Introduction to Classical Mechanics
Dr. Anurag Tripathi
Assistant Professor
Indian Institute of Technology, Hyderabad
Lecture - 42
General Motion of a Rigid Body

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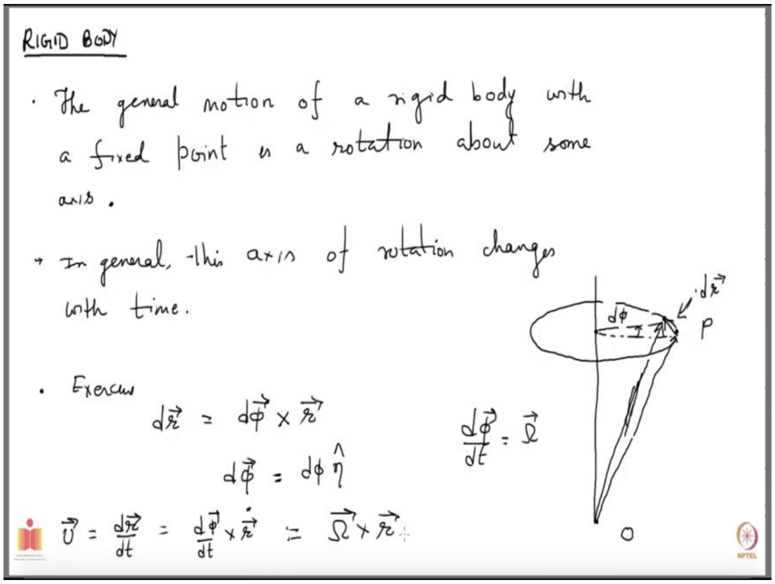

Summary of the last lecture is that any orientation of a rigid body with the fixed point can be reached by a rotation about an axis. And this is what is Euler's theorem. Equivalently, we can say that the general motion of a rigid body with the fixed point is a rotation about some axis. And of course, as the body is going to move that axis is also going to change with time.

Let me write down this version of it. So the general motion of a rigid body with a fixed point is a rotation about some axis. And, we should also remember that in general, this axis of rotation is going to change with time, rotation changes with time. Some students may face difficulty with this statement that they may have difficulty in realizing that this axis is in general going to change with time because most of us have experiences with rotation in the context of very symmetrical objects. And we are usually studying very special cases and things are usually rotating about a fixed axis. And that is what our early education about rotation tells us.

So, but you should realize that as things are going to move around with time, the entire distribution of mass is going to be shifted from one location to another location. So, it should not come as a surprise that this rotation axis is going to change with time. So now that we have established that this general motion is equivalent to a rotations, let us see how velocities of different particles which make the body are related to this direction of rotation or the axis of rotation and that is the thing which we want to look at.

So let us say at the instant I am looking at the body, it is rotating about this axis, that is the direction of rotation. And let us say this is point of the body, I call it point P and this is the origin of the body coordinate system or the system which I have taken, the origin of your system.

Now, let me draw a vector. Radial, radius vector from here to the point P. And because the body is rotating about this axis, the velocity of this point will be perpendicular to this plane if, for example, this this vector is in the plane and this is also in the plane. So these two these two make the plane and the plane is the same as that of your screen. Then this, the velocity vector will be perpendicular to the screen. And it is because it is a rotation, is going to be rotating like this.

But remember, it is not going to rotate it, rotate like this because as I told at the next moment it in general this axis will not be here it will shift to somewhere else. But let us say, we are looking at a very small interval of time. So in that interval, it will be moving along this this circle of fixed radius. So it will just cover some infinitesimal distance and let us say it moves to here, which is what is your new location. It is difficult to, so that is your new location and this is your displacement. So that is the vector dr.

And let us look at the center here. This is the new location, this is the original location, the body is moving counter-clockwise and it covers an angle d phi. Okay, this is the angle d phi and this is your origin here. So if your body is rotating counter-clockwise and covers an angle d phi then you can check, it is not a difficult exercise. You please check that the displacement dr is given by d phi, I have put a vector sign here cross r. And what is d phi, d phi is defined to be a vector, vector on quotes. You can you can look up any textbook and you will find more discussion on these vectors.

So this vector has a magnitude which is equal to the angle that has it has covered d phi and then it has a direction eta and the direction eta is given by right-hand rule. So if your, your thing is your particle is going counter-clockwise then the, your thumb is pointing towards the axis of rotation. And that direction is the direction which is a sign to phi, d phi. Okay, that is good.

Now, I want to look, find out what the velocity of this particle which is this point P which is in the rigid body, what is the velocity of that particle. I hope it is clear what I am doing. I am, for me, this point P is one of the points of the rigid body. And the body is moving and this point P is moving with that. That is what I am looking at.

So your velocity V is the following. V is your dr you divide by the time interval dt so you get d phi over dt cross r. And I define d phi over dt to be Omega. So which gives me Omega cross r. So that is the velocity of this particle at the instant considered. That is good. Now, I would like to consider motion of rigid body not necessarily with a fixed point. So I want to get rid of this constraint that one of the points is fixed and I want to consider the general body, general motion of a rigid body.

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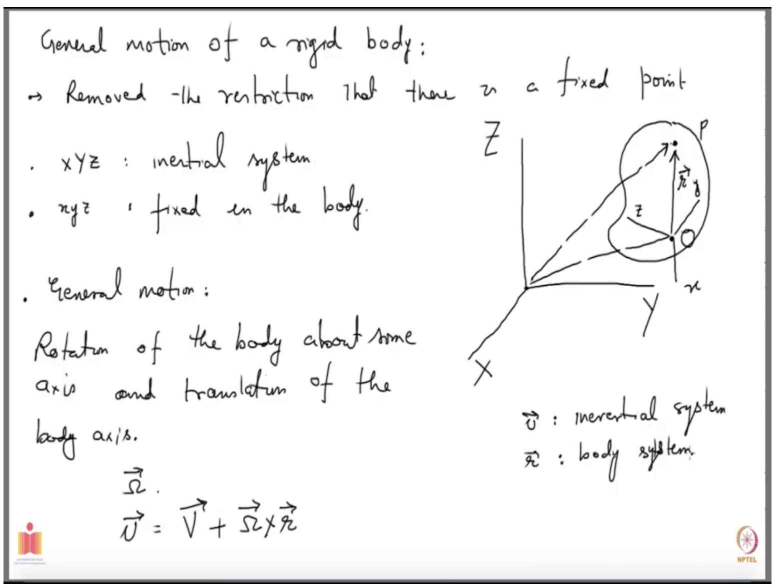

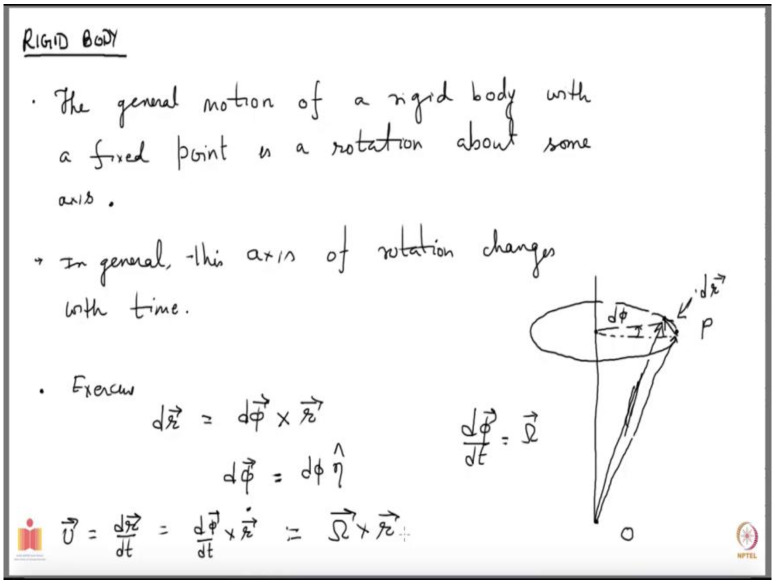

So by that, I mean I have removed the restriction of having a fixed point in the body, restriction that there is a fixed point. So if that is the case, I mean, that is not a case that is the general case. What is the most general motion? Obviously, the most general motion would be the following. You can think of the system moving in the following ways. So you take the origin of the body axis maybe I should draw it, I should draw it.

So let us say these are your space co-ordinates or inertial co-ordinates, which I will label by capital X, capital Y, and capital Z. Then here is your body and as you see, the body is not located at the origin, I mean the origins are not, it looks like a mango anyway. So this is the origin of the body axis. So I am saying I have attached a co-ordinate system here. Sorry I should have put in, let me say I put it here like X, yes, it is not coming out nice.

Let me draw it the way I have in my notes, let me draw it this way. Okay, this is better. X, Y, and Z, and this one is fixed with the, with the body. So this is going to rotate as the body rotates and here is the origin of the body system and here is the origin of the space system and they do not coincide and I am not saying any point is fixed in here.

So the most general motion will be the following. So you can say that this point O is translating in space and on the top of it, the body is rotating with that point. The co-ordinate system of, the co-ordinate system that is fixed in the body that is rotating about some axis. So that will be the most general motion.

So let me write down here. First, I am going to emphasize that X, Y, Z is an inertial system. It is inertial system. And then your X, Y, Z is of course non-inertial and it is a system fixed in the body and it rotates with the body or moves along with it. And as I said, the general motion is the following. Rotation of the body about some axis plus a translation of the origin of the body system, of the body axis or body system. That is what the most general motion would be.

Now, because the body is going to be rotating about some axis, let me say that it has an angular velocity Omega which we defined here. So that is the angular velocity, which I escribe to this system. And now, if I look at some point P here, let me say that from the origin it is a distance or it has a radial vector r and this vector r is in the body system. This is rigidly attached in the body. But this guy is going to move around with the body because of the motion of the entire, entire system because it is going to rotate and translate also.

So if you look at the velocity v, of this point P in the body, in the inertial system, it has some, it has some location in the space system or inertial system and some some radial vector here. And I am looking at the time rate of change of this vector, which gives you the velocity v, which will be given by the velocity of the origin which I call capital V plus the rotation of this entire system, which I have said it has a angular velocity Omega and then as I saw a minute ago it has to be crossed with r. So that is what the velocity of any point P would be in this system. Let me emphasize that V is measured in the inertial system and your r is in the body system.

Now, I am saying that Omega is the angular velocity of this body and I have chosen to tell the locations of different points in the body from the origin O. Now, what if I choose a different point as an origin? Will my angular velocity change? Will the vector Omega change? Will I get a new direction or a different magnitude for Omega? So that is what we want to ask.

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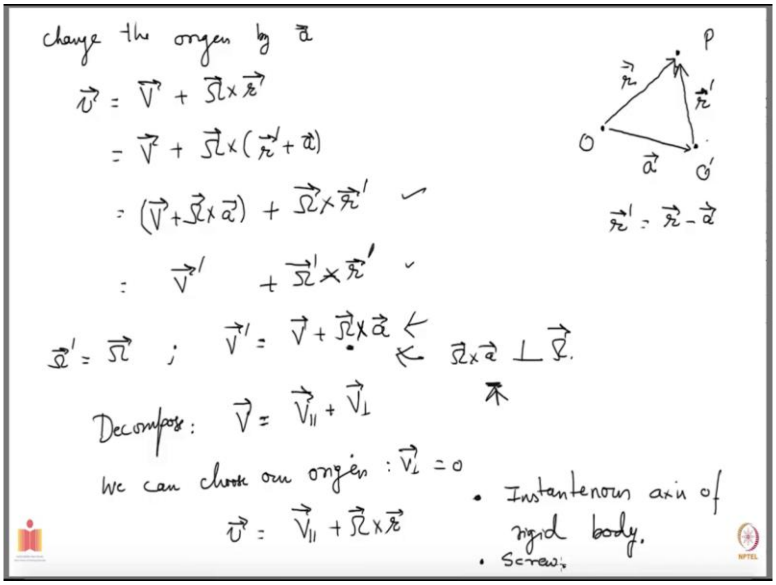

So let us say, we change the origin. Let us change the origin by, you can shift it by vector a. So let us say it was located here initially, you go to, we chose a different location in the body and you say this is located vector, the separation between these two is given by vector a which is a constant and you are looking at any point P. If this was originally called r vector as we wrote earlier, then from the new origin O prime, it is r prime. Its radius vector is r prime.

And I want to know what happens to this relation, V is cross r. So what I do is I substitute. So here if you see, your r prime, this vector is r minus a. So if I substitute in here, I get V plus Omega cross r is r prime plus a, right. Now, I write this as V plus Omega cross a, now a is a constant remember. So I am clubbing that piece is that piece with V and then I have the piece which has Omega cross r form.

So as far as the velocity of the particle is concerned in the in the space system or in the inertial system that is not going to change of course. It is what it is. But the decomposition has changed because you have chosen a different origin. So what you have now is, this is what is your V prime plus Omega prime cross r prime. If you have started with O prime as the origin, this is what you would have written.

So if you compare these two, you realize the Omega is same. So Omega does not change if you change the origin of the body system. And also you realize that the velocity of the origin of this body axis changes to the following. And this is interesting because it says that the change in the velocity of origin is only in the component of V perpendicular to Omega. So if you decompose V into a perpendicular and a parallel component, let me write it down, it will be clear.

So let us decompose V as V parallel. And by parallel, I mean parallel to Omega so the direction of your rotation axis and hen V perpendicular. So you look at the plane perpendicular to the axis and your V is getting decomposed in these two. So one is parallel to Omega and then the remaining components lie in the plane perpendicular to Omega.

And what is this relation telling you? This relation is telling you that under a shift of origin, only the perpendicular components change because this is perpendicular to Omega. Meaning the parallel component is not going to change, because Omega cross a is perpendicular to Omega. So let me just write a line here. So V perpendicular does not develop, that is not a, I think it is not a very nice line.

Anyway, I have said what I wanted to say that your only V perpendicular is going to change, which means, which means that your V parallel is not going to change of course. And it also means that you will be able to choose a vector a such that you can get rid of the V perp, V perpendicular. So because of this, because of this thing we can choose, choose our origin which is equivalent to saying we can choose our vector a such that V perpendicular is 0. And what does that mean then? It means that the velocity V would have the following form, which means that each point in the body because this is a generic point, each point in the body is rotating about some axis which is given by Omega and it is also moving along the same axis.

Remember V, this thing is just the velocity along the direction Omega. And this is clearly the motion of a screw. You know what is a screw? A, if you look, if you look at a screw, when you turn a screw all the point on the screw are rotating about the screw axis and the screw is also moving forward or backward, depending upon what you are doing. So all the point are rotating about the screw axis and of course, all the points are also moving along the screw axis.

So we see that the general motion of a rigid body is that of a screw and it also makes sense to say that the entire body is rotating about that axis. Because if you look at the screw, it is rotating about it is about it is axis. So we define instantaneous axis of rotation of rigid body. And what is that, that axis is the axis about which the rotation is happening at this moment and of course, the body may also be translating along that axis and this axis is called instantaneous axis of rotation.

And the most general motion of a rigid body is the motion of a screw. Very good, we will continue with more on rigid body in the next video.