetic energy and potential energy matrices.

So we will stop here for this class. In the next lecture we will pick up some examples and start working on them.

Thank you.