Classical Mechanics: From Newtonian to Lagrangian Formulation
Prof. Debmalya Banerjee
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 57
Small oscillation – 5

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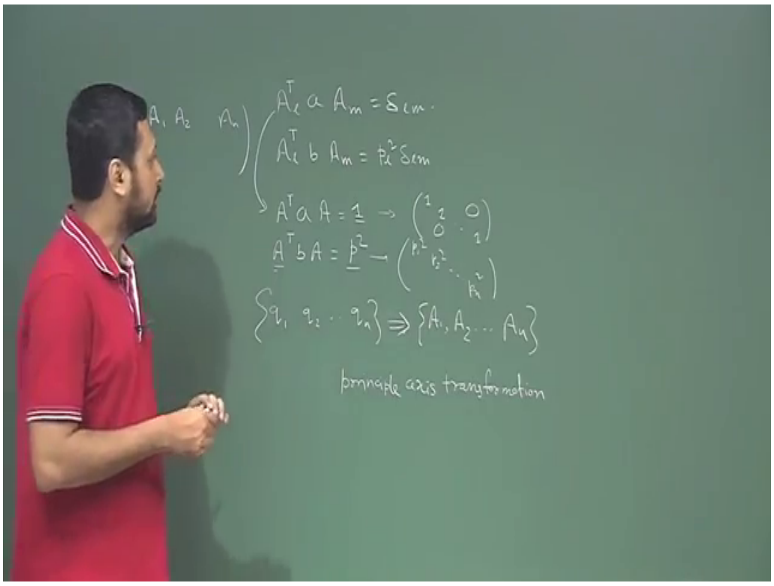

So, we are back with our discussion on small oscillation, now we ended the last lecture by describing this relation which was given by a l what was it a A m equal to delta l m and using this relation we could also prove that A l b A m will be p l square delta l m now both this actually it is a so I have we have written right now we have written it for one particular Eigen vector, but if we construct this combined Eigen vector A which will be putting so it will be like if we club all the Eigen vectors along their rows and from this combined Eigen vector matrix A, then this relations will translate to A sorry there is the transpose here and transpose there so this relation this will be a A equal to identity matrix and the second one will be A t b A equal to let us call it p square matrix so p square matrix is nothing, but this one is where you have only the diagonal elements as 1 and in this case you will have diagonal elements as the Eigen values or Eigen frequencies.

So, this actually signifies that the process or the procedure what we are following in order to get the Eigen values and Eigen vectors they are sufficient the or rather the Eigen vectors which we are getting they are able to diagonalize both kinetic energy and potential energy matrices at the same time so we are essentially we are going into a space after this or rather this vectors A or rather set of vectors A 1, A 2 up to A n they represent a set of orthogonal normalized Eigen vectors which determines the which let us say initially the coordinates what we have taken q 1, q 2 up to q n, they were not orthogonal definitely and so this describes a coordinate transformation process or principle coordinate transformation process which from which we move from q 1, q 2 q n to from here we move to A 1, A 2, A n so this side is not orthogonal not normalized this side is both orthogonal and normalized and this process is called a principle axis transformation.

So, this is the principle axis transformation and it is not unique in any sense I mean we have used it for so what we did for rigid dynamics rigid body was slightly different we actually went from one orthogonal set to the other orthogonal set when we transformed or we diagonalized the inertia tensor. Inertia tensor was initially obtained in some orthogonal coordinate system, but that that was not the principle axis, but later on we moved it into a diagonal frame in which it was diagonalized as well as I mean it was diagonalized so, but that axis system was also a orthogonal axis system. So we are going from one orthogonal to another orthogonal, but here we are going from one non orthogonal non normalized system to one orthogonal and normalized system.

Now, this relation for normalization actually this relation actually or rather this relation actually gives you a normalization condition which will give you the normalized values of the amplitude matrix A, we will take an example of this quickly and then we will move forward. So let us take this example; example of Triatomic molecules stretching.

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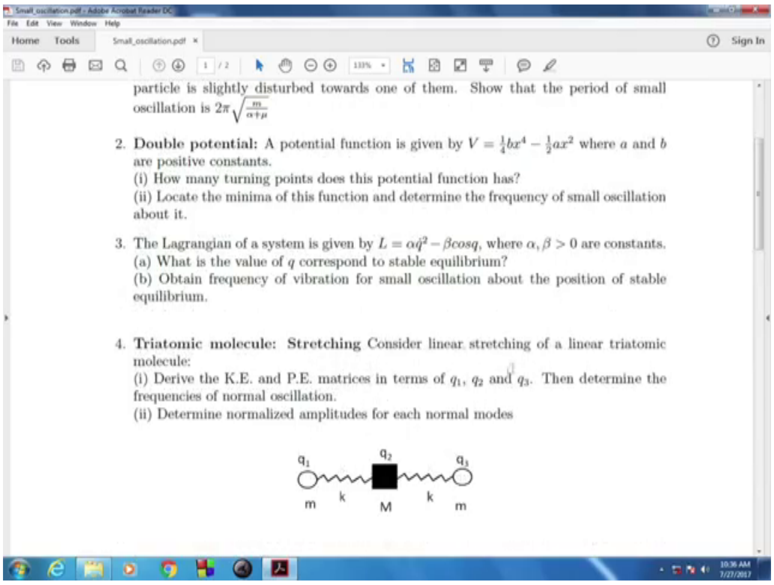

So, we have a linear Triatomic molecule here and this linear Triatomic molecule has 1 mass which is capital M, which is slightly bigger than the other 2 masses small m which are in the sides and let us assume that the we can model this linear Triatomic molecule in by a system which we by which we connect 2 springs between these 3 masses. First we have to derive the kinetic energy and potential energy matrices in terms of q 1, q 2 and q 3 and then we have to determine the frequencies of normal oscillation and then we have to determine normalized amplitude for each normal mode, right now we have not formalized the definition of normal mode we will do that, but first let us to give you a feel let us do this first part at the beginning.

So, we start with this linear Triatomic molecule and please remember that here we are discussing only the stretching of linear Triatomic molecule, now in a Triatomic molecule we can have different types of motion.

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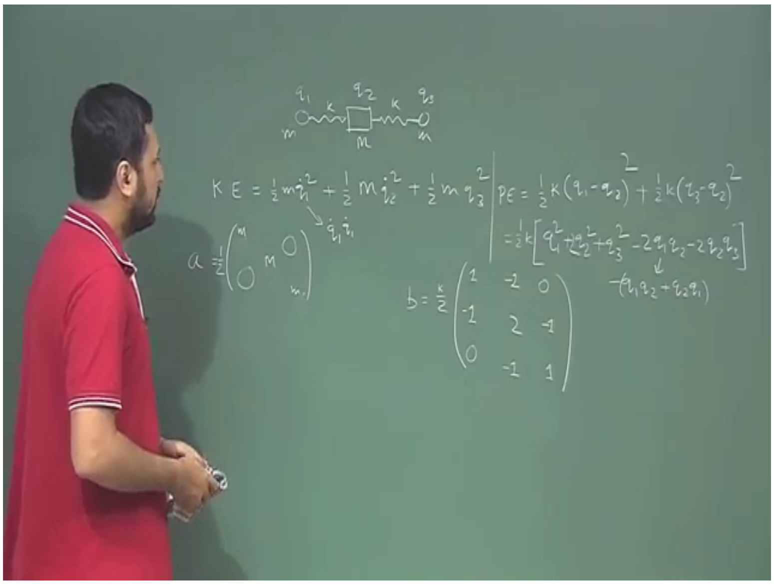

So, let us say this is this is a molecule based on Triatomic molecule we have m m capital M, so it can either stretch so that means, these 2 springs can you know oscillate like this also this mass can also oscillate in this plane whatever happens sorry in this line along this line.

Now, but apart from that also there are modes of vibration possible let us say in 1 mode these 2 can come down and this 1 can go up yeah or yeah so this is also one possible mode of vibration of course, there are translational modes along both the x and y and z axis rotational mode around y axis rotational mode around z axis we are not considering that at the moment, but just for vibration we can have both stretching and for example, this 1 is bending when this 1 goes up and this 2 goes down, so the molecule will looks something like this instantaneously and then once again this will go up and this will come down, but this is something which we will discuss later, right now we are discussing the stretching vibrations so that means, let us say in one instance these 2 you know these 2 atoms are moving out or moving in and this one might be stationary might be also moving means we will see that in a moment, question is first how to tackle this problem how to take the problem of writing kinetic energy and potential energy matrices.

So, we start with q 1, q 2 and q 3 as the normal coordinates please remember that q 1, q 2, q 3 they are the instantaneous position not the equilibrium position or you can take them as the displacement from equilibrium position so when the system is in equilibrium as in it is there is no vibration then all q 1, q 2 and q 3 you can take them as 0 oh that is your choice and whenever they move away from little bit slightly from the equilibrium you can take them as q 1, q 2 and q 3, right.

Now, if this is the case then let us try to write the expression for kinetic energy now kinetic energy will be half m q 1 dot square plus half capital M q 2 dot square plus half m q 3 dot square so these are 3 expression these are the expressions for kinetic energy, now if I want to decompose I mean if I want to construct the matrix out of it how will you do that so the a matrix will have elements only along the diagonal because there is q so q 1 1 q 1dot square actually corresponds to a term which is we can decompose this term into by writing q 1 dot q 1 dot so it is a 1 1 position so m by 2 will come here.

Similarly, for this 1 m by 2 will go there and for this 1 m by 2 will come here and we will have 0 here 0 here because there were no terms like q 1 2 q 2 3 q 3 1 like that we do not have such terms typically in a for almost all the systems you will find that kinetic energy is a diagonal matrix, but we will take up examples of course where kinetic energy is not diagonal we can have coupled systems, but right now we are dealing with the system where kinetic energy is diagonal. So, what we can do is we can just for simplicity we can take the m by 2 out and put it here, similarly we can write the potential energy which is see if from equilibrium this mass is displaced to by q 2 and this mass is displaced by q 1 then what is the net stretch of this spring, spring constant is k here and k here it is the identical spring, right.

So, the net displacement of the spring it does not matter that is a good thing about when we are writing the potential energy we do not care whether it is q 2 minus q 1 or q 1 minus q 2 please remember recall when you had to write it in the through Newton's law the equation of motions for a coupled spring oscillated system you always have to take care whether it is q 1 minus q 2 or q 2 minus q 1 and that problem we do not have to face here that is the beauty of Lagrangian.

So, for the first spring it is half k q 1 minus q 2 whole square and for the second spring it is q 3 minus half k q 3 minus q 2 square very good and open the brackets and we see once again there is a half k we will just keep it outside open the bracket and we will see that q 1 square plus q 2 2 q 2 square because 1 q 2 square will come from here plus q 3 square and from there first term we will get a minus 2 q 1 q 2 minus 2 q 2 q 3 very good we got this and now when we write this matrix b we have a k by 2 outside right and we have 3 diagonal terms q 1 1 term will be 1 1 2 2 term will be 2 and 1 3 3 term will be 1 what about the other terms we have a term q 1, q 2 we can put minus 2 here yes or no no we cannot do that because please remember that both these matrices are symmetric so we have to write them in a symmetric manner so this 2 q 1 q 2 we have to write it as q 1, q 2 plus q 2, q 1 (Refer Time: 13:05) you wrote it.

So, once you do that we have 1 we have minus 1 at q 1, q 2 position and minus 1 at q 2, q 1 position please keep in mind that these matrices are symmetric so we cannot have we cannot just write 2 here and 0 here we have to write minus 1 here minus 1 here similarly this term q 2, q 3 it has to be coming now row number 2 column number 3 so this is a minus 1 here and row number 3 column number 2 minus 1 here.

Now, there is no connection between q 3 and q 1 in terms of potential energy so there is no; no third spring so that is why this 1 3 and 3 1 position will be 0 so this is the shape of your potential energy matrix so we have a kinetic energy matrix we have a potential energy matrix please see if it is clear so I will repeat again kinetic energy is straight forward we have 3 diagonal terms rest at 0, for potential energy we have 3 diagonal terms 1, 2 and 1 and for the off diagonal terms we have to symmetries this cross terms so if it is q 1, q 2 you have to write q 1, q 2 q to q 1, q 2, q 3 you have to write 2 q 2 q 3 we have to write q 2, q 3 plus q 3 q q 2 so that is how we got minus 1 here minus 1 here very good. Now once we have these matrices it is time to diagonalize them as in time to find out the frequencies we need some space.

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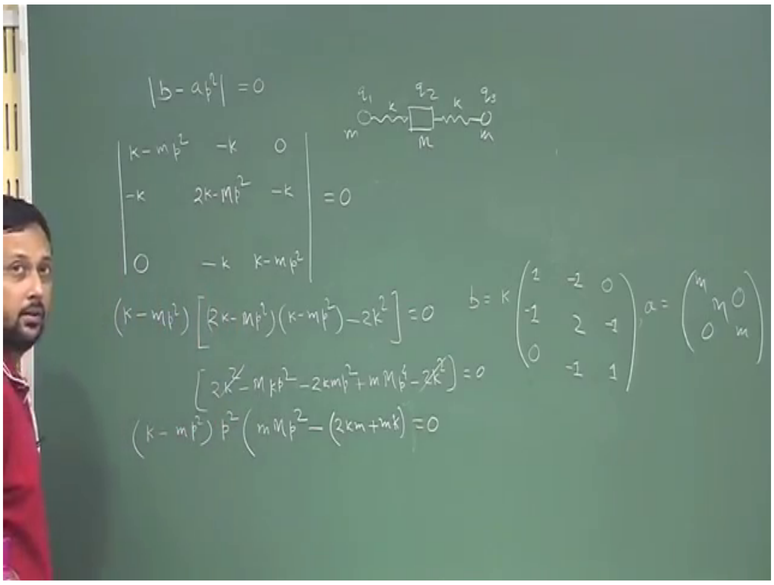

So, what I am going to do is I will use this space for solving, now what do we have one more thing is there is a 2 here and a 2 there we can get rid of this two's because it is a common factor it is just a common factor we can just get rid of this so if we do that recall that determinant b sorry b minus a p square determinant of this will be or rather I should put a symbol b minus a p square equal to 0.

So, we have to do a term by term substitution using this relation so it will be k minus m p square so it is b 1 1 minus a 1 1 p square second position it will be b 2 1 2 which is minus k, but a 1 2 is 0 so it will be simply minus k 0 and here also it will be minus k here it will be 2 k minus capital M p square here minus k 0 minus k k minus m k square this is equal to 0, I hope it is clear the first term coming from so these 3 diagonal terms only have p square because that is coming from the form of a if a would have had cross terms of diagonal terms we would have had p square in other positions as well right we do not have that.

Now, what do we have what do we have we have this so for once we open the bracket it will be k minus m p square, p square minus k square minus k so this will be 0 minus this will be 0 minus plus so it will be basically plus k square k minus m p square fine which is equal to 0, check the determinant a I know I am pretty sure you will find it yourself also.

So now, what we can do is we can take k minus m p square common once we do that actually so what will happen is k minus m if I take a k minus m p square common this term will cancel out. So finally equation will be this equal to 0 right the second term so sorry there is no minus k square term so this equal to 0 just a minute I think I have made some mistake which is which always happens oh yeah definitely I did sorry there was a yeah there was a minus sign plus sign so it will be 2 k square actually yeah I think just give me a second ha this minus k square minus k square so it will be minus k yeah so it will be 2 k square; now if it is 2 k square just keep this in my hand, right.

Now, if I open the bracket here inside it will be 2 k square minus m k p square capital m and minus 2 k m p square plus m m e to the power 4 minus 2 k square equal to 0 and this one this one cancels leaving behind so we have we can factorize it as k minus m p square and between this 3 terms we can take a p square out right that we can do which will leave us with m M p square minus 2 k m plus m k equal to 0 right so this is the final factorized form.

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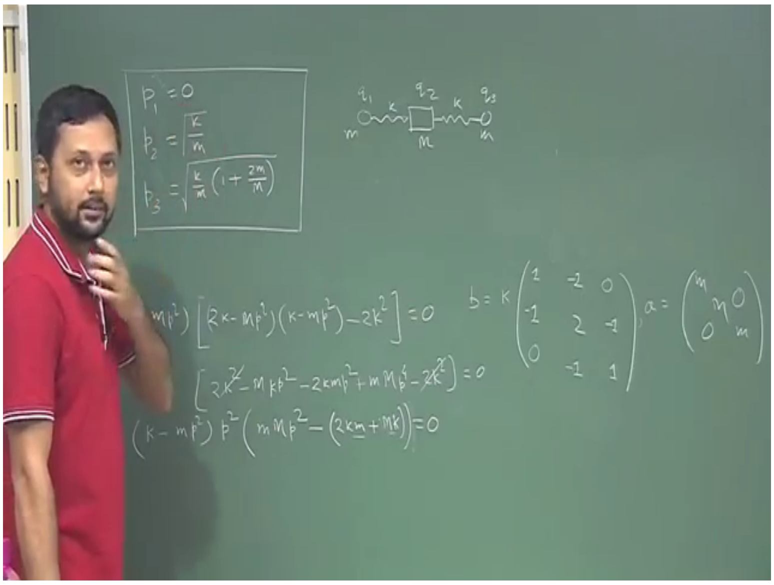

Now if I now write the values of this so there so each of this will be individually equal to0 each of this term so this term will give you p will have 3 values it is a third order equation third order polynomial to begin with so one value of p square will be 0, the other value of p square let us call it p 1 p 2 from this relation will be simply p 2 square will be k by m and p 3 square from this relation will be 2 k m or m will m k will come out between this 2 so it is m capital M, right; so it will be k by m 1 plus capital M will go from there 1 plus 2 m by m right am I right plus 2 m by m very good yeah I did it right great so p 1 will be 0, p 2 will be root of this and p 3 will be this so these are 3 frequencies very good I am happy with myself because I can never make this I mean I can never calculate this on board I will definitely make some mistakes anyway this time I did it.

Now what we so this is one part of it so these are your normal frequencies so you see there is 1 frequency which is a pure translation 0 frequency means it is a pure translation we will see that once we calculate the normal mode normal amplitudes what about p 2 and p 3 this also represents 1 type of vibration mode right and this modes we will see once we will once we calculate the Eigen vectors so I will just remove this because we will need this space, but we can always to draw it later, right.

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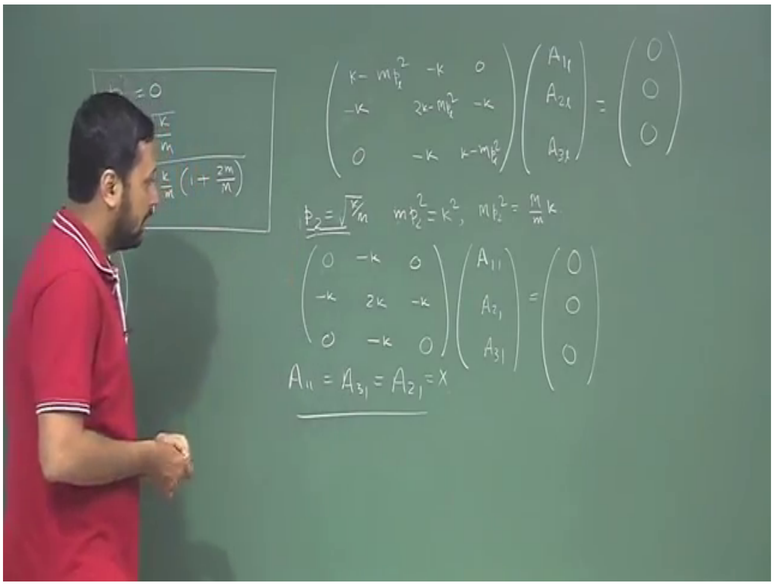

Now, if I go back to the original equation it is k minus m p square minus k 0 minus k 2 k minus m p square minus k 0 minus k k minus m p square so this is a matrix equation now if you recall it was the matrix equation so A 11 let us say or let us put an index here l l l so A 1 l; A 2 l; A 3 l; which will be equal to 0 00 so this is a matrix equation from which we wrote the determinant if this matrix equation to I mean if there has to exist a non trivial solution for A the determinant of this term will be equal to 0 that is how typically we solve any Eigen value problem right so this one is a master equation. Now we put l equal to 1 so that means, we first do it for p 1 which is equal to 0 so once we put p 1 equal to 0, here this equation will reduce to k minus k 0 minus k 2 k minus k 0 minus k k and we have A 1 1, A 2 1, A 3 1 which will be equal to 0 0 0.

Now from the first and the last relation we see that A 1 1 is equal to A 3 1 this has to happen also if we put it in the second relation second line so basically if I open I mean if I put it back into terms of simultaneous equations the first equation will be A 1 1 k minus A 2 1 k plus 0 equal to 0 second equation will be minus A 1 one k plus A 2 k times A 2 1 minus k A 3 1 equal to 0 like this so first and third equation will gives us A 1 1 equal to A 3 1 and from second equation we will see that this A 2 1 is also equal to them so it is a situation where all this 3 will be equal.

Now if these are all 3 are equal; that means, the displacement of this Triatomic molecule for 0 frequency mode is equal that means, they are all either moving along this line towards this direction or towards that direction which is a pure translation without any surprise this is what we have expected right so, this is there is no confusion about it and we got this.

Now when we put so this is 1 what we will do is we will just keep it so we can put them some equal to some A 1 so what I will do is I keep it like this it will be the your first vector which we will call let us call this some x so A 1 is equal to x x or rather what we can do is we can take this x common we can write 1 1, now if we put p 2 which is root k by m into this equation so if it is k by m then p m p 2 square will be k square and capital m p 2 square will be m by m k, right, so if I plug it in into the second equation we see that the first term readily vanish right because these 2 will be equal so we will have a 0 here ones also the last term because these two are identical so this is one more thing from the just from the symmetry of this equations you can always comment that whatever be the value of A 1; this value of A 3 will be same these 2 components will be identical by the construction of this matrix.

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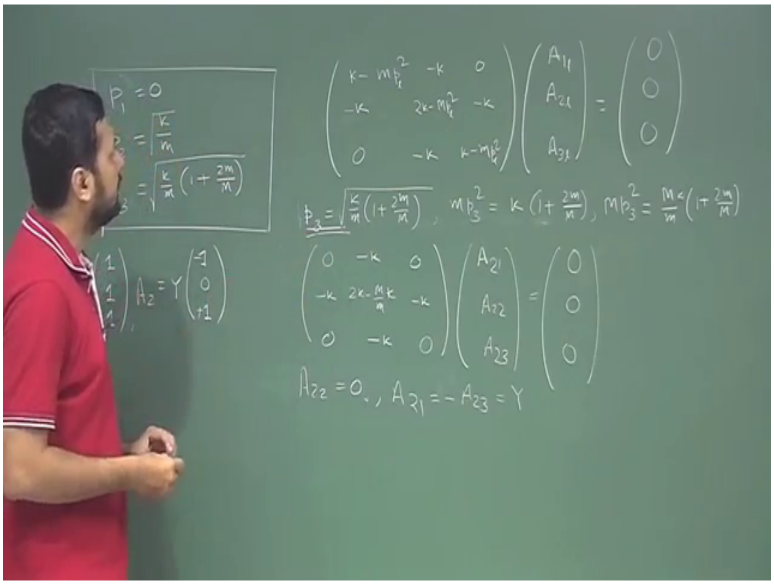

And for the second term it is 2 k minus m by m k right and this will be A 2 1, A 2 2, A 2 3 right great now we will see something directly from here that from the first and the third equation you get A 2 2 equal to 0 is it not and if you put 2 equal to 0 in the second A 2 2 equal to 0 in the second equation you see that A 2 1 is equal to minus A 2 3 and we can put equal to some y so that means, A 2 will be some y multiplied by if this is this I take 1 this will be 0 this will be minus 1 or if I put a minus sign here so this will be plus does not matter it is a relative thing so either this is minus this is plus or this is plus this is minus this means that at when the system is oscillating with this particular frequency middle mass the heavier molecule heavier atom is fixed and the 2 side atoms they are oscillating in opposite directions.

So when they are going out they are going out together when they are coming in they are coming in together so that means, their motion is always out of phase as we see form this minus sign and if you recall during our discussion of theoretical discussion on the amplitudes a we have stated that it will be either the phase will be either 0 or 180 degree this is an example of phase 180.

Now, similarly we can compute the third one so we have to put this huge monster there if you do that it will be let us do it 2 m by m and so m p this will be p 3, p 3 square will be equal to k 1 plus 2 m by M, right and M p 3 square will be equal to M k by m 1 plus 2 m by M right.

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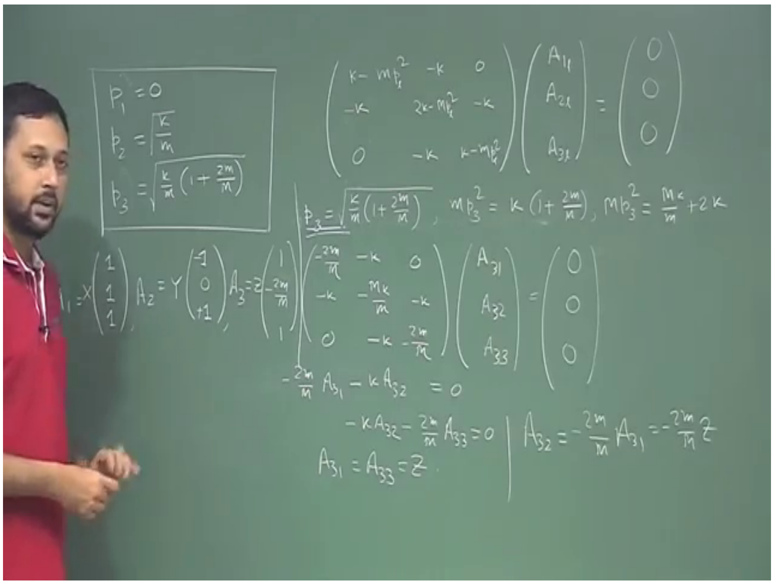

So, if you put that in here it is a slightly tricky it will be the first term will be minus 2 m by M this last term will be minus 2 m by M and the middle term will be yeah will be simply so it will be 2 k 2 k, what is it no I think I oh yeah right so this can be simplified to this will cancel out it will be plus 2 k right so the third term will be simply minus m k by M right excellent and this will be 3 1; 3 2 and 33.

So we have once again a situation where we see from the first and the third equation that A from the first and third equation we can see that minus 2 m by M, A 3 1 minus k A 3 2 equal to 0 and from the third equation we have minus k A 3 2 minus 2 m by M A 3 3 equal to 0 so if I subtract one from the other what I readily get is A 3 1 is equal to A 3 3 let us call it some set and now if I put that in the second equation what I get is A 3 2 is equal to I am just write the final expression A 3 2 will be equal to minus 2 m by M A 3 1 or 3 3 which is minus 2 m by M Z.

So we can write A 3 as Z 1 1 minus 2 m by M so these are the expressions for 3 Eigen vectors we have to normalize them so we will take up take it up in the next lecture and we will continue with our discussion on normal modes.

Thank you.