Quantum Mechanics - I
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Lecture - 15
Composite Systems

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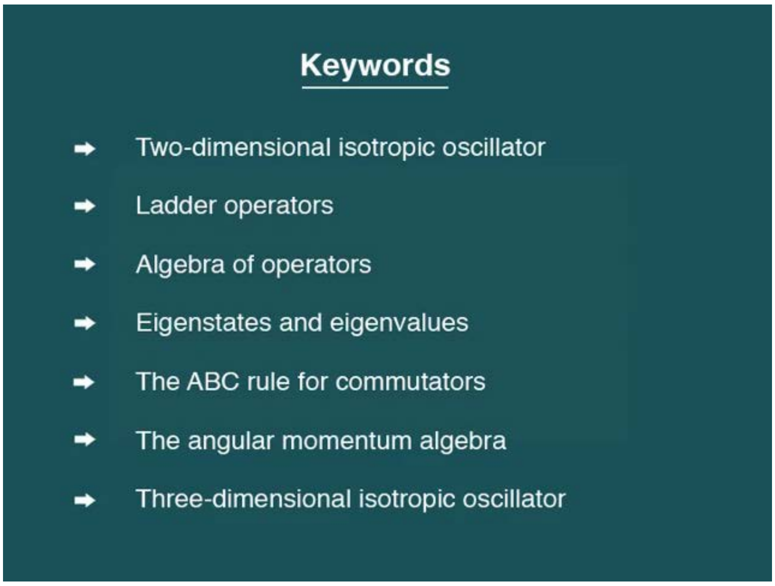

In today's lecture, I will discuss a simple composite system. A composite system made up of 2 sub systems and you already know the sub systems. I am going to consider a 2 dimensional isotropic oscillator: One oscillator along the x axis and the other oscillator along the y axis. Since, we have already attempted the problem of the simple harmonic oscillator, it should be possible for us to now consider 2 such oscillators not interacting with each other. But, we are interested in looking at the system as a whole.

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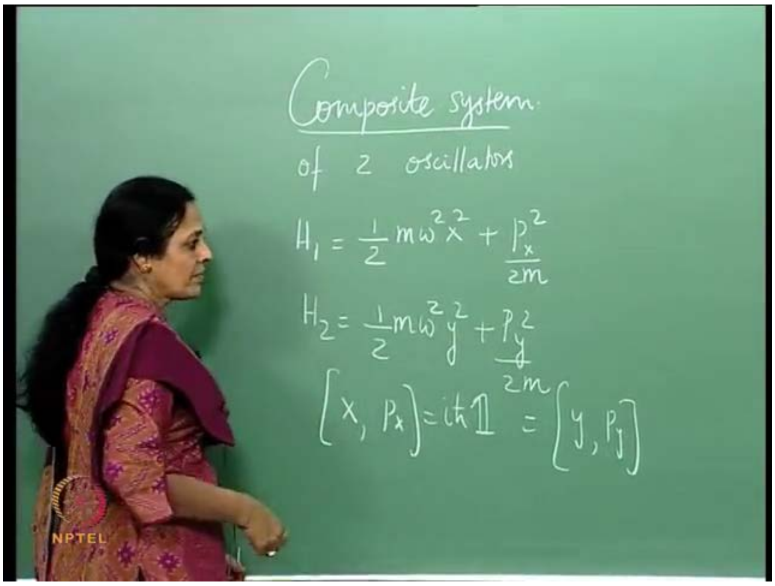

So this is the composite system that, I am going to consider. It is a composite system of 2 oscillators. As such therefore, the Hamiltonian for the 1st oscillator which I will represent by H 1 is simply half m omega squared x squared plus P x squared by 2 m. Where m is the mass of the oscillator x is the coordinate and P sub x is the linear momentum corresponding to the coordinate x. Then there is the other oscillator which too has the same mass the same omega but now, it is along the y direction and therefore, I have y and P sub y. So these are the 2 independent Hamiltonians and I have a situation with X P sub x commutator is i h cross identity and that is the same as the commutator of Y with P sub y.

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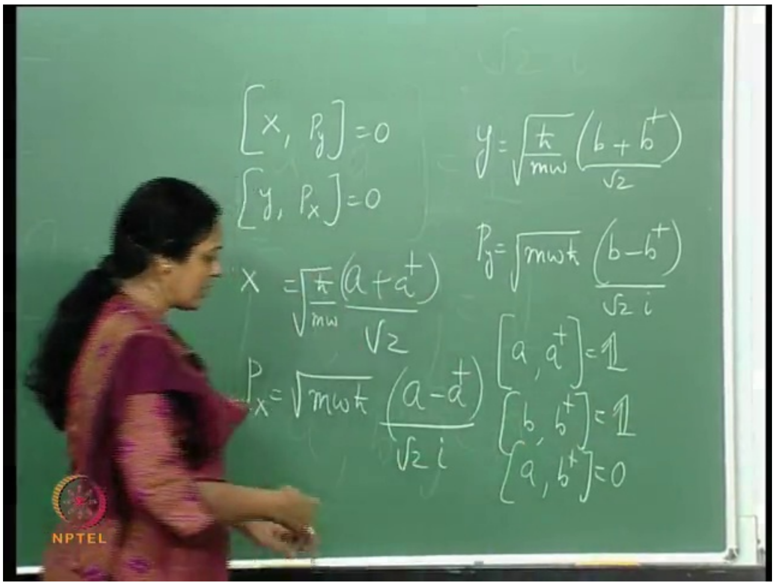

Since, these are independent oscillators X with P sub y commutator is 0. Similarly, Y with P sub x commutator is 0. That is how I take care of the fact that these are absolutely independent of each other. I could write this in terms of the raising and lowering operators. You will recall that we defined X as a plus a dagger by root 2 except, that there was a root of h cross by m omega multiplying that taking care of the right dimensions. And P sub x is simply root of m omega h cross a minus a dagger by root 2 i.

I would do the same thing for the oscillator along the y axis. And I would write, y is equal to, root of h cross by m omega b plus b dagger by root 2 and P sub y as root of m omega h cross b minus b dagger by root 2 i. In other words, the ladder operators that is the, raising and lowering operators for the oscillator given by (Refer Slide Time: 00:57) this Hamiltonian h one are a and a dagger.

The corresponding ladder operators for the oscillator with Hamiltonian H 2 those operators are b and b dagger. So, I would have a, a dagger equals identity b, b dagger equals identity. Every other commutator vanishes a b dagger is 0, b a dagger is 0, b with a dagger commutator is zero and so on. So, these are the 2 independent oscillators.

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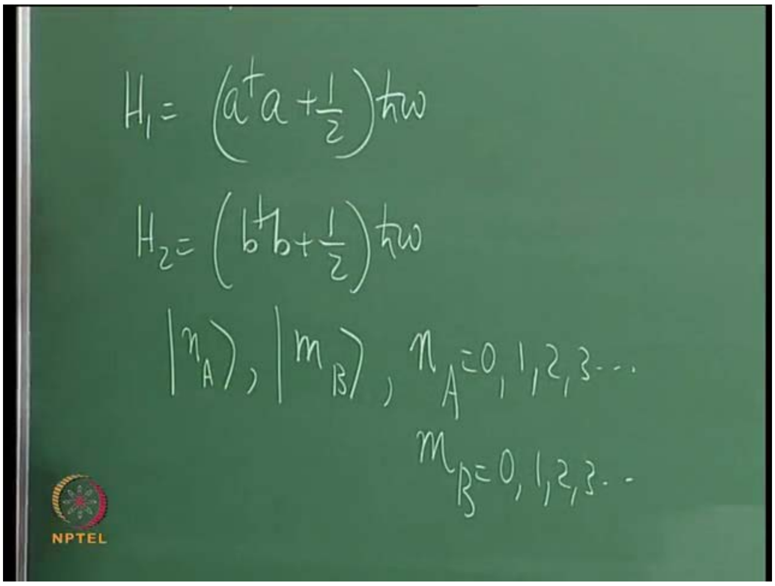

What is the composite system? The composite system has a Hamiltonian which is going to be the sum of H 1 and H 2. I could write H one in terms of a and a dagger as a dagger a plus half h cross omega. Similarly, H 2 is b dagger b plus half h cross omega. The number operator corresponding to the 1st oscillator is a dagger a and the number operator corresponding to the 2nd one is b dagger b. Therefore, if the states corresponding to the 1st oscillator are represented by this ket n sub A by a I mean the 1st oscillator by b i mean the 2nd oscillator. So the states, which are the energy Eigen states of the 1st oscillator, are represented by the label n sub A. So, my composite system is composed of system A and system B system A B in the 1st oscillator and system B B in the 2nd oscillator. Represent the states this way; where n A can take values 0, 1, 2, 3 and so on and m B can take values 0, 1, 2, 3 and so on.

I reiterate that A and B are simply, used as subscripts to show that these are the labels corresponding to one of the oscillators and these are the labels corresponding to the 2nd of the oscillators. And since the energy Eigen state labels take value 0, 1, 2, 3 I have these values for n A and m B.

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So given this it is clear, that a dagger a acting on the state n sub a simply pulls out this number and we have this Eigen value equation. Similarly, b dagger b acts on the state m sub B to give me this. The raising and lowering operators you will recall act in the following manner. The lowering operator simply pulls out root n sub A and lowers this state label by 1 and I represent it in this manner.

Similarly, b pulls out the label m B under the square root and I have a lowered state m B minus 1. The corresponding raising operators are a and a dagger and b dagger so that, I have the following equations and b dagger acting on m B gives me root of m B plus 1 with a higher energy state whose label is m B plus 1. So this is what I have by way of equations.

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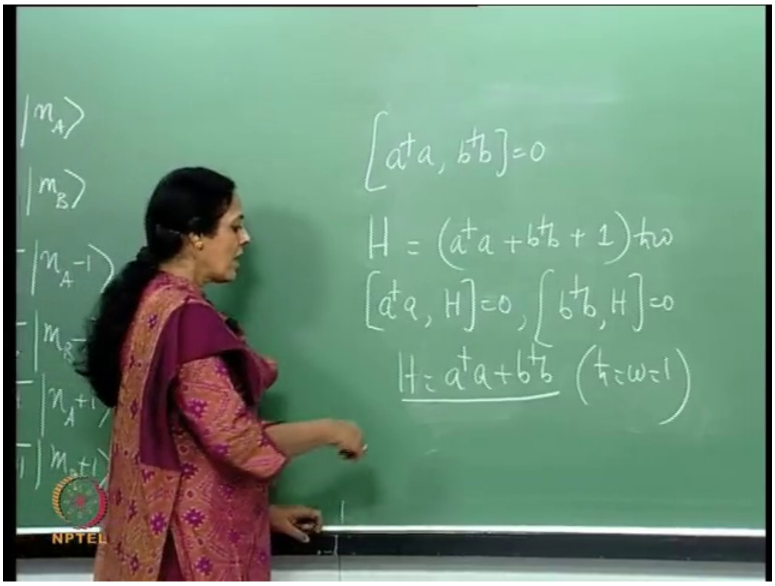

I now, find out what are various operators in this problem which commute with each other? Since, the commutator of a with b dagger vanishes and they are independent oscillators, it is clear that a dagger a with b dagger b is 0.

I have the total Hamiltonian H which is a dagger a plus b dagger b plus 1 h cross omega. Because, there was a half h cross omega from the 1st oscillator contributing here and another half h cross omega from the 2nd oscillator. So, this is the total Hamiltonian for the system. It is evident that the commutator of a dagger a with H is 0. And the commutator of b dagger b with H is 0. So, I should be in a position to simultaneously diagonalize the Hamiltonian H corresponding to the composite system H 1 and H 2. Suppose, I forget the 0 point energy 1 h cross omega for the moment and consider the Hamiltonian to be just a dagger a plus b dagger b.

I can always add 1 h cross omega later and here I have set h cross equals omega equals 1. For convenience, we can always put that back adjusting dimensions appropriately. So, i will now consider this Hamiltonian a dagger a plus b dagger b and find out the Eigen values, the energy Eigen values and the corresponding Eigen states corresponding to this Hamiltonian and of course, a dagger a and b dagger b which I already know.

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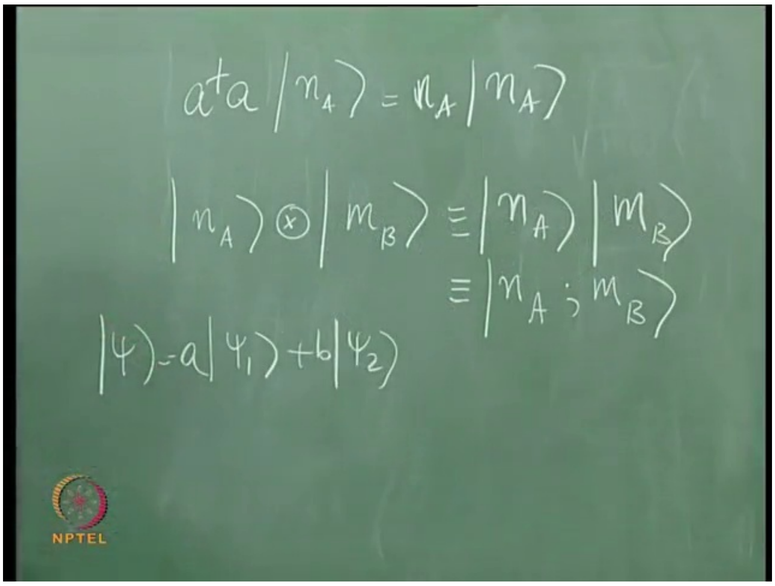

It is clear that when a dagger a acts on the combined state of the system, it is nearly going to act on the label n a leaving the other ket corresponding to 2nd oscillator alone. So, now we have reached a state where, we need to define the state space of the full Hamiltonian. So, we postulate the following: The state or the state space of the full Hamiltonian is made up of states which are, tensor products of the states of the sub systems. In other words, the next state of the composite system is given by the tensor product of ket n A with ket m B. This could be treated as a postulate. It is clearly inspired by the fact that even in a single system there is a super position principle. And if you have basis states psi 1 and psi 2 a psi 1 plus b psi 2 is also a state of the system.

Extending this and inspired by this you could treat this as the postulate which says, that the combined space has states which are tensor products of the subsystems Eigen states. We have to clearly understand, how exactly these operators act on these states? I could use a shorthand notation for this. I could call this n A, m B dropping this sign I essentially mean by this notation this object. I could make it even simpler and I could write this. By this notation I would mean this, I would be using all these notations interchangeably as I go along. But, any of this would represent the state of the full system.

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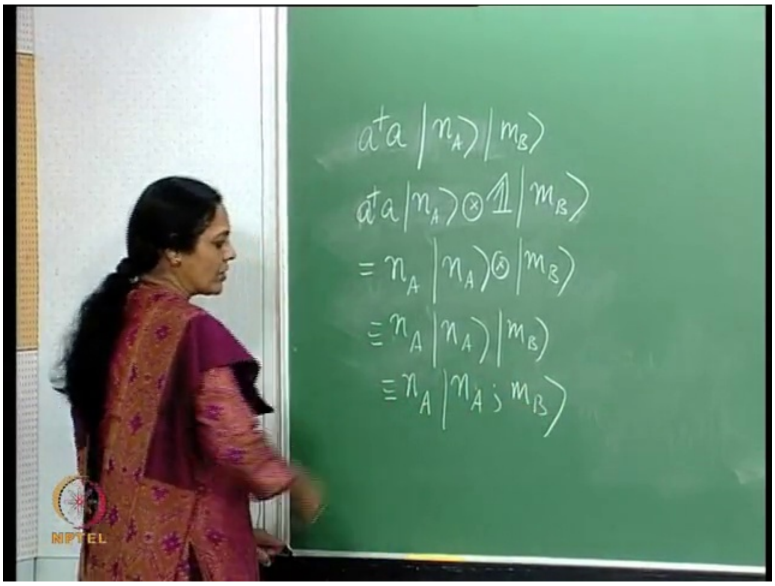

Now, we have the following situation. When a dagger a acts in the state n A, m B. It is clear that, it only acts on the states corresponding to system A leaving m b as such. Therefore, it is like saying that a dagger a acts on the ket n A and the identity operator acts on the state m B. So this gives me n A n A m B, in my notation this would simply be where this is a number labelling the state and this is the next state of the system that is identical to n A and the state written this way.

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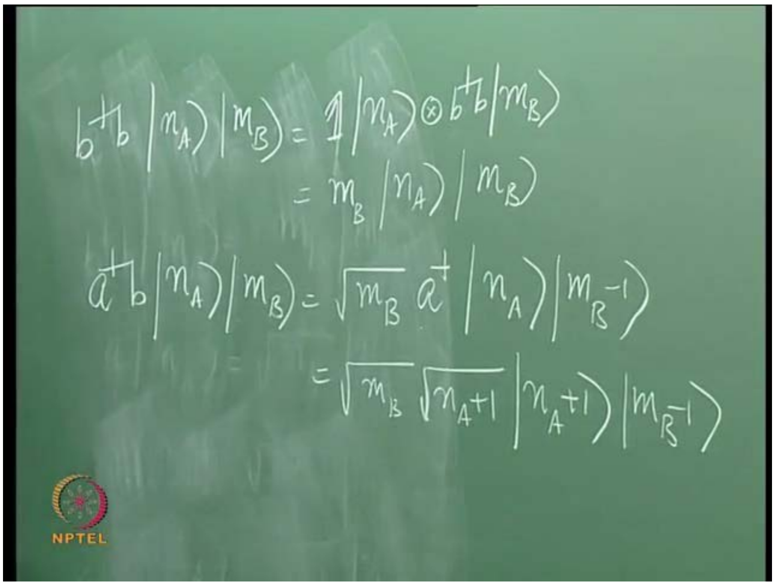

Similarly, when b dagger b acts on the state of the system it works in the following manner. It is like the identity operator acting on the states corresponding to system a tensor producted with this. And therefore, b dagger b pulls out an m B from here, that is a number leaving me with the state n A m B. Now, let us look at operators like a dagger b. So, I extend this 1st of all b acts on the state n A, m B leaving n A alone and pulling out root m B which is a number from here. Of course, a dagger has to operate on the net state and the net state is now n A m B minus 1.

The action of b on the ket m B was to pullout the value root m B, and lower the state by 1. Leaving n A as such the ket n A as such and then a dagger acts on the state n A. When a dagger acts on the state n A pulls out a value root of n A plus 1, raises the state label by 1 leaving the system be untouched. So, this is the net effect of the operation of a dagger b on the state n A m B. Similarly, we can work out the action of various operators on these states.

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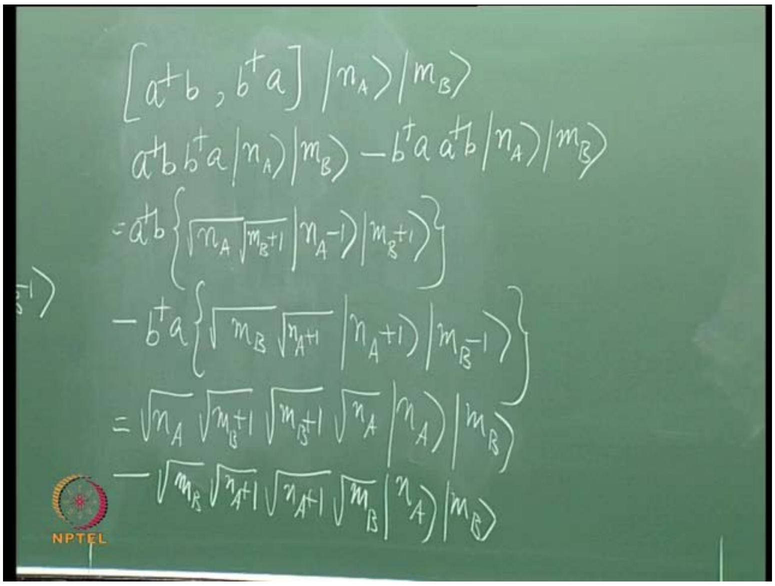

Let us look at the algebra of these operators. Now, that we have described how exactly the operators act on the states. Let us find out the commutator of the non Hermitian operator a dagger b with its counterpart b dagger a its Hermitian conjugate b dagger a. Now, this acts on state n A m B of the system. And what does it do? When I expand the commutator I simply have 2 terms a dagger b b dagger a acting on the state, minus b dagger a a dagger b, acting on that state I use the rules that I have been defined earlier.

So this is a dagger b, b dagger a, acts on the state n A m B, first of all, a simply pulls out root n A reducing the state label by 1 b dagger acts on m B increasing it by 1. So that is the effect of b dagger a on n A m B it is clear that, b dagger is the raising operator on this state and therefore, the label went up and a is the lowering operator on this state and therefore, the label went down. That is the 1st term minus b dagger a, which acts on the net state a dagger b ket n A ket m B.

So once more when b acts on m B it pulls out root m b lowering the state to m b minus 1, leaving n A untouched. And when a dagger acts on n A it takes it to root n a plus 1 and the state label is raised by 1. So, this is how the commutator expands. So, let me proceed and complete the simplification. I have in the 1st term the numbers root n A root m B 1 one which, I can pullout and then a dagger b acts on these states. So b acting on m B plus 1 and a dagger acting on n A minus 1, that is what I have here. Here a, so this would simply give me an m B and a dagger acting on n A minus 1 would take it to n a.

So the net state would be ket n A ket m B, the coefficients are obvious b acting on m B plus 1 pulls out root of m B plus 1 and a dagger acting on n A minus 1 pulls out a root n A leaving behind ket n A ket m B. That is from the 1st term similarly, the 2nd term gives me the following coefficients: a on ket n A plus 1 gives me root n A plus 1 ket n A. And b dagger acting on ket m B minus 1 pulls out root m B increasing the label by 1 so this is what, I have for the commutator of a dagger b with b dagger a acting on the state n A m B.

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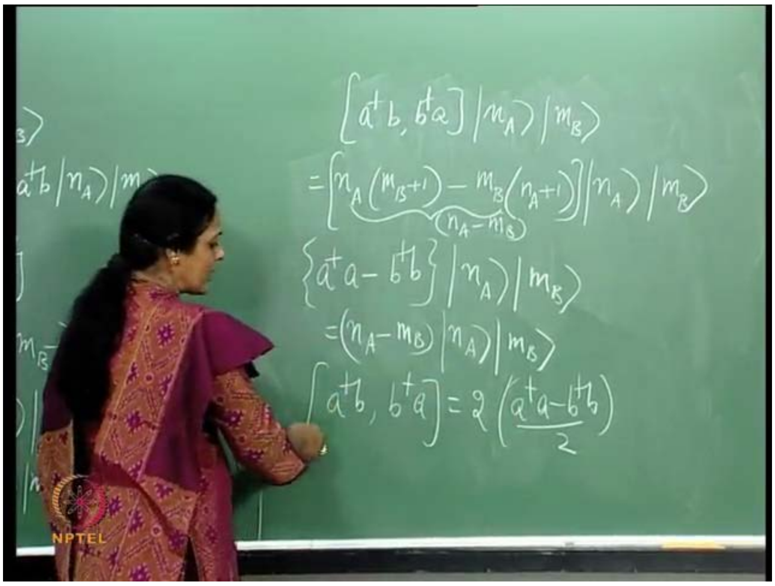

Let me simplify this. I have a dagger b with b dagger a acting on this state n A, m B of the composite system to be simply n A, m B plus 1 minus m B n A plus 1 multiplying the state n A m B. So this is what I have; this is the commutator that I have. Now, consider the effect of the operator a dagger a minus b dagger b acting on the state n A m B.

Now this, clearly simplifies n A m B cancels out and I just have n A minus m B out here. Consider the action of the operator a dagger a minus b dagger b acting on the state, this is simply n A minus m B, n A m B. So, indeed it looks like I have proved that a dagger b commutator with b dagger a is twice a dagger a minus b dagger b by 2, this is the relation that I have. I have very deliberately put a 2 out here and divided by 2 so that, I may define this as a hermitian operator where I have a commutation relation which is similar to what I knew from the S u two algebra I had s plus with s minus is 2 is S z.

So now, I would like to check if indeed I can mimic the angular momentum situation using these 2 oscillators and in fact that is the purpose of my talk today. This composite system was selected by me, to see if I can reproduce the angular momentum algebra using 2 harmonic oscillators. If indeed I can identify this with the S z, I should be able to check out the analogue of the commutator of S z with s plus and S z with s minus, which is what I will proceed to do now.

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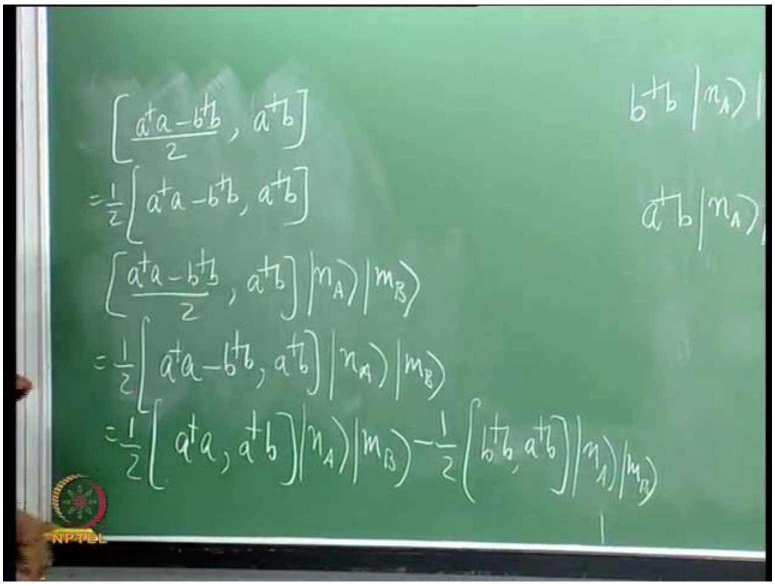

So, let me consider the commutator a dagger a minus b dagger b by 2 with a dagger b. That is the same as the following: I could do this for instance by just applying the a b c rule. In order to make things very clear I will do it explicitly by attempting to find out its affect on the states. In other words, I would like to repeat the procedure and find out the action of this commutator on the state n A, m B. So that is half a dagger a minus b dagger b commutator with a dagger b n A, m B. So, this is the same as finding the commutator of a dagger a with a dagger b, acting on n A, m B minus half commutator of b dagger b with a dagger b acting on the state n A, m B.

It is pretty clear, here that a dagger commutator with a dagger b is 0, I could use the a b c rule. And then I find that the only non vanishing contribution comes from the commutator of a with a dagger and that is 1. Similarly, here the only non vanishing contribution in the commutation comes through the commutator of b dagger with b and that is minus 1. And therefore, this becomes a straight forward matter.

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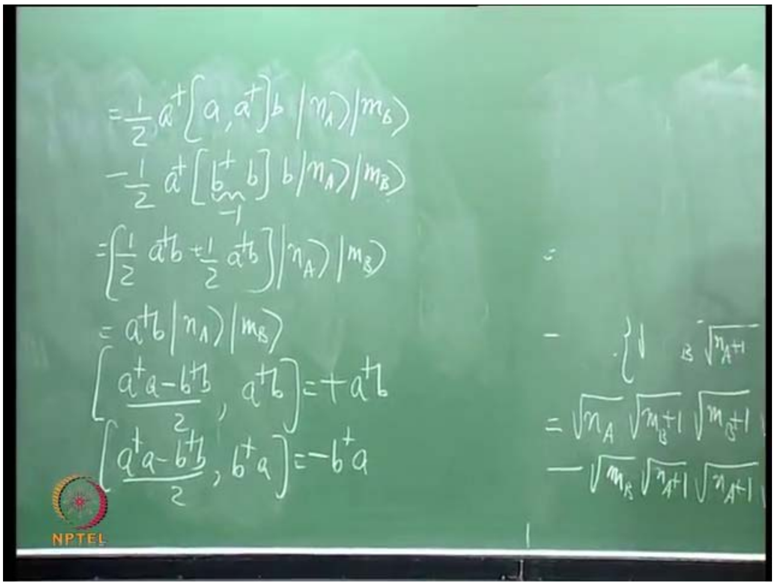

So this is the same as half (Refer Slide Time: 24:04) I can pull out a dagger on this side commutator of a with a dagger b n A, m B minus half a dagger commutator of b dagger with b b, which is the same as half a dagger b from the 1st term and this is minus 1 and therefore, plus half a dagger b from the 2nd term acting on n A, m B, which is the same as a dagger b acting on the state n A, m B. In other words I have shown, that a dagger a minus b dagger b by 2 commutator with a dagger b is plus a dagger b. In a similar way in we can find out the commutator of a dagger a minus b dagger b by two with b dagger a and check that this is minus b dagger a.

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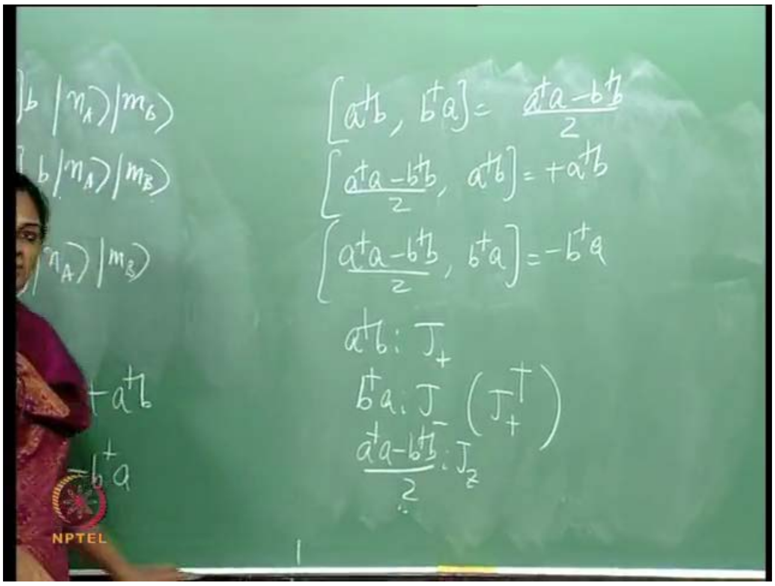

I have therefore, got the s u 2 algebra. I have a situation where, I have identified 3 operators 2 of them being Hermitian conjugates of each other and the 3rd being a Hermitian operator, satisfying the following commutation relationship and this is simply the S u 2 algebra.

So, I identify a dagger b as the raising operator, I could call it J plus b dagger a as the lowering operator and you would recall that this is the same as the Hermitian conjugate of J plus. And I can identify a dagger a minus b dagger b by 2 as J z. I would like to call this J plus J minus and J z by way of notation. Because, I am looking at a particular spin system or orbital angular momentum. This is the general angular momentum algebra which I have generated. And therefore, I refer to these as J plus J minus and J z.

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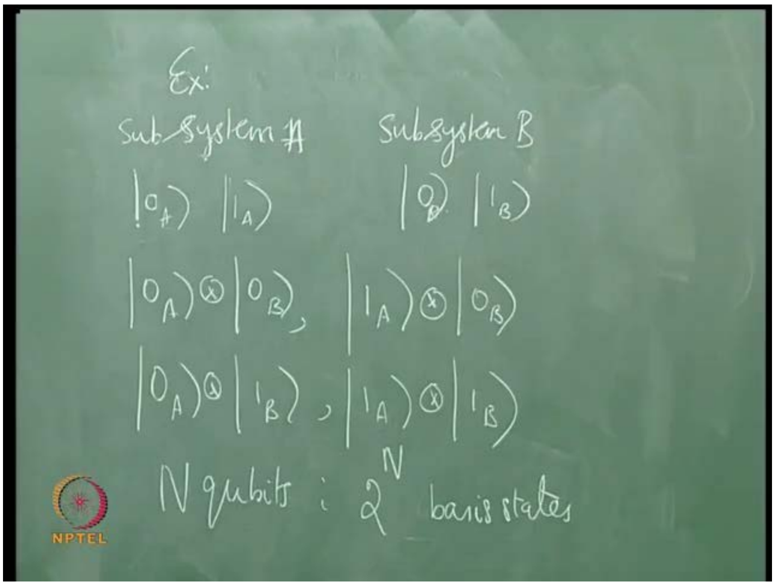

Before I proceed, I would like to comment about the dimensions of the full space. Once we do a tensor product of the individual subsystem space the dimension increases in a certain fashion, for instance, if system 1 subsystem 1 and subsystem 2, have just 2 basis states. This is subsystem A and that is subsystem B. Suppose, they have just 2 basis states it is an example that I am considering. I refer to them as 0 A and 1 A and 0 B and 1 B respectively. The tensor product will give me the following basis states. And the last is 1 A tenor producted with 1 B. So, there are 4 basis states. It is pretty clear, that if subsystem A had 3 basis states and subsystem B had 3 basis states then you would have a total of 9 such states.

Now, if I just look at qubits each qubit is like a spin doublet. There are 2 states as in this example and if I put a collection of N qubits, the composite system on the N cubits has 2 to the N basis states. Now, in general if I want to deal with a large system I could take a single system with 2 to the n basis states n being sufficiently large or I could work with the set of N qubits each qubit having a 2 dimensional linear vector space, as in this case. In other words, I can have small subsystems in this case each is a qubit a doublet. I could combine a large number of them and provide 2 to the n basis states for the composite system.

Instead of choosing a single system with a very large linear vector space in the sense that, it is a single system with 2 to the n basis states. Now normally if it is a single system where the basis states are the Eigen states of the Hamiltonian the 2 to the n basis states are the energy Eigen states. And therefore, reaching out to the higher energy states would mean expending that much energy it means going to higher energy levels of a system. Instead, I could work with 2 to the n qubits with n cubits and provide 2 to the n basis states. That could be cheaper for me in terms of an experimental arrangement. That is merely a digression. It is a point worth remembering for our future lectures when we will talk about interacting systems.

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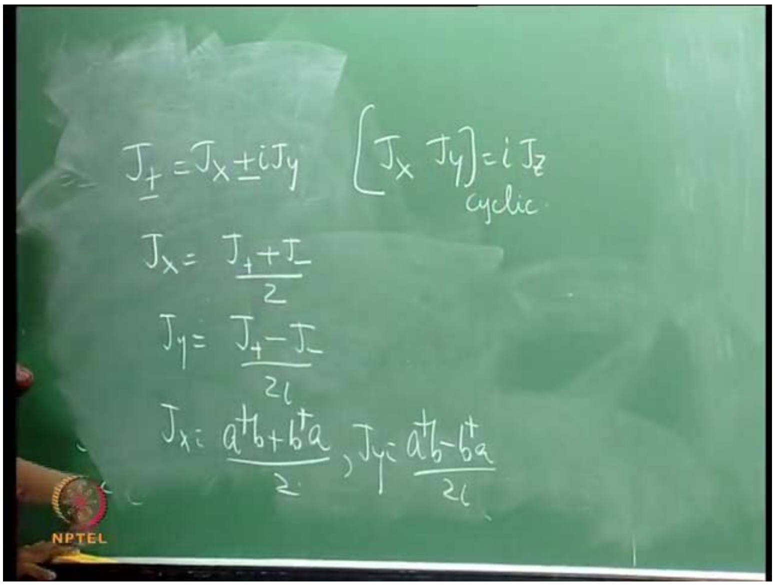

But right now, getting back to this point I can now define the analogs of J x and J y. You will recall that J plus was J x plus i J y and J minus was J x minus i J y. Therefore, J x is

J plus plus J minus by 2 and J y was J plus minus J minus by 2 i. And therefore, I can write J x as a dagger b plus b dagger a by 2 and J y as a dagger b minus b dagger a by 2 i. I, emphasize the J y is Hermitian because J minus dagger was J plus but there is an overall negative sign and that is taken care of by the fact that i and its complex conjugate is minus i.And therefore, I have these Hermitian operators. It will automatically follow that J x J y is i J z and this is the cyclic relation. This can be explicitly checked out but it will work out simply because I have obtained the S u 2 algebra already in terms of J plus J minus and J z.

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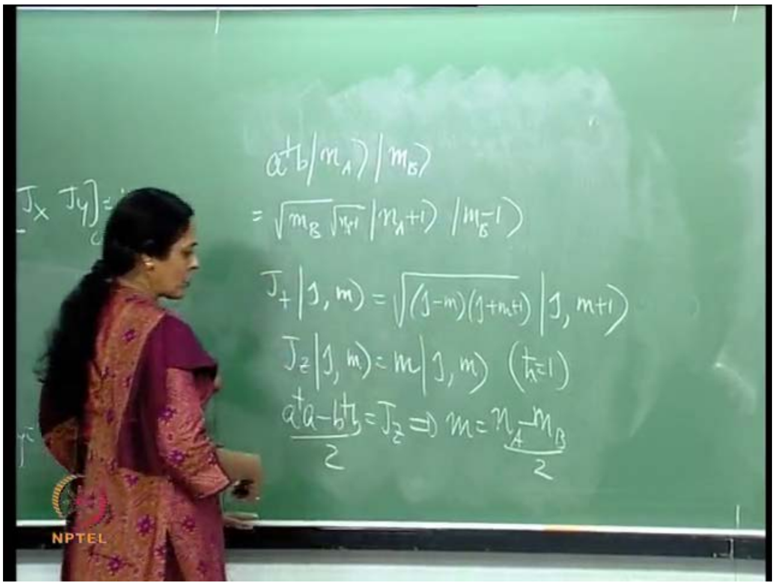

Now, let me look at the action of a dagger b that is J plus on the combined state of the system. So a dagger b acts on the state n A, m B. As I said before, b acts on m B to pull out root m B lowering this to m B minus 1. Leaving this state ket n A, alone and a dagger acts on the state ket n A pulling out a root of n A plus 1 raising the label to n A plus 1.But I know that when J plus acts on the state j, m of the composite system, I should get root of j minus m times j plus m plus 1 j, m plus 1. So I would like to now identify j, I know m, m is clearly the Eigen value corresponding to the state j, m corresponding to the operator J z. You will recall that I have set h cross equals 1 and since J z is a dagger a minus b dagger b by 2. This implies that m is simply n A minus n B by 2 and therefore, I have identified m. It is clear that the values of m can be either integer or half integer, because when n sub a is 0 and n sub b is 0 m is 0.

But when this is 1 and that is 0 for instance I get m equals half when n A is 0 and m B is 1 that is an m. When n A is 0 and m B is 1, it is clear that m is minus half and so on. So this quantity m can take integer and half integer values and these could be positive or negative. That is as I would expect from angular momentum algebra.

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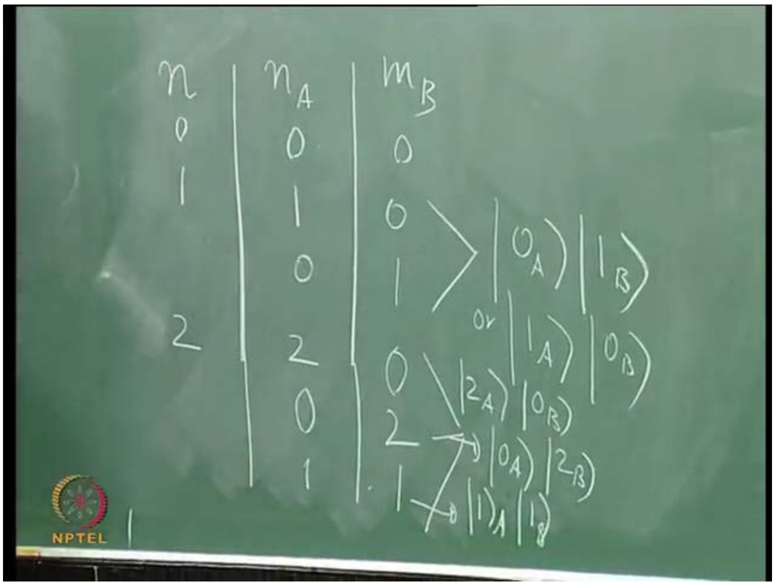

In order to identify (Refer Slide Time: 35:14) and make the connection between root m B root n A plus 1 with root of j minus m j plus m plus 1. Let me consider the total number operator n which is a dagger a plus b dagger b. Its Eigen values are clearly the sum of the Eigen values of a dagger a and b dagger b. So, let me call the Eigen value corresponding to the operator n as n that is, n A plus m B.

So, let us make a neat column tabular column and what do we see? When n A is 0 and m B is 0 n is 0, n A and m B can be only be integers and therefore, n can only be an integer. So, let me start and since these are 0 or positive n can only be 0 or positive. So, let me put n equals 1, this could come from n A equals 1 m B equals 0 or n A equals 0 m B equals 1. In other words, the Eigen value n equals 1 could come from the state 0 a 1 b or 1 a 0 b. So, as far as the composite system is concerned n equals 1 is a doubly degenerate state.

There are 2 Eigen states; distinctly different Eigen states which correspond to the Eigen value n equals 1. Suppose n equals 2 n A could be 2 and m B could be 0 n A could be 0 m B could be 2 or n A could be 1 and m B could be 1. So, what are these states this would be a 2 a 0 b the 2nd one is a 0 a, 2 b the 3rd one is a 1 a 1 b. So there are 3 states corresponding to n equals 2. It is obvious, that the degeneracy for a given n is n plus 1 there is an n plus 1 fold degeneracy corresponding to a given n.

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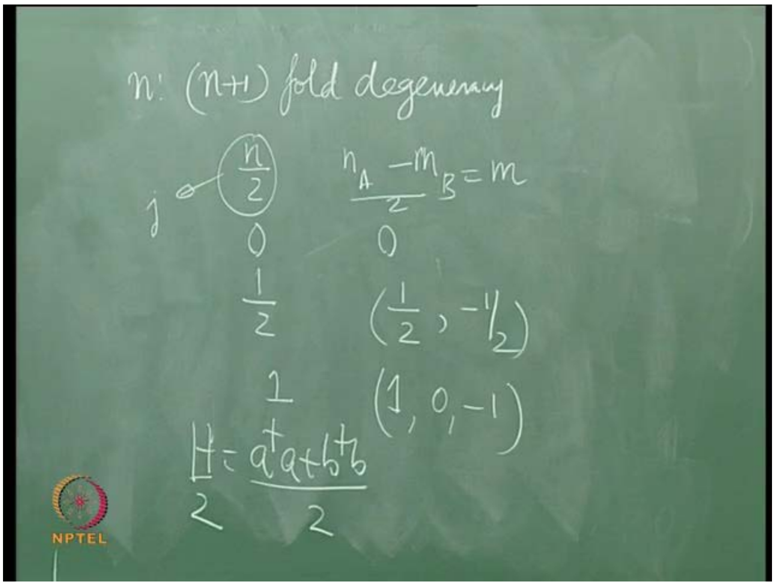

Coming from various combinations of n A and n B. So, for a given n there is an n plus 1 fold degeneracy. Consider the object n by 2. So, let us make a column n by 2 n A m B, when n is 0, n by 2 is 0, n A is 0 n b is 0 we might as well make a column n A minus m B by 2. So when n is 0, n A is 0, m B is 0 and n A minus m B by 2 is 0 but this object is simply m. When n by 2 is half, I find that n A could be 1 and m B could be 0 giving me m equals half or n A could be 0 and m B could be 1 giving me the value minus half. So you see, it looks like n by 2 is to be identified with j and we will check it out with n by 2 equals 1. In other words when n is 2, n by 2 is 1 and what do I have? I have m values 1, 0 and minus 1.

I would like to call n by 2 as j, j takes positive values 0 half 1 and so on. And for a given value of j, m takes values minus j to plus j in steps of 1. And therefore, I have now identified j in terms of operators h was a dagger a, plus b dagger b and what we are looking at is the operator h by 2 which is a dagger a, plus b dagger b by 2.

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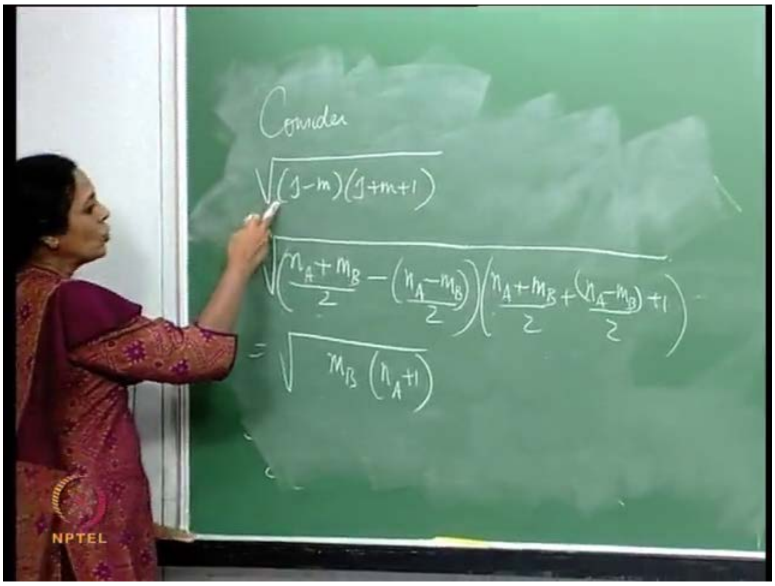

So now returning to our problem, we had a situation where (Refer Slide Time: 35:14) a dagger b acted on n A m B to give root m B root n A plus 1, changing the state to ket n A plus 1, ket m B minus 1. So, let us look at what this is going to be? Consider, root of j minus m into j plus m plus 1 and expand this in terms of n A and n B. This is n A plus m B by 2 minus n A minus m B by 2 that is the 1st term. The 2nd term is n A plus m B by 2 plus n A minus m B by 2 plus 1 and all this within the square root, under the square root.

So, this is just going to be root of m B from here and this just gives me an n A plus 1 which is what I have here. And therefore, it is clear that (Refer Slide Time: 35:14) when a dagger b acts on this it acts precisely in the manner in which the angular momentum raising operator acts. I have identified root m B times root n A plus 1 which is what comes out (Refer Slide Time: 35:14) when a dagger b acts on the state as the coefficient I have identified that with root of j minus m into j plus 1 m plus 1, where j itself is n A plus m B by 2 and m is n A minus m B by 2. Therefore, the angular momentum algebra is complete.

We have shown that starting with 2 oscillators which do not interact with each other. We can produce the angular momentum algebra by suitable combinations of a a dagger b and b dagger we have identified J x J y and J z. We have identified J plus and J minus and also checked out that j can only take values (Refer Slide Time: 40:45) 0 half 1 and so on. And for a given j there is a 2 j plus 1 degeneracy in m, we have also checked that in terms of energy Eigen values given n there is an n plus 1 fold degeneracy in the case of the 2 dimensional oscillator.

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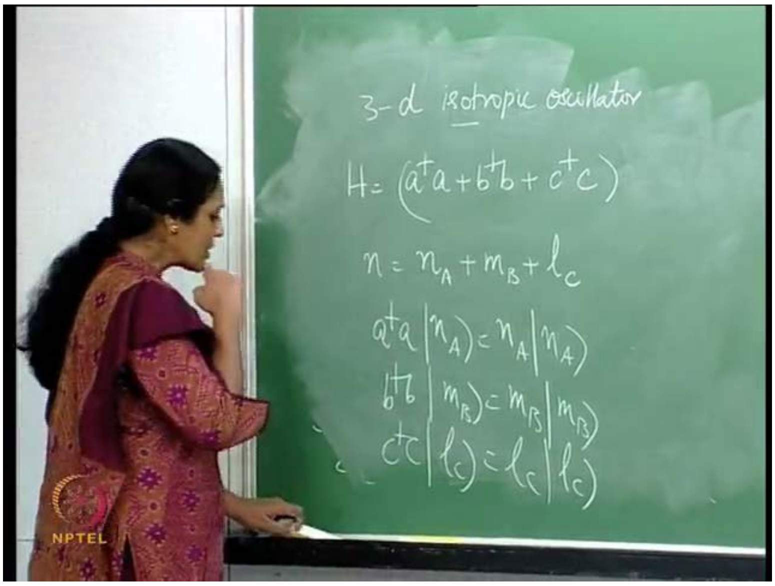

Let me now extend this to a 3 dimensional isotropic oscillator and at least check out the extent of degeneracy. So, in the case of the 3 dimensional isotropic oscillator, I have these 3 independent oscillators. The notation would now be evident, h is a dagger a, plus b dagger b, plus c dagger c. Of course, there would have been a 3 by two h cross omega half coming from each of these oscillators. But, we will choose to forget that for the moment our aim is merely to find out the degree of degeneracy of every Eigen value. And therefore, the analogue n in this case is given by n A plus m B plus l c.

Where a dagger a acts on the state n A to give me Eigen value n A, b dagger b acts on the state m B to give me this Eigen value equation and I have a corresponding equation for the subsystem c. So let us look at what would happen here? Each of these could take value 0 1 2 3 and so on, n also would take values 0 1 2 3. And therefore, we make a table in the case of the 3 dimensional oscillator.

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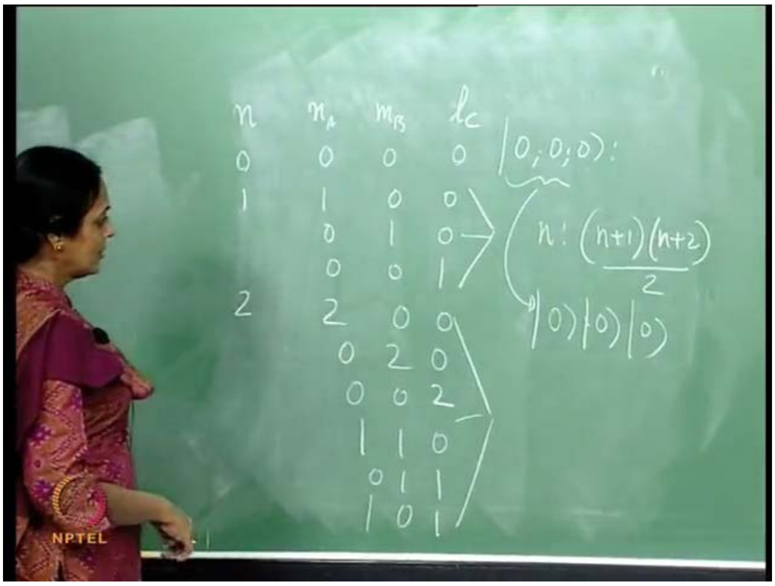

I have n n A, m B, l c when n is 0 they are all 0,when n is 1 I have the following 3 possibilities. All these correspond to n is 1, when n is 2 I have these 6 possibilities and so on. It is pretty clear what is the extent of degeneracy? For a given value of n, so when n is 0 there is exactly 1 state the label is 0 0 0, if you wish. When n is 1 there are 3 states these states would carry labels: 1 0 0 or 0 1 0 or 0 0 1. By this I mean this state would mean n A equals 0, m b equals 0, l c equals 0 and so on. So I have an n plus 1 times n plus 2 by 2 fold degeneracy for a given n. So, both in the case of the 2 dimensional oscillator and the 3 dimensional isotropic oscillator there are states which are degenerate in contrast to a simple harmonic oscillator. This is the simplest composite system but we will consider for the moment subsequently in my lectures I will talk about interacting systems which are composite systems.