Fluid Dynamics for Astrophysics
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Lecture - 02
Continuum hypothesis, distribution function and stress-viscosity relation

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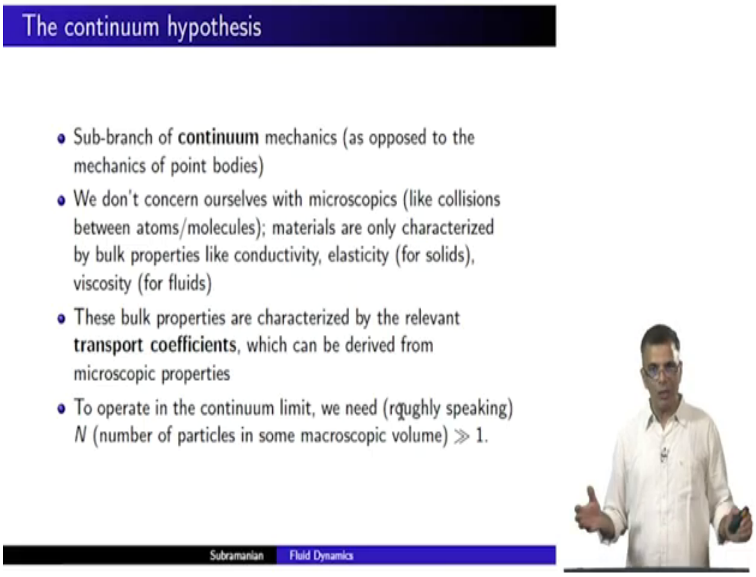

So, let us start the actual contents of the course by starting to talk about the Continuum Hypothesis. As I mentioned earlier, the reason this is because we are the conservation laws that we are generally familiar with a conservation of say momentum or conservation of energy when we learn classical mechanics.

We normally you know apply these laws to point objects, extended objects alright. But extended objects are merely a collection of point objects. And there are so, you most of the time you idealize you know for instance the trajectory of a stone to the trajectory of a of the center of mass of the stone yeah. So, this is what we are generally used to.

However, here in this course, we are talk we are not talking about point objects, we are talking about a continuum say consider the flow of air or water. So, this is a aggregate, this a this an aggregate of many molecules yeah. So, this what I mean by saying that this is a essentially a sub branch of continuum mechanics as opposed to the mechanics of point bodies.

And in particular, we do not concern ourselves with microscopics like collisions between atoms and molecules yeah. This is what one does in the kinetic theory of gases when one wants to understand concept like pressure of a gas. You know for instance you have a gas in a container so, what causes I mean the gas to exert pressure on the walls of a container?

That is because the gas is a finite temperature and therefore, the molecules are constantly you know buzzing about and they are constantly striking the walls of the container and then bouncing back. And so, the impulse you know manifests itself as pressure on the walls of the container.

However, we will not be concerning ourselves with those microscopics, we will instead be you know concerning ourselves with a macroscopic effect of these microscopic collisions aka pressure for instance. So, pressure will be one quantity will be invoking quite frequently.

So, everything of course, derives from collisions, but we will not bother so much about the you know we will have a you know a sandpaper to smooth over, we will be resorting to what is called you know coarse grain description of these things.

We will be you know looking at a fuzzier description and we will be concerning ourselves mostly about the macroscopics. Macroscopic quantities such as pressure, density, velocity and so on so forth. Pressure, density, velocity of the continuum not of the constituents of the continuum per se yeah.

Materials also are characterized only by the bulk properties such as conductivity, elasticity for solids elasticity and viscosity applied for fluids and elasticity applied for solids. Of course, you know there are complex fluids which are visco elastic, we will not go into those complexities.

But roughly speaking we will be concerning ourselves only with these bulk properties which some people like to characterize, like to describe as emergent properties. Properties that emerge from microscopic such as collisions.

These bulk properties of course, for instance conductivity or viscosity, they are characterized by the relevant transport coefficients. You can think of the coefficient of conductivity yeah, conductivity of what? Say heat.

So, the heat conductivity of copper is more than that of aluminum for instance which is why we think of you know copper is a better conductor of heat than aluminum. Well, where did this come from? You know concept of conductivity which is a one transport coefficient, examples of other transport coefficients are viscosity coefficient and elasticity modulus so on so forth.

So, the this transport coefficients came from microscopic properties which we will not bother about, but it is important to keep in mind that they come from microscopic properties yeah. Instead of bothering about the microscopics per se, we will simply you know restrict ourselves to understanding the transport.

The fact that there are transport coefficients and this is this value for this particular material or this particular gas. We will take that to be a given and go forward from there. So, these are the three things, these are the three major things to keep in mind.

The other thing is so the natural question that comes to mind is we have been talking about the distinction between microscopics and macroscopics, but what is the boundary line yeah? So, turns out the boundary line is a little fuzzy. The boundary is just that in order to operate in the continuum limit, we need again I emphasize roughly speaking the number of particles in some macroscopic volume to be much larger than one.

Notice the two vague you know adjectives in this sentence in some macroscopic volume yeah. So, what do you mean by some macroscopic volume? Say this room, when I am standing in the size of this room is say about 10 feet or maybe a little more.

So, representative macroscopic volume would be say 1 cubic foot certainly not 1 cubic centimeter right maybe 2 cubic foot, 2 cubic feet or 5 cubic feet, but certainly not 1 cubic centimeter or 1 cubic inch yeah. So, this is what one I mean by saying some macroscopic volume, some you know reasonably you know large volume, exactly how large it is depends upon the situation.

In astrophysics of course, 1 cubic foot does not make much sense, you would have to be talking about say of the order of a parsec or so, it all depends upon the situation at hand exactly what you call a macroscopic length scale or a macroscopic volume yeah. So, you are justified in using the continuum limit if the number of particles in some representative macroscopic volume is much greater than 1.

So, here is the other fuzzy thing in this definition what exactly they mean by much greater than are you talking about 10? Are you talking about 100? Are you talking about a million? Turns out again the answer is a little fuzzy.

Ideally, you would have you would like to have as many particles as possible in a macroscopic volume so that when you are reasonably far enough from it, you do not you are not able to discern the dynamics of each particle per se. But you are able to discern the dynamics of the of the collective ensemble yeah.

You are able to say for instance if I make a statement saying the number density of particles or the number density of molecules in the room is so much. Well, you know you are you are mentioning the word density which means that the implicit in this statement is the fact that there are many molecules yeah. Only then the concept of density makes sense exactly how many?

Are there more than 10 molecules in which case you know do there have to be more than a million molecules? Well, that in fact, depends and again in astrophysics astrophysicists like to slide on thin ice. If there are situations where you know the number density is very small, outer space contrary to what is commonly stated is not vacuum. It is you know there is matter out there it is just that the matter is extremely tenuous, extremely thin and yet, one resorts to continuum description of the matter.

And so, many times, you know the continuum limit becomes a little questionable astrophysical an applications be that as it may we will come to it when we deal with it, but in everyday situations such as you know the flow of air or some something like this the continuum description is very much you know valid.

In the following sense, the number of particles in some macroscopic, some representative macroscopic volume is typically much larger than 1. Because of which it is meaningful to talk in terms of to invoke macroscopic properties such as number density, pressure and so on so forth to characterize a continuum.

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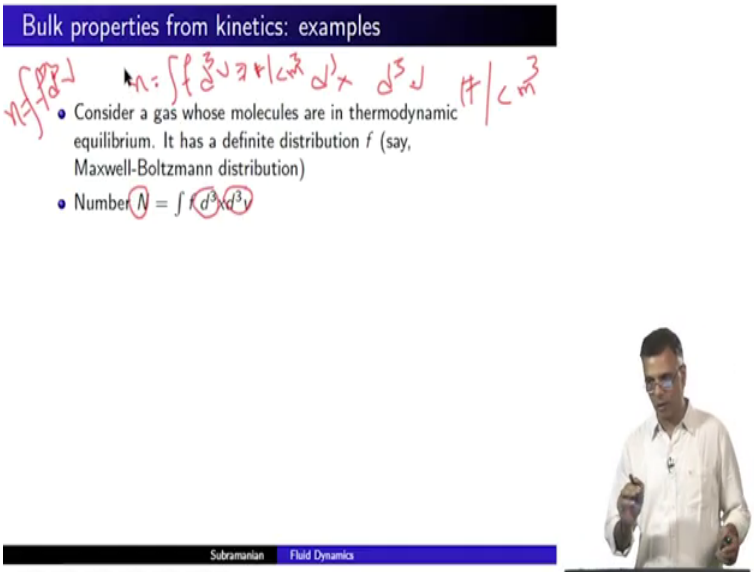

So, to explain on that just a little bit here is an example of bulk properties from the microscopic kinetics yeah. So, here is one example. Consider a gas whose molecules are in thermodynamic equilibrium in that you can ascribe a definite temperature to it.

The molecules have been bouncing around each other for a long time and have established an equilibrium amongst each other. And a consequence of this is that the molecules have a very definite distribution f by distribution I mean say the Maxwell-Boltzmann distribution.

So, on the x axis say you might have speeds, on the y axis you might have the probability of finding particles in a certain speed range yeah. So, I am sure you have you have come across this and so, that would be you know the distribution would be characterized by you know this distribution function f and for Maxwell-Boltzmann distribution this is just an exponential as you know.

Now, what is an example of a bulk you know property that is derived from this distribution? Say the total number of molecules in a box yeah, the molecules are characterized by Maxwell-Boltzmann distribution and you integrate this distribution over all space d cube x and all velocities yeah. So, you integrate the distribution function over all space and all velocities, and the normalization works out to be the total number ok.

So, this is the first you know most basic kind of you know illustration of how you derive bulk properties such as the total number from kinetics. So, as I was saying, the total number is given by the this you know integral where you are integrating over all space as well as all velocities yeah. So, d cube x as well as d cube v yeah.

What if I was not interested in the total number per se, but I was only interested in the number density say the number per centimeter cubed yeah. So, in or if I wanted that, then I would say, I would simply write n is equal to integral of f d cubed v this is supposed to be a 3.

Let me write this again n is equal to integral of f d cubed v and this would be, this would give me the number per centimeter cubed. That is because I am not integrating over d cubed x. If I integrate over d cubed x, then the per centimeter cube goes away and I get just total number here which is a dimensionless quantity.

So, here is a is an example as I said of you know a macroscopic quantity such as the number or for that matter the number density derived from a microscopic quantity not quite microscopic I would call it a mesoscopic quantity which is the distribution function f.

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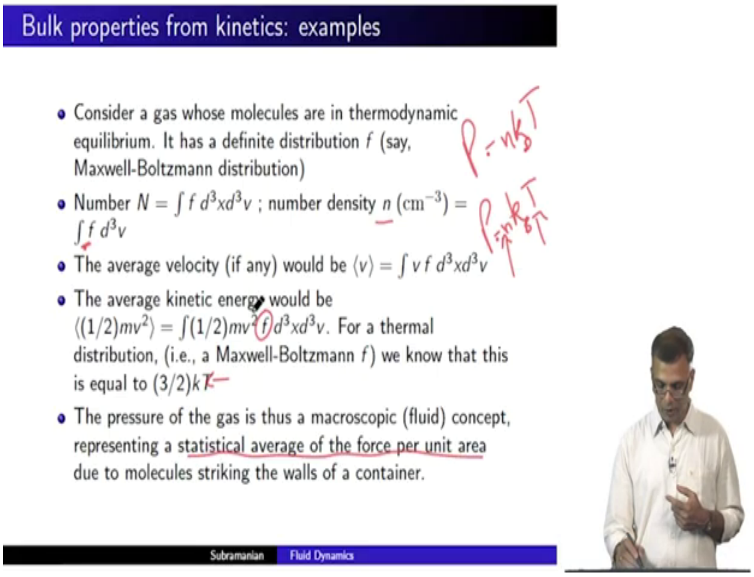

Yeah, here is an example of what I was just saying. The number density n in centimeter per centimeter cubed would be just you know this quantity f you know integrated over d cubed v. If you are interested say in the average velocity yeah so, what you do is you just stick the velocity in here and you integrate over the distribution function.

So, the basic normalization of the distribution function is given by f d cube x d cube v which gives you the number and if you wanted to find out an average velocity, if there is any. For instance you know is there an average velocity to the air in this room, in other words is there a breeze flowing through it or not yeah so, there may or may not be an average velocity. If there is you could find that out by doing this to the distribution function.

The point I am trying to make here is this is how you derive a macroscopic quantity like the velocity from a microscopic quantity which is the distribution function. This is the basic point

I am trying to make yeah and now, so, the average velocity is one thing. But something that you might be a little more familiar with would be for instance the kinetic energy.

The average kinetic energy, average of half m v squared well, what do you do? Instead of velocity here, instead of v here, you just say you just stick half m v squared into in into the same integral, in into this very same integral. Look this and this are the same, the integrand are the same except you have stuck a half m v squared here yeah.

So, you perform this integral, I mean I have just shown one single integration is really a six-dimension integral, 3 dimensions for space and 3 dimensions for velocity yeah. So, you perform this integration, and you get the average of the quantity half m v squared. And we know that for a thermal distribution, in other words, the distribution that is characterized by a Maxwell-Boltzmann distribution f, we know that this half m v squared this, this is equal to 3 halves k T.

So, here is an example of how you derive the temperature of a gas T from the microscopic distribution function yeah. You see you are going from the microscopic distribution to the temperature yeah. So, the microscopics concern themselves with kinetics things like collision and so on so forth.

But if you wanted to get a macroscopic quantity like the temperature which will be used in our study of fluids in the study of the of the continuum, this is how you do it yeah. So, this is an example of how you get a bulk property such as the temperature from the microscopics.

And you know the pressure is simply equal to n k T many people like to say k B sorry P equals n k Boltzmann T. So, the k B is simply the Boltzmann constant. So, the pressure so, this is a microscopic quantity and this is a microscopic quantity and the product of them with the Boltzmann constant of course, gives you the pressure.

So, you see the pressure of a gas which is essentially you know represents a statistical average of the force per unit area due to molecules striking the walls of a container. It is just this it is really a microscopic concept and that can be derived from and so, it is related to these two microscopic quantities density and temperature.

And it can be linked to the microscopics in this way, the density is related to the microscopic like this yeah. The distribution function is a microscopic quantity and you integrate that to get the density and the temperature is linked to the microscopics in this manner yeah, we just did that. This is how the temperature is linked to the distribution function here.

So, here we have about how many examples about three examples of how you squeeze out macroscopic quantities such as pressure, number density, temperature and things like this from the microscopic quantities which are from microscopics of the gas which are characterized by the distribution function right.

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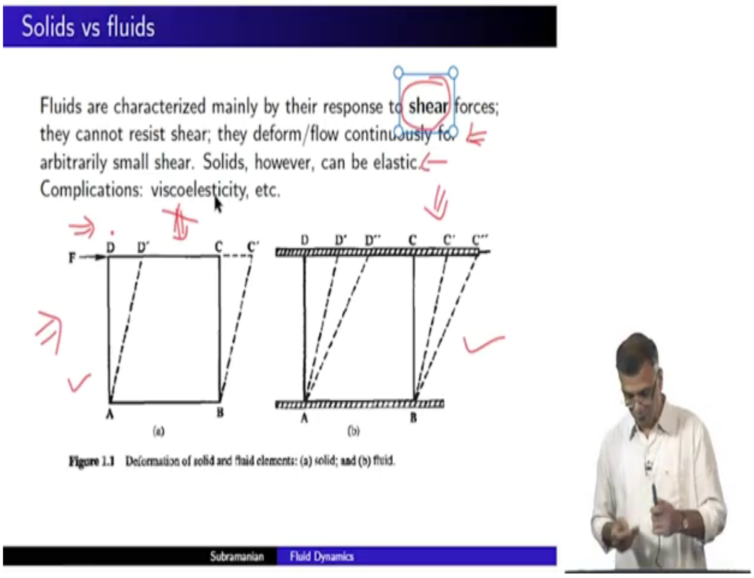

So, having done that, so the whole point of this slide was how you derive was to justify our description of fluids as a sub-branch of continuum mechanics. As opposed to the mechanics of discrete point bodies right and so, that was there.

The other basic question you might have in your mind is we keep talking about this thing called fluids and you might know that fluids comprise. When we normally say fluids, it includes both liquids as well as gases there are differences, but nonetheless, they are both considered fluids.

So, on the other hand in everyday life, we come across solids you know a piece of rock or so, what really is the distinction yeah? The distinction is kind of subjective, but one central difference between solids and fluids is that fluids are characterized mainly by their response to what are called shear forces. And what I what one means by shear is a force that is not normal to a surface.

Say you have a surface like so and you are pressing against it, there is a normal force whereas, a shear force is something that is trying to deform the surface like so ok. A shear force is something that can be tangential to the surface either in this way or in this way ok.

So, consider for instance, you know this kind of a situation where you have an element of fluid like so and you are trying to deform it with a shear force like this. Notice the force is not normal it is not normal like this, it is not like so on the it is shearing like so yeah.

So, you are trying to distort this element so that you know the square which is A, B, C, D is you are trying to make it a parallelogram A, B, C dash, D dash yeah that is what you are trying to do. In this case, this is a solid element and this is a fluid element as such this diagram does not tell you that much.

But the main thing to focus is that here, fluids cannot resist shear. When you apply a shear force like so, a fluid element flows, it you know the top surface here it simply keeps the top surface simply keeps sliding over the bottom surface ok.

When you apply a shear force to an element of fluid, it flows whereas, you apply a shear force to an element of a solid like here, it tends to snap back to its original position. This is what elasticity is all about. You try to deform, and it snaps back whereas, you know fluids on the other hand. The fundamental difference between a fluid and a solid in some sense is its ability to withstand shear ok.

You apply a shear force being any force that is not normal to the surface, any force that is parallel to the surface either this way or that way or a combination of these yeah. So, here you see here is an example of a force that is parallel to the surface and so, you are applying this kind of a shear force. A solid tends to withstand shear, it tends to snap back whereas, a fluid cannot do that. It the surface simply slides continuously over the bottom surface and the fluid flows ok.

So, of course, there are so there is what I mean by saying solids can be elastic. So, whereas, you know a fluid deforms or flows continuously in response to an arbitrarily small shear. The shear can be arbitrarily small and yet the fluid will start flowing yeah.

There are complications, there are materials which display you know characteristics of both viscosity as well as elasticity. Viscosity you know refers to fluids, elasticity refers to solids in some sense they are the same, but for instance, nano molecules of nano particles of water. You think of water as a fluid right turns out that when you confine water to about a nanometer scale you know droplet, it starts behaving like a solid very strange, but true yeah.

We will not be going to these complications; we will restrict ourselves to what are called Newtonian fluids for which this is not true. We do not bother about complications like viscoelasticity. So, the main point of this slide is that the boundary line between solids and fluids is its capacity to withstand the fact that fluids cannot resist shear in response to even an arbitrarily small amount of shear it flows. So, this is the main thing.

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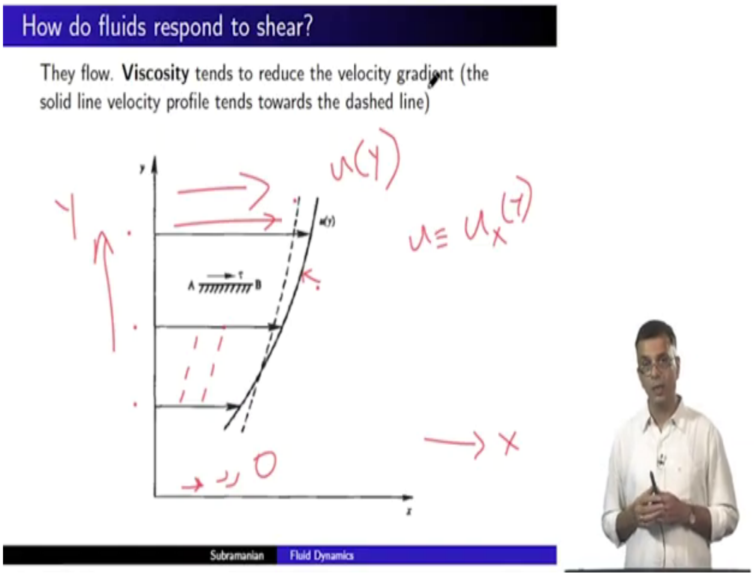

And how exactly do fluids respond to shear well here you go. So, consider what this is these arrows here these represents vectors that show the magnitude of X directed velocity as a function of Y here so, this is Y and this is X ok.

So, the velocity is only in the X direction. So, what is written as u Y here is essentially the X component u is the X component of velocity and that is all, that is the only component of velocity there is. So, what this is saying, what this you know diagram is saying here is that the X component of velocity varies with Y. So, this for instance could be representative of a fluid like say honey.

Consider you know honey in a box yeah and you apply a shear force, you apply a force to the top layer of honey. You can imagine that the top layer will flow smoothly, the top layer here will flow smoothly whereas, that is not true of the next bottom layer here.

That will not flow with the same velocity as the top layer, the velocity of the next bottom layer is slightly smaller and the velocity of the layer below that is even smaller so on so forth. Until the time you come to the very bottom of the vessel where the velocity might well be almost 0 yeah.

So, what it is almost as if these different layers are connected by little rubber bands like this ok. So, when you are trying to slide the top layer over this, there are these imaginary rubber bands which prevent, which are trying to prevent the sliding and this is what viscosity is all about.

In other words, if there is a velocity gradient, if there is a u X as a function of Y yeah, you see the u X is large here, not so large here even smaller here so on so forth. If there is a gradient in u X as a function of Y, viscosity tends to reduce that velocity gradient yeah.

So, this solid line, the this is the this would be have been the original profile and that tends towards this dotted line here and so, this is what viscosity is all about. Viscosity is that phenomenon which tends to reduce the velocity gradient. And this is something that is peculiar to fluids.

We will come to a more formal definition of viscosity in due course, but for the time being, you know since, we talked about the you know distinction between solids and fluids. And we sort of hinted at the fact that fluids are characterized by viscosity whereas, solids are characterized by elasticity without going into definitions of elasticity. Since, we are mostly concerned with fluids, I thought I would sort of give you a general idea broad definition of what viscosity is all about and this is it.

Viscosity is that property for fluid which makes it resist shear although it does flow. A fluid cannot resist viscosity sorry a fluid cannot resist shear and how much it flows depends upon the viscosity inherent in the fluids. And you can think of viscosity as that property which makes little rubber bands connect different layers of the fluid yeah and these rubber bands are preventing the top layer from sliding arbitrarily over the bottom layer.

The stronger this rubber band is the more the viscosity which is why say you think of honey as a more viscous fluid than water. By the time you reach the bottom layer here, the velocity would be almost 0 yeah. So, the gradient is very large whereas, that is not so with in case of what we normally think of is a less viscous fluid which would be water right.

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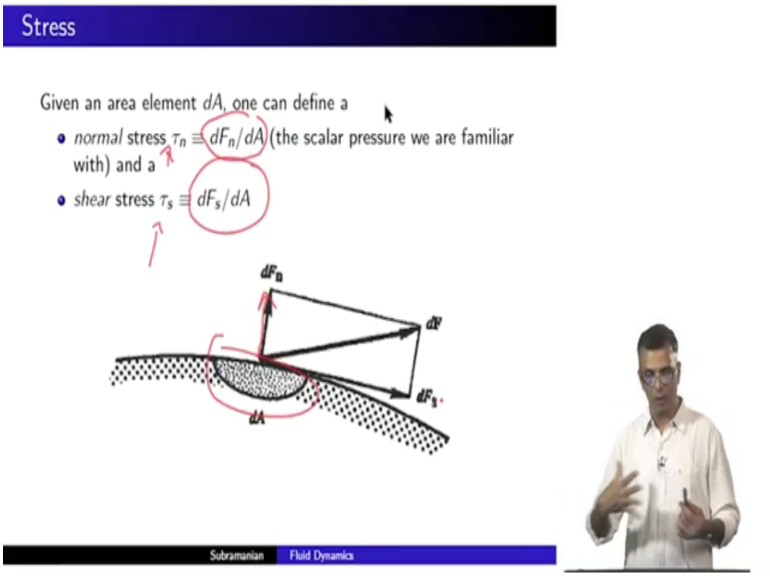

So, the next concept I would like to talk about is something called stress. The kind of stress that we are more most familiar with is just pressure yeah and that would be what is called a normal stress. And so a normal stress would be basically a normal force d F n d A where F n is a normal force. Normal meaning here you have a surface and you have a force like this which is normal to the surface yeah.

So, this is the scalar pressure that we are familiar with, but this need not be the only kind of you know and this is one kind of stress, you can also you guessed it by the amount of emphasis I was placing on the word shear in the previous slides. We will be concerned not only with normal stresses, but also with shear stresses yeah. So, the kind of stress that arises from shear forces.

So, suppose you have an area element like so, like this yeah and you have a shear force that is not necessarily always normal. This would be the normal force, but you can also have a tangential force, a force that is tangential to the surface and this would be what is called the shear force. The tangential force can be like so or also into the you know plane of the screen. So, either way it is parallel to the surface.

And you can also, you can always you know define this quantity d F s d A in analogy with this quantity d F n d A right. So, the only difference being we call this a shear stress, we call this a normal stress simply because this arises from shear forces and that arises from normal forces.

So, as such, there is no big distinction between these two kinds of is just that we are a little more familiar with normal stresses than we are with shear stresses. Simply because we are more familiar with the concept of pressure turns out that a normal pressure is not all there is. There are also these shear components and these two kinds of stress can be combined into one unified kind of description.

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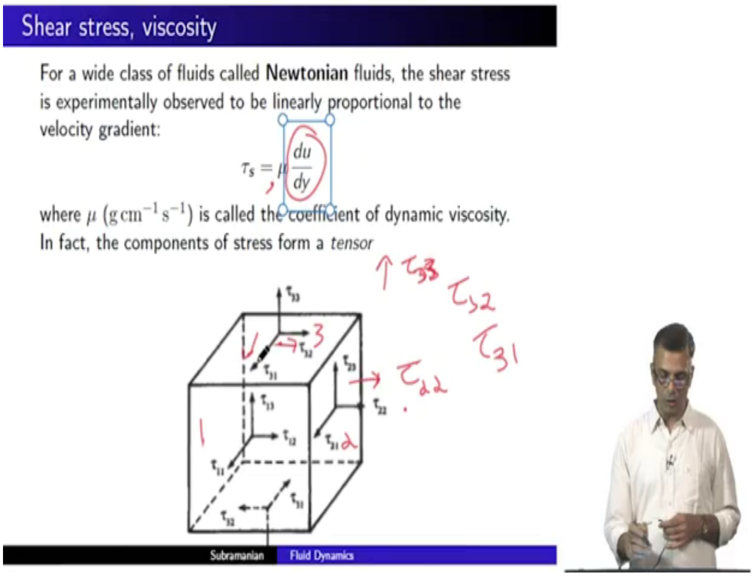

So, what is it got to do with viscosity? Well, for a wide class of fluids called Newtonian fluids which is the only kind of fluid that we will be concerned with in this course. The shear stress, the quantity tau s here which we define here which is essentially the ratio of d F s to d A or d A is the you know differential surface area is proportional to is observed to be linearly proportional to the velocity gradient.

So, the velocity gradient we encountered here you see, there is a gradient of the Y of the of the X component of the velocity as a function of Y right. So, that is what this is. This is the d u, d y that we encountered turns out that the shear stress is linearly proportional and the proportionality constant is this mu yeah and this mu is called the coefficient of dynamic viscosity ok.

So, you can see how intimately connected the concept of viscosity is with the concept of stresses yeah and the stresses need not only be normal consider an imaginary cube like this. So, the normal stresses are say you know the this cube let us see yeah so, this is phase 1, this is phase 2, this is phase 3 and so on so forth.

You know the normal stresses would be what are called the normal stress on phase 3 is the one which is pointing that way, the normal stress on phase 2 is the one which is pointing this way and I call this tau 33 just a notation yeah and I call this tau 22. What these subscripts mean are the stress on phase 3 yeah, which is arising out of a force that is also directed along the direction 3.

Here, the 22 means the stress on phase 2 which arises out of a force that is also in the direction of the outward normal characterizing phase 2 ok. But as you can see tau 33 is not the only kind of stress there can be. There can also be a tau 32, there can also be a tau 31; there can be a tau 32 and there can be a tau 31 yeah.

So, and the tau 32 it would be this and the tau 31 would be this and these two are what are called shear stresses because these stresses arise from forces that are tangential to the surface. Whereas, tau 33 is a normal stress and there is no real need to you know strongly differentiate between these three kinds of stresses. It is just that one is normal, the other two are tangential, you can consider them in all in one unified kind of description.

And we will see why the concept of shear stress well, the concept of shear stress is central to viscosity because of this yeah. It is experimentally observed that the shear stress is linearly proportional to the velocity gradient and the constant of proportionality is this coefficient of dynamic viscosity and as it so yeah.

The components of these of all the stresses from what I would call a tensor, the tensor is simply a matrix. The tau, the elements of which are tau 31, tau 32, tau 33 so on so forth ok. Some of them are shear stresses, some of them are normal stresses. The normal stresses are the ones in which these two you know indices are the same. For instance tau 1; tau 11 or tau 22 or tau 33 all the other off diagonal elements are called shear stresses right.

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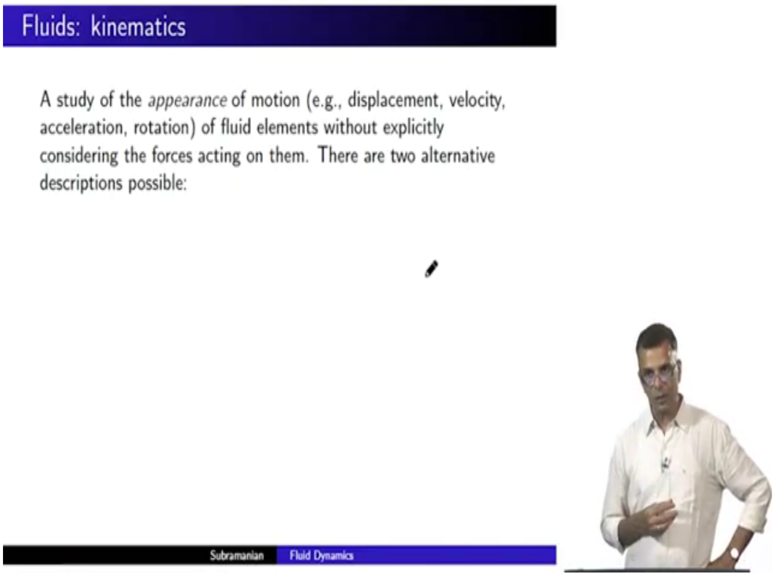

So, from now on, we will start talking about having talked a little bit about how fluids are characterized? How they are different from solids and what is the main dividing line between solids and fluids?

We will start talking about the study of fluid kinematics which is essentially a study of the appearance of motion. For instance, displacement, velocity, acceleration, rotation and so on so forth of fluid elements without explicitly considering the forces that are acting on them.