Foundation of Quantum Theory: Relativistic Approach

Spinor Field quantization 1.

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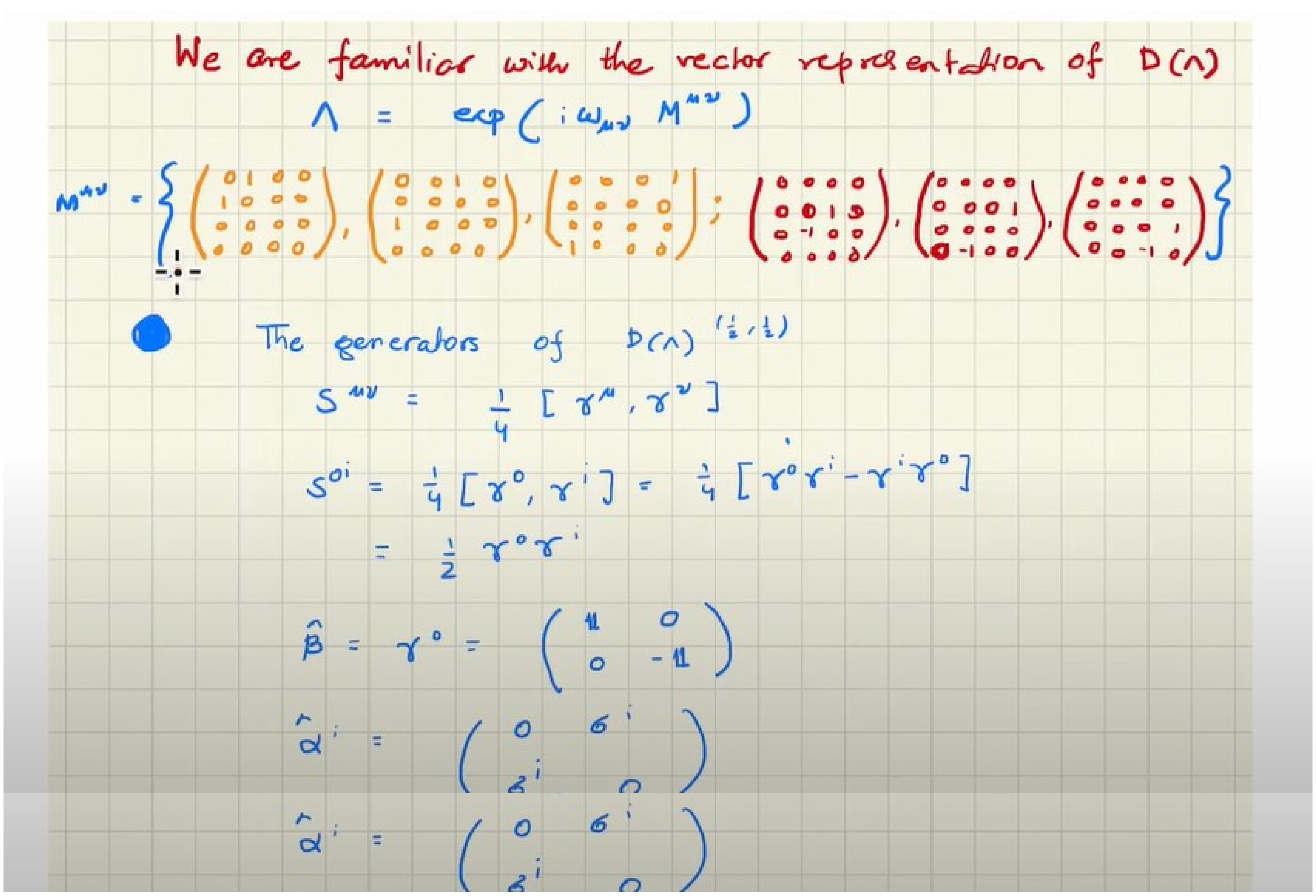
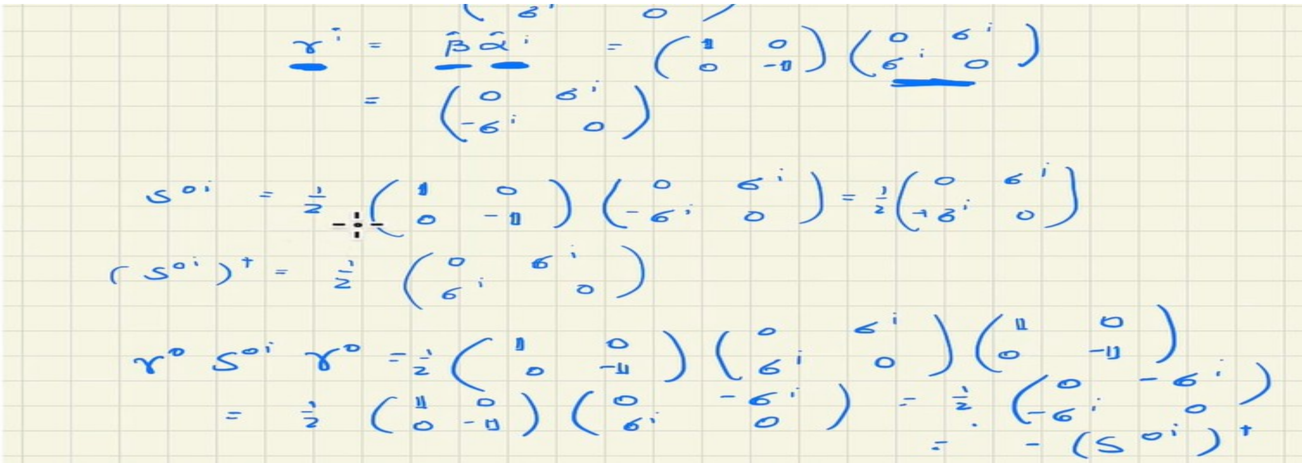
Spinorial Generators

Lecture- 22

So today we will discuss about the spinomial generators of Lorentz group. In the previous class we have seen that the Lorentz group is basically identified with the relation amongst its generating operator or the algebra. So we had seen previously that the low range transformation matrix come out three different rotations and three different boosts. So we had seen previously that the low range transformation matrix come out three different rotations and three different boosts. So ultimately there are three rotation generators and three boost generators. So ultimately there are three rotation generators and three boost generators.





We are familiar with the vector representation of D(λ)

$$\Lambda = \exp\left(i\omega M^{\mu\nu}\right)$$

$$M^{\mu\nu} = \left[\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix};\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\right]$$

The generators of $D(\lambda)^{(1/2,\ 1/2)}$

$$S^{\mu\nu} = \frac{1}{4}[r^{\mu}, r^{\nu}]$$

$S^{\mu\nu}$, = 1 over 4 [r^0, r^1] ,= 1 over 4 [r^0 r^1 – r^1 r^0] = 1 over 2 r^0 r

$$\hat{\beta} = r^0 = \begin{pmatrix}\varepsilon & 0\\0 & -\varepsilon\end{pmatrix}$$

$$\hat{\alpha} = r^0 = \begin{pmatrix}0 & \sigma\\\sigma' & 0\end{pmatrix}$$

$$\gamma' = \hat{\beta}\,\hat{\alpha} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & \sigma'\\\sigma' & 0\end{pmatrix}$$

$$= \begin{pmatrix}0 & \sigma'\\-\sigma' & 0\end{pmatrix}$$

$$S^{01} = \begin{pmatrix}0 & \sigma'\\\sigma' & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & \sigma'\\\sigma' & 0\end{pmatrix}$$

$$(S^{01})^{\dagger} = \frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & \sigma'\\\sigma' & 0\end{pmatrix}$$

$$\therefore \gamma^0 S^{0i} \gamma^0 = \frac{1}{2}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sigma' \\ \sigma' & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \quad \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -\sigma' \\ \sigma' & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma' \\ -\sigma' & 0 \end{pmatrix} = -(S^{0i})^{\dagger}$$

We have seen the examples of rotation matrices in the vector representation which are these three matrices which are doing rotations about x-axis, y-axis and z-axis.

And similarly there are three boosts matrix of transformation which are boost along x-axis, y-axis and z-axis and corresponding to these there are generators which will tell us along what direction the transformation has been made for example if i want to know what is the generator of this matrix the rotation about the x-axis what I will do I will take the matrix itself do the derivative with respect to the parameter of transformation which is θ in this case then I will obtain a new matrix after the derivative and then in that matrix I will put the parameter to zero ,so that will give rise to the generator of the rotation along the about the x-axis and you we are supposed to get after this operation which you can verify as an exercise the first matrix over here sorry not the first matrix over here first matrix over here the red matrix is the rotation about the x-axis. Similarly if I do the rotation about the y-axis or the z-axis take the corresponding transformation matrix. Do the derivative with respect to the parameter and put the parameter to 0. I am supposed to generate these remaining two generators of the rotations. These are three generators of rotation matrix acting on vectors. This is the vector representation generators. Similarly, for boosts, there are three boosts along x-axis, y-axis and z-axis. Do the following operation. Take D(λ) upon d parameter of boost β. Obtain the new matrix and put the parameter to 0. And that will give rise to the 3 boosts generator which are these 3 matrices. So, this overall gives you 6 new generators, 6 generators of Lorentz transformation, 3 boosts and 3 rotations which collectively are written as a $M^{\mu\nu}$. ,$M^{\mu\nu}$is supposed to be some anti-symmetric rank 2 tensor. An anti-symmetric rank 2 tensor has 6 independent components in 4 dimensions. As you might have seen for electrodynamics there was an anti-symmetric rank 2 tensor $F_{\mu\nu}$ if you are familiar with that. That had 6 components 3 magnetic fields and 3 electric field. Any anti-symmetric rank 2 tensor has 6 components. All the diagonal entries are 0 because anti-symmetric tensor diagonal elements are 0. So M_{00}, M_{11}, M_{22} , M_{33} are 0. Surviving entries are M_{01}, M_{02}, M_{02}, M_{03} , , M_{12}, ,M_{23}These are the 6 components which are surviving. And they are anti-symmetric that means m10 is – of M_{01} ,M_{21} is – of M_{12} and so on so there are six independent components and these six are the six matrices ,And as we have seen previously that these six matrices are related to each other in terms of the Lorentz algebra. So three of them are rotation, three of them are boosts and boosts and rotations talk to each other through the commutation relation which we had written in the previous class. So these objects rotations, rotations commutator is supposed to generate another rotation generator. Boost, boost commutator is supposed to generate a rotation. Generator and boost and rotation commutator is supposed to generate a boost generator. So this is the relation between the six, three Js and three Ys. And you can verify the relation what we had written above are indeed satisfied by this 6 elements. So you can verify the algebra which we have written call it call this one as ,Y_1, ,Y_2, ,Y_3, ,Y_1, ,Y_2, ,Y_3 and this 3 as J_1 , J_2, J_3 . You will realize that the algebra which we had written previously above are exactly satisfied by these 6 matrices. So therefore they are generators ,of the Lorentz group. They are the generators of the Lorentz group in four-dimensional space. They are vector representation. Now, in order to go to the spinors, we are looking for a spinor representation of the Lorentz group. That means I want a matrix $D(\lambda)$ which acts on spinors. These matrices, the Lorentz transformation generated by these matrices ,are obtainable from these generators will act on the four vectors Pxyz or $P_0 P_x P_y P_z$. These are the vector representation. So, these Lorentz transformation matrices which are generated by this $M^{\mu\nu}$. Remember the λ we had written as $e^{i\theta j}$ and –, so let us say –, let me write it cleanly. Remember, we had written it as $e^{-i\theta j - \eta y}$. This is how a general low-inch transformation matrix is written. Three j's come with three θs or call it β this one and three y's come with three βs collectively in terms of ,$M^{\mu\nu}$there can be written as exponential of $\omega_{\mu\nu}$ and ,$M^{\mu\nu}$ where $\omega_{\mu\nu}$ is also six component object made up of three θs and three βs and ,$M^{\mu\nu}$ is made up of three j's and three y's. , So collectively it is written like this sometimes with i just to account for the group structure or algebra where we had written. So therefore I have written the λ like this.

So the game is clear you have to find out the generators acting on your space for example for vector space for vector representation these are the six$_\mu$ news which we have obtained and they satisfy the same algebra which we had seen for rotations and boost matrices which we saw in the previous class the question is how to write the same thing for the spinor space or D(λ) half of representation. Previously we had seen that the spinor representation can be factored into double copy of the rotation group which is written as spin half representation. So, double copy of spin half representation is equivalent to spinorial representation and therefore I am calling it a D(λ) half of representation. This is also a 4 cross 4 dimensional matrix. But this is acting on a spinner, the spinner which has two copies of spin half objects, x_{i1}, ,x_{i2} , x_{i3} , x_{i4}.. So, these two are spin half objects put together they generate a spin spinner of half half rank. And this still being a 4 cross 4 dimensional, 4 cross 1 dimensional vector like a t x y z, they do not transform as vector, they transform as spinors. That means they transform with a different matrix D(λ) half half. And what this D(λ) half half will be obtainable from their generators. ,So, therefore in the spinorial space we are looking for again 6 generators of the Lorentz algebra because the algebra has to remain same. So, therefore there should be 6 generators and which act upon the spinorial objects. Now, again since this is not a course on a full field theoretic and representation, so therefore I would not be deriving those properties for you, but I would just be lifting down in a spinorial space what are those 6 generators. So, we are familiar with the γ^μ matrices, γ_0 was supposed to be β and γ_i's were supposed to be β times α_i. These were the matrices which we have written down. The statement is if I take the collection of γ_0 and γ_i as γ^μ and take the commutator between the two γ_s, γ^μ and $\gamma^{\nu\nu}$ can take any value,$_\nu$ can take any value. Then I obtain a two index object, one with μ, one with ν because I can take any value of$_\mu$ and I can take any value of ν. So this is a rank two object because two indices are appearing. And this is anti-symmetric as well because commutator is anti-symmetric. Therefore I have generated a rank 2 anti-symmetric object and my claim is that this satisfies exactly the same algebra which these ,$M^{\mu\nu}$ satisfy so remember this has six pieces in it three rotations and three booths and they satisfied a relation amongst themselves which are suited which was suitable for low inch transformation the claim is that if i take this set ,$S^{\mu\nu}$ that also has six component S^{0i}, S^{01}, S^{02}, S^{03} and $S_{ij} S_{12} S_{23}$ and S_{31} just like what M had. So those six elements of ,$S^{\mu\nu}$ also exactly satisfy the same algebra which six components of ,$M^{\mu\nu}$ satisfy. So therefore this is also a valid representation of low-density transformation generators. And therefore, these are the things which will act on spinors. So this acts on a spinner in the sense of learning generation of matrices. So ,$S^{\mu\nu}$ are the generators of transformation. The D(λ) matrix, which is given by half-half, will be obtainable from exponential of I $\omega_{\mu\nu}$, the same $\omega_{\mu\nu}$, rotation or boost, whatever amount I am doing. And then the generator will change. The amount will not change. If I am doing rotation by 30 degree, the parameter will take the value 30 degree. But what matrix will do the 30 degree change will depend upon which representation I am talking about. So in spinors, this ,$S^{\mu\nu}$ will go and sit. In vectors, ,$M^{\mu\nu}$ will go and sit. In function space as we saw, there was this X cross P kind of operator which was going and sitting. So fine, so in spinous space, this $S^{\mu\nu}$ is our generator, exponential of that with $\omega_{\mu\nu}$ is the Lorentz transformation matrices for spinors. And let us try to do some simplification to see their structure clearly. ,So first, the three components S^{0i} S^{01} S^{02} S^{03} are to be written as 1 by 4 times γ_0 and γ_i commutator. So 0 and *i,* so here 0 and i were appearing,$_\mu$ and$_\nu$ were 0 and i, so therefore the commutator will have a 0 and I which is if I open it up, it is just $\gamma_0\ \gamma_i - \gamma_i\ \gamma_0$. Now, we know that γ^μ and γ^ν satisfied the something called Clifford algebra if you remember, – of $2\eta_{\mu\nu}$ identity operator. So, if I take$_\mu$ is equal to 0 and$_\nu$ is equal to i, I will get a $\gamma_0\ \gamma_i$ and right hand side I would have a η_{0i} which is 0. So, therefore $\gamma_0\ \gamma_i$ is equal to – of $\gamma_i\ \gamma_0$ and use this fact to write the commutator into one half of one pair $\gamma_0\ \gamma_i$. The second pair is the negative of the first pair. So, they add up. So, 1 by 4 becomes 1/2, all right. So, let me clean it up. So, we have this structure S^{0i}. The first three operators of the low-range transformation generators for spinors are half times $\gamma_0\ \gamma_i$. Where γ_0 if you remember was the β matrix itself which is identity cross identity and – identity along the diagonal and γ_i matrix was β times α i matrices where α i had sigma matrices along the anti-diagonals. So, as a result γ_i has a sigma and – sigma along the anti-diagonals. So, that is the S^{0i}. So, we can write down ,fully what is S^{0i}, half times γ_0 which is this and γ_i which is this. So ultimately you will get S^{0i} is half times $\sigma_i\ \sigma_i$ along the anti-diagonals. What more if I take the† of this S^{0i} ,σ_i, this σ_i and σ_i will be done complex conjugation with, Hermitian transpose with and will be replaced along the anti-diagonals. But since poly matrices are Hermitian, I can convert it back to sigma a σ_i. Therefore, the generators are Hermitian matrices. Just like M0i's were also Hermitians. Further there is a unique identity which connects S^{0i} with its† in some sense. So what I do, I justmultiply $\gamma_0\ \gamma_0$ from left and right to $S^{0iP.}$ So S^{0i} was half which is here times $\sigma_i\ \sigma_i$ along the anti-diagonal which I am writing in the middle and then I squeeze it between γ_0 from the left and γ_0 from the right which is here and here. So once I do that, this is a simple algebra. You can just pay attention to matrix multiplation and you will see that once you do that, you will get – σ_i and – σ_i along the anti-diagonals. Okay, you have to just be careful. This is 2 cross 2 dimensional matrix identity. This is also 2 cross 2. This is also 2 cross 2. This is also 2 cross 2. 0 is also 2 cross 2, empty 2 cross 2 matrix, dimensional matrix. So everything is 2 cross 2. You have to do it slightly carefully. This means 1 0 0 0 0 1 0 0 0 0 – 1 0 0 0 – 0 – 1. Similarly, for this you can write the 0 you have to write for, let us say we are writing sigma z, sigma z is, sigma z would be 1 0, there is a – outside present, so – 0 1 here. And then here the same thing 1 0 0 1 and then 0 0 0 0 that is this so you have to just do it carefully and you will be able to recover that if i put γ_0 from left and right and squeeze s 0 i with γ zeros what I will get is – $\sigma_i\ \sigma_i$ along the anti-diagonals which is the – of the† itself. The† was $\sigma_i\ \sigma_i$, so was S^{0i} itself, but I am writing in a convenient way because that will become clear, region of that will become clear. So if I squeeze γ_0, if I squeeze S_{0i} with γ_0, what I am getting at the end is its† with a – sign. This is for the first three generators of the nodal transformation on spinors. What about the remaining three? The remaining three are Sij, again by the definition it is $\gamma_i\ \gamma_j$ and same Clifford algebra which we had discussed for i not equal to j, I will obtain $\gamma_i\ \gamma_j$ multiplied together with a half factor. The similar way the 1 by 4 was converted into one half for the previous case. It has been writen down what is γ_i and what is γ_j put together you will get now you will get a diagonal entry – $\sigma_i\sigma_j$ along the first diagonal and – $\sigma_i\ \sigma_j$ along the second diagonal. So that is the S^{ij} for you For example S^{12} will be – of ,σ_1 σ_2 *ie* ,20 0 – of ,$\sigma_1\ \sigma_2$. Again you can see that it is also hermitian you take the† you will see that it will come back to itself so sorry if I do the fij† you do this you will see that $\sigma_i\ \sigma_j$ will be replaced by $\sigma_j\ \sigma_i$ so you see here ,– sign is there $\sigma_i\ \sigma_j$ is there and here also we have a $\sigma_i\ \sigma_j$ so you see that it is sigma this was $\sigma_i\ \sigma_j$ this is $\sigma_j\ \sigma_i$ so again you see they have not up to sign this is fine but they have not come back to itself $\sigma_j\ \sigma_i$ is not equal to sigma a σ_j so again we have not been able to generate a Hermitian transformation. Remember Lorentz transformations were not being generated by Hermitian matrices. Even for vectors if you recall we had in the previous class we had seen that the Lorentz transformations ,boosts were not unitary matrices that means they were not being generated by Hermitian matrices. The same thing is happening even for a spinorial representation that the boosts generated by Sij are not Hermitian matrices. Okay fine with this realization we again do the same thing which we had done for S_{0i}. S^{0i} we had squeezed it between γ_0 and γ_0 do this again ,Put a γ_0 to the left and a γ_0 to the right.

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{2}\gamma^i\gamma^j$$

$$= \frac{i}{2}\begin{pmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^j \\ -\sigma^j & 0\end{pmatrix} = \frac{i}{2}\begin{pmatrix}-\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j\end{pmatrix}$$

$$(S^{ij})^\dagger = \frac{i}{2}(\gamma^j)^\dagger(\gamma^i)^\dagger$$

$$= \frac{i}{2}\begin{pmatrix}0 & -\sigma^j \\ \sigma^j & 0\end{pmatrix}\begin{pmatrix}0 & -\sigma^i \\ \sigma^i & 0\end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix}-\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i\end{pmatrix}$$

$$\gamma^0 S^{ij} \gamma^0 = \frac{i}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}-\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

$$= \frac{i}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}-\sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j\end{pmatrix}$$

$$D(\Lambda) = \exp\left(+\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

$$D(\Lambda)^\dagger = \exp\left(-\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right) = \exp\left(+\frac{i}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0\right)$$

$$= \mathbb{1} + \gamma^0\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\gamma^0 + \frac{1}{2!}\left(\gamma^0\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\gamma^0\right)\left(\gamma^0\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\gamma^0\right)$$

$$= \gamma^0 D(\Lambda)^{-1}\gamma^0$$

$$S\mu v = \frac{1}{4}[r^\mu, r^v] = \frac{1}{2}r^i, r^j$$

=

$$\frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma' \\ \sigma' & 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$\gamma^0 S^{-j}\gamma^0=\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

$$= \, , \; \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$D(\lambda)=\exp(\frac{+1}{2}\omega_{\mu\nu}S^{\mu\nu})$$

$$D(\lambda)^{\dagger}=e^{\frac{-1}{2}\omega_{\mu\nu}(S^{\mu\nu})^{\dagger}}=e^{\frac{1}{2}\omega_{\mu\nu}\gamma^0(S^{\mu\nu})^{\dagger}\gamma^0}$$

$$= \; 1+\gamma^0\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\gamma^0+\gamma^0(\frac{1}{2}\omega_{\alpha\beta}S^{\alpha\beta}\gamma^0)(\gamma^0\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}\gamma^0)$$

$$= \; \gamma^0 D(\lambda)^{-1}\gamma^0$$

$$= \; r^{\mu}-i\,\omega_{\alpha\beta}[S^{\alpha\beta},r^{\mu}]+\ldots$$
$$= \; r^{\mu}-i\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}_{\nu}r^{\nu}+\ldots$$
$$=, \; [S^{\mu}_{\nu}+i\omega_{\alpha\beta}(M^{\alpha\beta})^{\mu}_{\nu}+\ldots]r^{\nu}$$
$$= \; \Lambda^{\mu}_{\nu}r^{\nu}$$

Do this again matrix multiplation with care. This time it will become clear what we were trying to do. It will still is Sij† with a – sign. What have we done to recapitulate? We had previously seen S^{0i} and $S_{0i}^{\dagger}$ are equal to each other. They are Hermitian. So rotations are generated by Hermitian transformation. Therefore rotations are unitary transformations and rotation generators squeezed between γ_0 became the – of its Hermitian conjugate. Boost generators are not Hermitian but still boost generators squeezed between γ_0, γ_0 do become – of their Hermitian conjugate. This is a very important relation which has a implication for the next line which we are going to write. So, now for spinal representation we want to know the matrix which operates on spinal straight to do a rolling transformation. And with the prescription of the generators they have to be exponentiated with the parameters of transformation. I am going to write this D(λ) as this. So okay so there are a couple of notational issues one should be very careful in order to make sense of the next steps so I have written *i* upon two that two can be ignored so this is just a bookkeeping parameter many textbook use so in principle you can forget about the two so let me erase the two out from here. And one more thing apart from this extra 2 I had initially written in notes which I should could have avoided. So, let me just emphasize first let us forget about 2 that is that was immaterial. Secondly, when I am writing this the transformation matrix as generators times the parameter times *i*, then this *i* whether *I write* it explicitly here or I absorb in ω depends upon the notations we are writing. So, here when I have written the transformation generators as 1 by 4 commutator γ_i γ_j, typically in that sense the Lorentz transformation matrix *D(λ)* is written as just exponential of $\omega_{\mu\nu}$S$^{\mu\nu}$. However, if I want to write down in terms of $i\ \omega_{\mu\nu}$, then many of the times people would write a – *i* in front of this as well. So with any of these definitions, we can go ahead. So we should be slightly careful when we write in a consistent notation. Since I had chosen to write the Lorentz transformation for vectors as with *i* and trying to do the business with the same thing over here for $S^{\mu\nu}$ as well. So since I'm going to write as with *i*, let me supply back *i* here in the definition. Okay. So then effectively it would be this. And therefore you will see everywhere when we were taking a†, the *i* which was present will become a – *i*. So ultimately this will become a – *i*. So it will become a plus over here. Okay so there are two relations which you can obtain either with *i* or without *i* depending upon which way you are writing the transformation matrix so one way we can stick with the without *i* definition where we have derived all these properties or in order to make sense with the textbook notations where various textbook use I can go back to plus *i* then I will correct all these equations which are getting with a – sign they will become a plus sign either way you can do business. So the for the time being let us stick with the deviation meaning let me not use the textbook notations people use we use the notations we were using one over here one over here that means I can just forget the *i* over here as well so let me erase the *i* out as well so we will just use the notation which we have been using so let me first remove all the *i*s and then we will get back to how to generate transformations out of it. That does not matter. Only thing you have to be careful when you insert an *i* over here, in order to compensate for that *i*'s extra insertion, the generator should also accommodate a – *i* somewhere. So, but since I am going to work with this ,$S^{\mu\nu}$ without any *i*, without any *i* here, I am going to use the definition of transformation without *i* here as well. Okay so if this is fine let us go ahead and see what do we get. , If D(λ) is there $D(\lambda)^{\dagger}$ if I do I should get this $\omega_{\mu\nu}$ $S^{\mu\nu\dagger}$ which is fine but *I know* $S^{\mu\nu\dagger}$ is in general – of $S^{\mu\nu}$ squeezed between γ_0. So I will get a – sign here. ,So $S_{\mu\nu}^{\dagger}$ is equal to – of ,$S^{\mu\nu}$ squeezed between γ_0 from both sides which is fine. Now Taylor expand this quantity I will get a one identity – this term here which is γ_0 $\omega_{\mu\nu}$, $S^{\mu\nu}$ γ_0 then the square of this quantity so 1/2 factorial and this quantity twice appearing. So you see all the terms have a very similar kind of a structure where ,Identity can also be written a γ_0 from the left and γ_0 from the right because γ_0 identity times γ_0 is again identity. So, all these terms have γ_0 from in the left and γ_0 in the right. This term has a γ_0 in the left, γ_0 in the right. ,This term also γ_0 in the left and a γ_0 in the right. In between there is a γ_0 γ_0 as well which together combines and gives you identity. So, you see the structure is that there is a γ_0 then identity – ,$\omega_{\mu\nu}$,$S^{\mu\nu}$ plus 1/2 factorial the twice product of $\omega_{\mu\nu}$,$S^{\mu\nu}$ and γ_0 and all higher order terms. So, you see ultimately you are going to get a γ_0 and exponential of – of $\omega_{\mu\nu}$,$S^{\mu\nu}$ and γ_0 at the end as well. So, therefore you are going to get after taking the† γ_0 ,the inverse transformation this was plus the direct transformation matrix comes with a plus signature the inverse transformation comes with a – signature so you get a *D(λ)* inverse γ_0 γ so therefore the identity which we have landed up with is the identity which we landed up with is this that $D(\lambda)^{\dagger}$ is not D(λ) inverse had it been a unitary transformation then it was would have been true but all the low range transformation are not unitary transformation so therefore γ_0 γ_0 extra multiplation you have to live with only if it were rotations only then the† would have been equal to inverse of the transformation but since it is involving boosts as well which are not unity transformation therefore a γ_0 extra has been come about.

Defining $\bar{\psi} = \psi^\dagger \gamma^0$

$$\bar{\psi}\psi \longrightarrow \psi^\dagger D^\dagger(\Lambda)\gamma^0 D(\Lambda)\psi$$

$$= \psi^\dagger \gamma^0 D(\Lambda)^{-1} \underbrace{\gamma^0 \gamma^0}_{\mathbb{1}} D(\Lambda)\psi$$

$$= \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$$

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \longrightarrow \psi^\dagger D(\Lambda)^\dagger \gamma^0 \gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu (D(\Lambda)\psi)$$

$$= \psi^\dagger \gamma^0 D(\Lambda)^{-1} \gamma^0 \gamma^0 \gamma^\mu (\Lambda^{-1})^\nu{}_\mu \, D(\Lambda) \cdot \partial_\nu \psi$$

$$= \psi^\dagger \gamma^0 D(\Lambda)^{-1} \gamma^\mu D(\Lambda) (\Lambda^{-1})^\nu{}_\mu \partial_\nu \psi$$

$=$

$$\boxed{D(\Lambda)^{-1} \gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu}$$

For the vector representation

$$\Lambda = \exp(i\omega_{\mu\nu} M^{\mu\nu})$$

$$M^{\mu\nu} = \left\{ \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}; \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix} \right\}$$

$$(\mathbb{1} - i\omega_{\alpha\beta} S^{\alpha\beta} + \dots)\, \gamma^\mu \,(\mathbb{1} + i\omega_{\rho\sigma} S^{\rho\sigma} + \dots)$$

$$= \gamma^\mu - i\omega_{\alpha\beta} [S^{\alpha\beta}, \gamma^\mu] + \dots$$

$$= \gamma^\mu + i\omega_{\alpha\beta} (M^{\alpha\beta})^\mu{}_\nu \gamma^\nu + \dots$$

$$= [\delta^\mu_\nu + i\omega_{\alpha\beta} (M^{\alpha\beta})^\mu{}_\nu + \dots]\, \gamma^\nu$$

$$= \Lambda^\mu{}_\nu \gamma^\nu$$

Defining $\bar{\psi} = \psi^\dagger \gamma^0$

$\bar{\psi}\psi \to \psi^\dagger D'(\Lambda)\gamma^0 D(\Lambda)\psi$

$$= , \quad \psi^{\dagger} D'(\Lambda)\gamma^0\gamma^0 D(\Lambda)\psi$$
$$= , \quad \psi^{\dagger}\gamma^{0\gamma} D(\Lambda)\underbrace{\gamma^0\gamma^0}_{0}\psi$$
$$= \quad \psi^{\dagger}\gamma^0\psi = \bar{\psi}\psi$$
$$=$$
$$\bar{\psi} r^{\mu}\partial_{\mu}\psi = \psi^{\dagger}\gamma^0 D(\Lambda)^{\dagger}\gamma^{\mu}(\Lambda^{-1})^{\nu}_{\mu}\partial_{\nu}(D(\Lambda)\psi)$$
$$= \quad \psi^{\dagger}\gamma^0 D(\Lambda)^{-1}\gamma^0\gamma^{\mu}(\Lambda^{-1})^{\nu}_{\mu} D(\Lambda)\partial_{\nu}\psi$$

$$D(\Lambda)^{-1} r^{\mu} D(\Lambda) = \Lambda^{\mu}_{\nu}\gamma^{\nu}$$

For the vector representation

$$\Lambda = , \quad \Lambda = \exp(i\omega M^{\mu\nu})$$

$$M^{\mu\nu} = \left[\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix};\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}\right]$$

$$(1-\omega_{\alpha\beta}S^{\alpha\beta} = \dots)\gamma^{\mu}(1+\omega_{\rho 6}S^{\rho 6}+\dots)$$

So irrespective of which notions, which *i* version or without *i* version you work with, you will always land up on this property. So I have shown you without *i*, but if you work with *i*, you have to correct this and previous terms as well. And you will land up on this identity once more. So this is a well-defined, understood identity that Lorentz transformations on a spinorial index, the† of that is not inverse of itself, but a γ not squeezed version of the inverse. Why have we derived that?

Because if *I* write down ,the $\bar{\gamma\gamma}$ remember we had defined we were trying to find a inverse quantity we were trying to write down a Lagrangian but xi† xi was not a lowest invariant object now we can see why because when we do the transformation of Lorentz transformation xI will undergo with transform with D(λ) of xi xi† will transform with D(λ)† of xi and since these quantities are not inverses of each other I will not get identity back and it will not transform as invariant quantity. But now you can convince yourself that if I work with a quantity called $\bar{\psi}$ which is this is $\psi^{\dagger}\gamma_0$. So $\bar{\psi}\,\psi$ if I see how does it transform it has a γ_0 in between. So ψ will transform with a d matrix $\psi^{\dagger}$ will transform with a d† matrix and I have a d† γ_0 d in between.That in between I can insert so I have† *I know* already is equivalent to inverse squeezed between γ_0 and γ_0. So you see in between I get a double γ_0 a pair of γ_0 which will become identity. So what you have to do you have to just write down d† as γ_0 d inverse γ_0. There was already a γ_0 present which will become identity so ultimately you will get ,$\gamma^{\dagger}$ a $\psi^{\dagger}$ γ_0 d inverse d ψ and $D^{-1}D$ is identity again and you will get back γ not $\psi^{\dagger}\gamma_0\psi$ which is equal to $\bar{\psi}\,\psi$. So, this quantity is invariant this quantity $\psi^{\dagger}\psi$ was not invariant it was not going to $\psi^{\dagger}\psi$ but $\bar{\psi\psi}$ goes to $\bar{\psi\psi}$ under Lorentz transformation. So, this is an invariant quantity ,Similarly, if I look at this quantity over here, which is the first term in the Lagrangian we were trying to write down, $\psi^{\dagger}\gamma^{\mu}\psi_{\mu}\psi$. This was again not Lorentz invariant, not Lorentz invariant. But if I correct it to $\bar{\psi}\,\gamma\,\partial_{\mu}\psi$, then you will see that due to the fact that ,which will be clear in a minute that if I do the transformation this $\bar{\psi}$ will pick up a d† inverse. This ∂_{μ} will transform under Lohr's transformation as a covector. , So, it will have this transformation law and this ψ will transform as a matrix transformation is like this. So, overall you will see you overall you will see you can just combine you will have a ,$\bar{\psi}$ on the left, this $D^{\dagger}$ is equivalent to γ_0 D inverse γ_0. Then there was already a γ_0 which is present here, so this will be this, γ^{μ} which is already there which is there, λ inverse which will appear from the transformation of the derivative operator which will appear over here. And then there is a $D(\lambda)$ which is position independent, so it will come out of the derivative and then there is a derivative hitting the ψ. So all these terms will appear if I take the transformation of the first term of the Dirac Lagrangian generalized to $\bar{\psi}$. Now use the properties that two γ knots put together is identity and effectively you will get from this thing, this is just a number, this is a matrix element.$_{vvvv}$ element of λ inverse this is just a number and this is a matrix so $_{number}$ and the matrix can be flipped together so I can take this on the left hand side and bring it on the right hand side so this becomes here $D(\lambda)$ appears here and I have this structure d inverse γ^{μ} D(λ) so i had a γ^{μ} matrix what I am going to do I am taking a transformation matrix d inverse from the left and d on the right and one can prove which we will see in a minute that this operation is equivalent to something like a Lorentz transformation as a vector on γ_v. γ^{v} is not a vector, it is a matrix, but still this operation is equivalent to what a Lorentz transformation would have done to a vector. This is the vector representation of the Lorentz transformation that combines with γ^{μ}. Remember our genuine vector v_{μ} undergoes a Lorentz transformation as $\lambda_{\mu v} V_v$. The same thing is happening to γ^{μ}. Despite it is not a vector, it is a matrix. So, it transformed with a D inverse of matrix from left and D of the matrix from the right. End product is that it is almost the same thing as a vector would have transformed. It is 4 matrices, previously 4 components get combined with these numbers. This time 4 matrices will get combined with these numbers . So, this transformation is equivalent to this. And once I have proven the quantity in the box then you will see that this quantity will turn up a $\lambda_{\mu v}$ of γ^{v} and there is a was a λ inverse of$_{vv}$ coming from the derivative low range transformation they will cancel each other so first let us prove this identity and then be done with that that is one minute job you can just take the low range transformation matrix γ^{μ} you have to identify the four dimensional representation generators where these six elements which we have written previously as well. But to start with the spinorial representation do not work with ,$M^{\mu v}$ start with the ,$S^{\mu v}$ remember when I am going to do $D(\lambda)$ inverse from the left D(λ) from the right it is the spinorial representation matrix I have to work with. So you start with a spinorial representation from the left the spinorial representation from the right ,Do this algebra, this is fairly straightforward algebra, use the anti-symmetric properties, you will see that the quantities which will be appearing in this will be the ,$S^{\mu v}$ matrix and γ^{v} matrix commutator will appear as the first term. Now here is the crucial identity which is an exercise for you to prove that $S^{\alpha\beta}$ and γ^{μ} these are two different matrices γ^{v} is v-th matrix of the four $\gamma_s S_{\alpha\beta}$,is one of the generator $_S{}^{01}$ S^{02} ,S^{03} *or* ,S^{04} S^{23} S^{31} one of these put together it is equivalent to identifying one of the this two index identifies the $M_{\alpha\beta}$ out of these six which one is this. So if this is 01 then it will pick up this 01 and the μ component of this will appear over here and the new component gets summed over. So there is a nice identity which is there which I am just going to write it down and that is the crucial thing in proving the identity that s α β ,γ^{μ} can be proven to be $M_{\alpha\beta}$ and then its μ index remains free and the new index gets summed over. This is an identity which you can prove or we will see it in a minute. But if this is the case I can replace the commutator appearing over here by the $M^{\alpha\beta}$. Now this $M^{\alpha\beta}$ is the vector representation for crossword dimensional matrix. And therefore you can see that the ,first term was a γ^{μ} the second term now is the same parameter but with the m matrix now and the γ^{v} together it can be written as a identity matrixmultiplying γ^{μ} and then this matrixmultiplying γ^{v} ,and this is just a expansion of the exponential if you recall which is nothing but the vector representation of the low range transformation so I started with a spinal representation of low range transformation put the D^{-1} and D from the both sides. I use an identity which I have not proven, but it is easy to prove that the commutator of the spinor representation generators and γ^{μ} leads to a vector representation generator. S was the generator for the spinor transformation, γ^{μ} was the γ^{μ} matrix. Their commutator generates a matrix which is 4 cross 4, but this is a generator of the Lorentz transformation in the vector representation. So ultimately the whole thing over here becomes a vector representation of Lowell transformation. So this is a straightforward algebra. There is nothing *much* to look at. Only thing is that you should prove this identity which we will do in the next class to sum up with. But ultimately if this quantity is, if this is true, that this identity is true, then you can go back, use this identity over here ,Therefore, this will become a $\lambda_{\mu\nu}$ γ^{ν} and this $\lambda_{\mu\nu}$ anD(λ) inverse$_{\nu\mu}$ will give me an identity matrix. They are inverses of each other and you can prove from there that $\bar{\psi}$ γ^{μ} ∂_{μ} ψ ,again is a Lorentz invariant quantity like a $\bar{\psi}$ ψ. So, therefore, both the pieces previously both of terms were appearing in the Lagrangian of a Dirac particle with $\psi^{\dagger}$. Here it was $\psi^{\dagger}$ and here it was $\psi^{\dagger}$. Those things were not Lorentz invariant, but I corrected them to $\bar{\psi}$ rather than $\psi^{\dagger}$ and ultimately all these things fell into place and I have an invariant Lagrangian at hand. ,So, from the next class we will quickly wrap up the discussion on this identities which I have written and then we will proceed from the Lagrangian which is a Lorentz invariant Lagrangian. So, I stop here.

Defining $\bar{\psi} = \psi^\dagger \gamma^0$

$$\bar{\psi}\psi \longrightarrow \psi^\dagger D^\dagger(\Lambda)\, \gamma^0\, D(\Lambda)\, \psi$$

$$= \psi^\dagger \gamma^0 D(\Lambda)^{-1} \underbrace{\gamma^0 \gamma^0}_{\mathbb{1}} D(\Lambda) \psi$$

$$= \psi^\dagger \gamma^0 \psi = \bar{\psi}\psi$$

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \longrightarrow \psi^\dagger D(\Lambda)^\dagger \gamma^0 \gamma^\mu (\Lambda^{-1})^\nu{}_\mu\, \partial_\nu \left(D(\Lambda)\psi\right)$$

$$= \psi^\dagger \gamma^0 D(\Lambda)^{-1} \gamma^0 \gamma^0 \gamma^\mu (\Lambda^{-1})^\nu{}_\mu \quad D(\Lambda) \cdot \partial_\nu \psi$$

$$= \psi^\dagger \gamma^0 \; D(\Lambda)^{-1} \gamma^\mu \; D(\Lambda) \; (\Lambda^{-1})^\nu{}_\mu \; \partial_\nu \psi$$

$$=$$

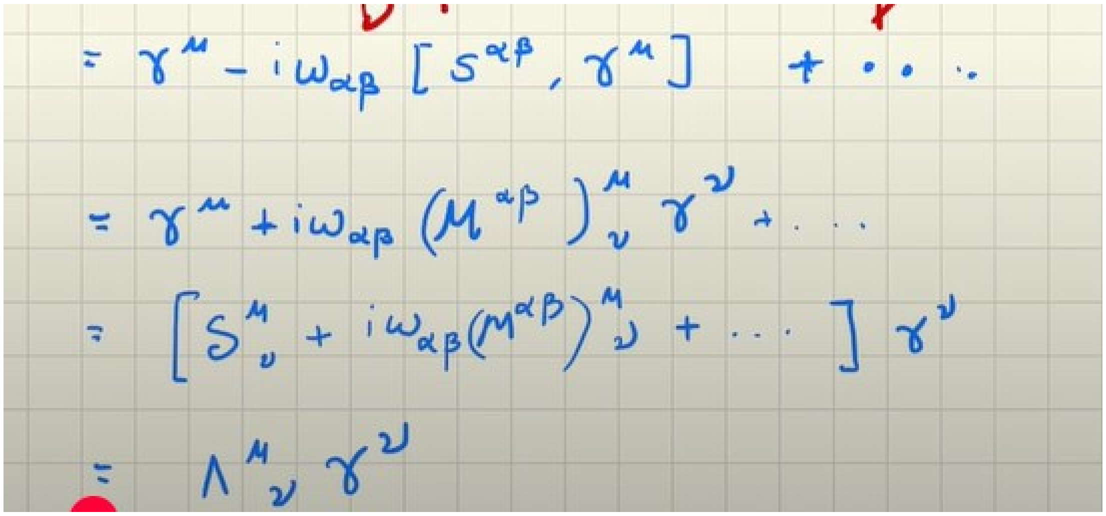
$$\boxed{D(\Lambda)^{-1} \gamma^\mu \; D(\Lambda) = \Lambda^\mu{}_\nu\, \gamma^\nu}$$

For the vector representation

$$\Lambda = \exp(i\omega_{\mu\nu} M^{\mu\nu})$$

$$M^{\mu\nu} = \left\{ \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}; \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix} \right\}$$

$$\left(\mathbb{1} - i\omega_{\alpha\beta} S^{\alpha\beta} + \cdots\right) \gamma^\mu \left(\mathbb{1} + i\omega_{\rho\sigma} S^{\rho\sigma} + \cdots\right)$$



So in the previous class we had seen that there was a necessity of changing from the Lagrangian form $\psi^\dagger$ ψ to $\bar{\psi}$ ψ because this object is not invariant under noise transformation while this object is this happened precisely because the transformation generated by matrices D(λ) for low range transformation are not unitary. That means $d^\dagger$ D(λ) are not unitary, are not identity. So therefore, I cannot use $\bar{\psi}$ ψ or $\psi^\dagger$ ψ as a invariant object in the Lagrangian. Remember the Lagrangian which we proposed initially was something like $\psi^\dagger$ i $\hbar$,γ^μ ∂_μ ψ – mc ψ. So, this kind of Lagrangian we thought would do our job of giving rise to the Dirac equation, which it does. Variation with respect to $\psi^\dagger$ indeed gives rise to the Dirac equation, but unfortunately the piece $\psi^\dagger$ ψ which is appearing with a mass over here is not Lorentz invariant. So we realized that we should not be using $\psi^\dagger$ ψ but we should use $\bar{\psi}$ ψ rather. So we realized that we should not be using $\psi^\dagger$ ψ but we should use $\bar{\psi}$ ψ rather. So if that is the proposition we should check whether this kinetic term is invariant now under low-inch transformation or not. And indeed we will see that it is invariant under low-inch transformation if ,the thing which is appearing inside so we try to see how does it transform inside there are things which appeared like that so you see $d^\dagger$ γ_0 γ^μ γ inverse of$_{\mu\mu}$ so these all these things will appear if I do try to do low inch transformation ψ will transform with a d ∂_μ will transform with a λ inverse γ^μ is a matrix ,$\bar{\psi}$ will transform with a $d^\dagger$ γ_0. So putting all these things together, I will get a $d^\dagger$ γ_0 γ^μ λ inverse and the d, all the items which will be generated from low-inch transformation. Now it so happens that this object inside which appears ultimately ,d inverse γ^μ D(λ) transforms γ^μ as a normal vector would have done.

γ^μ is not a normal vector, it does not have four components as a real numbers. A normal vector has four components as real numbers. γ^μ has are four matrices, so all its four components are four matrices. So therefore, it is not very straightforward that it will transform as a usual normal vector. But a matrix transforms under Lorentz transformation as a similarity side sort of transformation where youmultiply D inverse and D from the left and the right. The end result happens to be that it almost transforms like that it is a normal vector. Forgetting about its matrix structure, γ^μ transforms indeed like a normal vector. Only catch is that γ^μ are matrix objects not real numbers. So, once it this happens, if this happens then you can see what is happening inside. Inside if this transforms as a $\gamma^\mu{}_\nu$, γ^ν and outside there is a λ inverse. This λ inverse and this λ cancel each other and the object will become ,invariant under low instant formation. So, that is the goal we were looking to set. So, the essential criteria for that happening was this that the $\gamma^\mu\gamma^\mu$ transforms almost like a normal vector. In order to see whether this happens or not, we started doing the computation of the left hand side explicitly. ,I am going to write D inverse ,this way and d as this why can *I writ*e that because you knew that d the transformation matrix can be written as the exponential $i\ \omega\ \alpha\ \beta\ \omega\ \alpha\ \beta$ s $\alpha\ \beta$ depending upon where you observe i whether S contains i or not, in this notation it does not matter, ultimately the similar kind of structure will be set up whatever notation you use. So, I am using a definition where i right now is kept on the outside. Previously we had done business with i what was absorbed in the definitions of S as well. Both of these things are equivalent as discussed previously as well. So go ahead, do this computation, collect various matrices. One matrix multiplation will be this identity coming with this γ^{μ} and this identity. So there will be γ^{μ} like this. Another term would be thismultiplying with this and the identity. That will be the first term of the commutator. Similarly, this identitymultiplying the γ^{μ} over here and then combining with this will give me the second term of the commutator. And ultimately you will get these kind of things under the multiplation of both sides. Now there is a non-trivial claim which I am going to propose which was the commutator of S $\alpha\ \beta$ which is the generators of the Lorentz transformation for spinors and γ matrices ultimately is proportional to γ matrices back and the matrix which ismultiplying which is now a generator of Lorentz transformation for vectors. S were the generators of Lorentz transformation for spinors. M is the generators of Lorentz transformation for vectors. These are the M's. The six M's. Three rotations and three boosts. You see three boosts here and three rotations are in the red. So you can see this is the claim I am making. The commutator between S and γ gives me a result proportional to γ and M. M is the generator of the vector Lorentz transformations. If this happens then you can see that again the whole expansion adopts a form of a exponential. ,These are the different terms appearing in the expansion of the exponential and ultimately it will become exponential ,$i\ \omega\ \alpha\ \beta$ this time $M_{\alpha\beta}$ will appear because of the commutators being converted into m you see in the expansion commutator commutator second power commutators third powers and so on so forth will appear and if commutator of s and γ is m ultimately m m square and q kind of things will appear and this will become exponential of ω times m six ωs and six m multiplied together and then exponentiate it.And remember this was just the vector transformation λ. So therefore the end result will be $\lambda_{\mu\nu}\ \gamma^{\nu}$. So this result is indeed true subject to the condition I can prove that the commutator between S and γ becomes multiplation between M and γ. So$_{\nu}$ gets summed over, you see$_{\nu}$ is getting repeated no other index is repeated in one complete term. So as $\alpha\ \beta$ two index from here and one index from here appear in the following way. The first two index decide which $M_{\alpha\beta}$ I am going to talk about of the six and the$_{\mu}$ over here is going to talk about which row of the $M_{\alpha\beta}$ I am going to talk about. So, I am going to prove this now with couple of examples and then we will be having a faith upon this commutator turning into this m times γ.

Example

$\omega_{01} = 1 = -\omega_{10}$; $\omega_{\mu\nu} = 0 \quad \forall\ \mu\nu \neq 01, \neq 10$

$[S^{01}, \gamma^0]$

$\frac{1}{2}[\gamma^0\gamma^1, \gamma^0] = \frac{1}{2}[\gamma^0\gamma^1\gamma^0 - \gamma^0\gamma^0\gamma^1]$

$= -\gamma^1$

$(M^{01})^0{}_\nu \gamma^\nu = (M^{01})^0{}_1 \gamma^1 = \gamma^1$

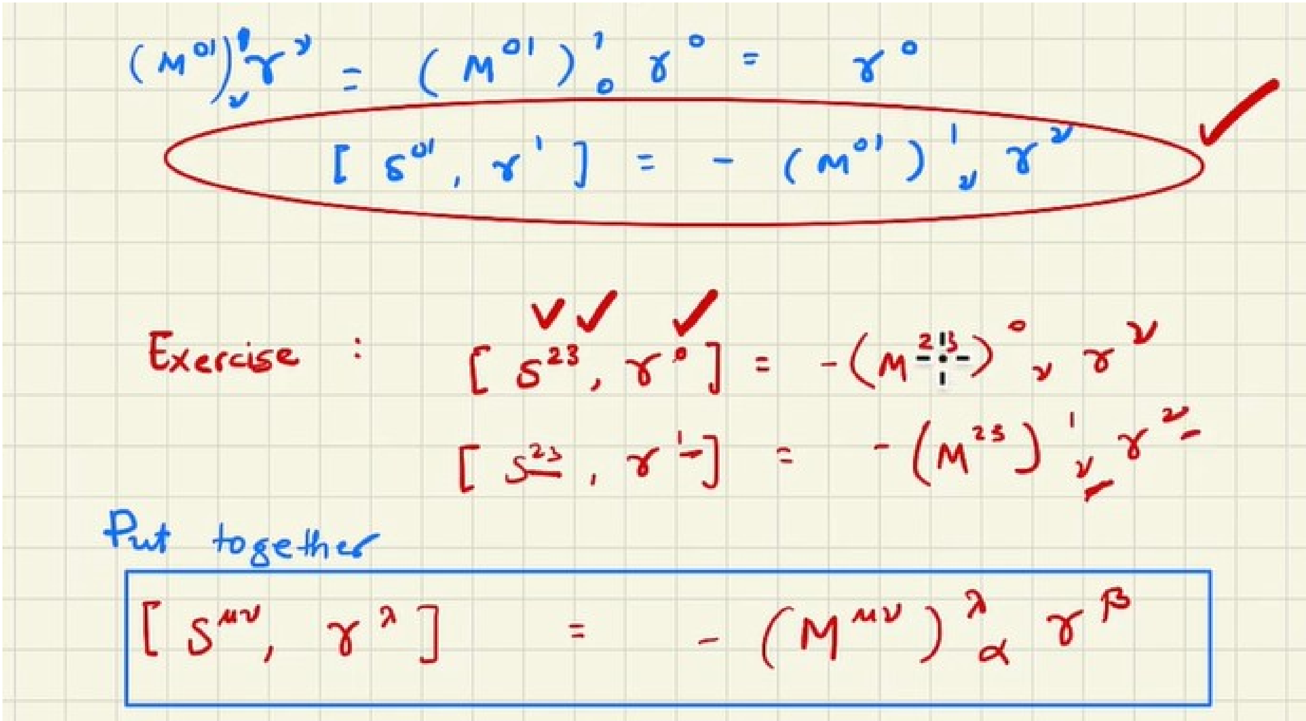
$$M^{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore [S^{01}, \gamma^0] = -(M^{01})^0{}_\nu \gamma^\nu$$

$[S^{01}, \gamma^1] = \frac{1}{2}[\gamma^0\gamma^1, \gamma^1] = \frac{1}{2}[\gamma^0\gamma^1\gamma^1 - \gamma^1\gamma^0\gamma^1]$

$= -\gamma^0$

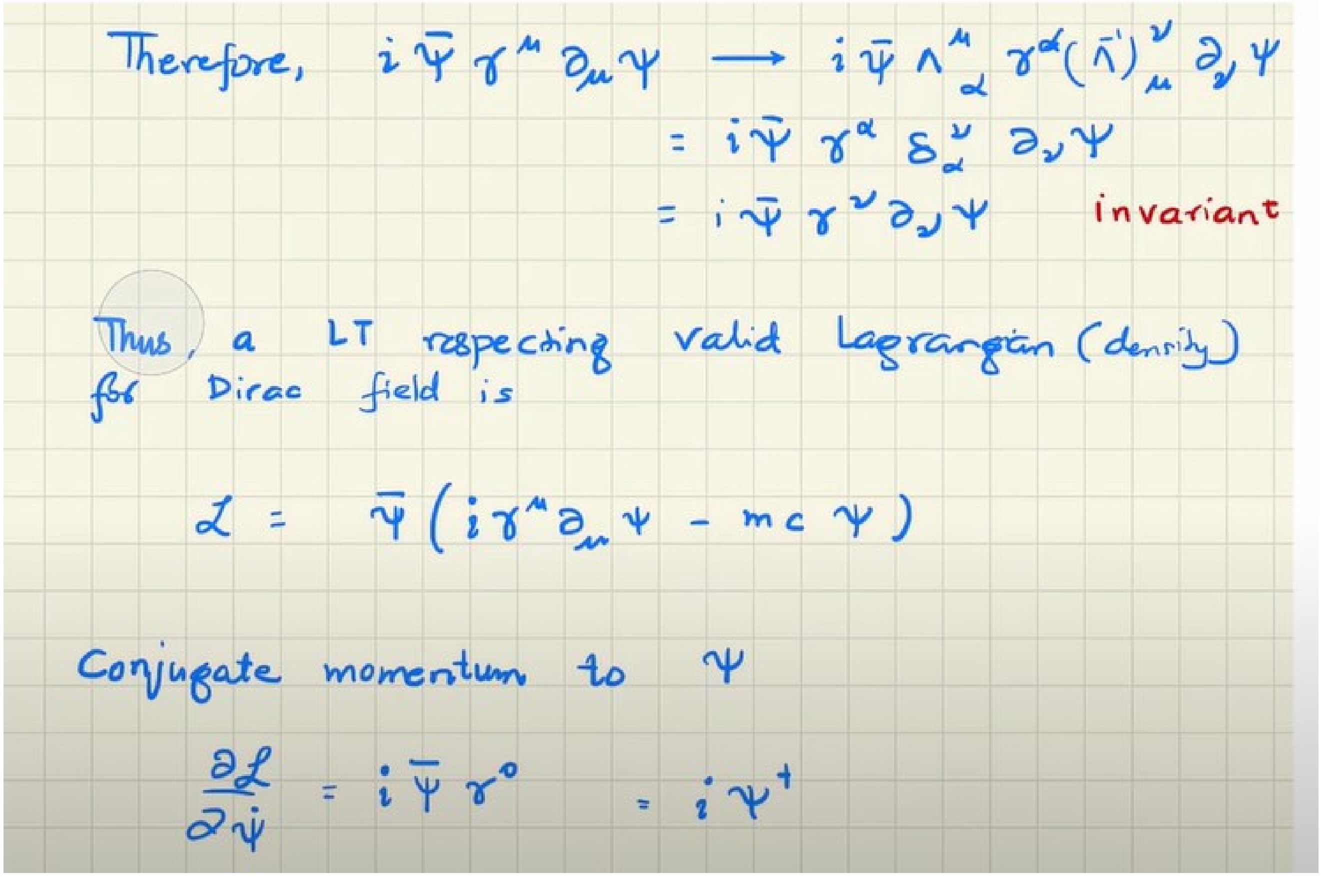
$$M^{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

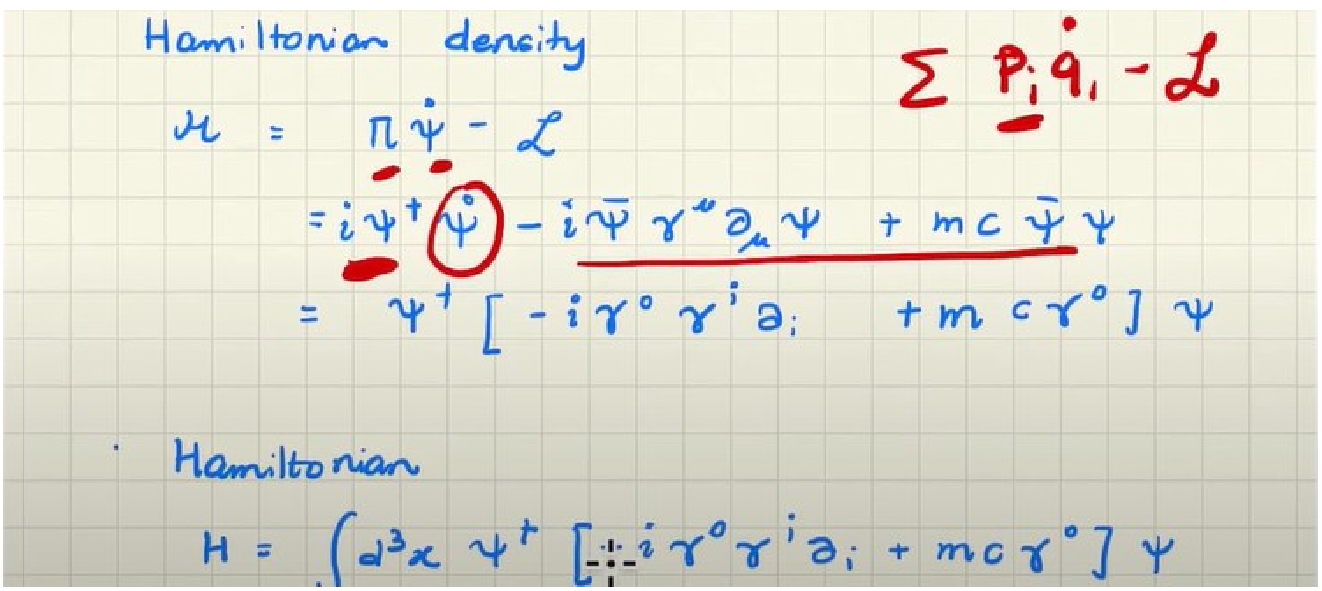


So, this will become true, therefore this will become true, this will also become true and hence the piece which is the kinetic term will become invariant. So, that is what we are going to see with the help of certain examples. So let us work it out and see whether we get the claimed identity or not. So the thing which we are trying to chase is whether S α β γ^μ commutator indeed becomes multiplation of $M_{\alpha\beta}$ and γ^μ. Where the row newth row this decides the row and the column this is newth column rather and new is being summed over along the row you sum new 0 0 1 1 2 2 3 3 and so on. So, let us try to see whether this emerges out or not. So, I am going to think of a low range transformation which is ,perfectly boost like. So, I am going to do a Lorentz transformation which is just a boost along the x-axis. That means all other parameters are 0, only boost along x-axis parameter is non-zero. I am going to choose that parameter ω 0, 1 which is 1 and being anti-symmetric it is also equal to – of ω 1, 0 and all other parameters are 0. Boost along y-axis is not being done, boost along z-axis is not being done, rotation along any of the x, y, z-axis is not being done. So, $\omega_{\mu\nu}$ is equal to 0 for all $_{\mu\nu}$ which are not equal to 0 1. Only ω 0 1 and ω 1 0 are non-zero all other $\omega_{\mu\nu}$ are 0. So, therefore, the B matrix which transforms the spinors which was being written as exponential of i $\omega_{\mu\nu}$ $S^{\mu\nu.}$ This will become equal to exponential of i ω 01 which is 1 and S01 that is all. Now, if that is the case ultimately the claim identity which we are going to look at should have a S01 γ^μ should be equal to M01.$_{\mu,v}$ and γ^v . This is what should emerge out under this Lorentz transformation. So that means I have to choose four$_\mu$'s differently. I can choose$_\mu$ is equal to 0, 1, 2, 3 and correspondingly right hand side will have 0, 1 or 2 or 3. So let us do one example for$_\mu$ is equal to 0. That means I want to evaluate what is the commutator between S and . Result says, looking at the result, it looks like the commutator should emerge out to be M01, 0 new. This should be the claimed result. Let us verify whether we do get this result back or not. So, let us write down the S01 matrix. S01 matrix is half of γ_0 γ_1 because that is the generator. γ_0 is just appearing over here and then let us do this multiplation, opening up the commutator. If I open up the commutator, I will get two terms γ_0, γ_1, γ_0, which is the first term and γ_0, γ_0, γ_1, which is the second term. Now two γ_0s are appearing over here, which will make γ_0 square and we all know γ_0 square is identity. So therefore, the second term will just become – of γ_1, – of γ_1. Look at the first term which is γ_0, γ_1, γ_0, γ_0, γ_1, γ_0. And we also know that γ_1, γ_0 over here is negative of – of γ_0, γ_1. So this is equal to – of γ_0, ,γ_0 γ_1 and again γ_0 γ_0 is identity. So, this first term is also – of γ_1. So, half of – of twice of γ_1 is equivalent to – of γ_1. This is the left hand side. Let us check what do we get on the right hand side.

Right hand side if this identity is true I should get ,So, sorry I am supposed to get a – over here. Look at this previous result. Here when I wrote down the commutator, it became m up to a negative sign change. So, the claim result is that commutator is negative of m. So, that is what we are trying to prove. Now, let us find out this object over here and see whether it is negative of – γ_1 or not. This is the row which is 0, this is the column which is row 0 and new gets summed over. The first entry, second entry, third entry and fourth entry. So m01 along this row has only one non-zero element which is m01.So when I sum over new, new will take value 00112233. Out of these four entries only M01 01 is surviving, M01 01 is surviving because M_{01} matrix is this. So, in this summation only one term is going to survive with value 1 and this will become γ_1. So, indeed it is negative of the commutators. So, S01 γ_0 is indeed negative of $-M_{01}$ 0_v γ^v . So, at least we have verified for this pair 0, 1 and$_\mu$ is equal to 0, it works out. You can do another example, you can take S01 γ_1 here, you will again find out that it is indeed coming out to be true that S01 γ_1 is indeed negative of – n01, 1_v and summation over γ^v . And actually you can keep doing this exercises for different values of α, β and$_\mu$. If you take α is equal to 2, β is equal to 3,$_\mu$ is equal to 0, you will get M_{23} 0_v and summation over$_v$. And similarly if you take 231, you will get 231_v and summation over$_v$. ,You can verify for all of the terms you are indeed getting this structure. So the claim which we are going to prove indeed turns out to be verified and we have not proven but verified this that put together we have a general structure that you take any of the ,$S^{\mu v}$ and you take any of the γ matrix. You will get the$_{\mu v}$ will identify which m we are talking about. The component of the γ will tell me the upper index of that m and the lower index will be summed over. So indeed this structure emerges out which we assumed if that emerges out that means the reverse map which we had talked about will work fine that if the claim identity is true then this commutator over here with a – sign will be replaced by a positive of generator m for vectors and higher commutators square of commutators will be squares of m's and what not ultimately will become exponential of ω times m which is the vector transformation matrix. So, therefore d inverse γ times d gives you ,gives you back λ times γ which was the claimed result which we are trying to obtain. And once that happens we can now see that the kinetic term is also invariant under Lorentz transformation. So, now we have a good Lagrangian.

So, as we can see kinetic term the D inverse γ^μ and B which was appearing now has been written in terms of ,this λ the λ_μ α γ α which is this λ_μ α γ α and already there was a λ inverse$_{\mu\mu}$ which was coming from the transformation of the partial derivative this and this they are just numbers they are not matrices these are component of matrices they can combine and give you a Kronecker δ_μ α because they are the matrices are inverses of each other that means componentsmultiply to each other such that the row of one combines with the column of other and so on gives you a identity matrix that means the component becomes the Kronecker δ unless they are the same they will give you 0 if they are the same they will give you 1.





This is the inverse transformation matrix which we had written. Once the Kronecker δ appears, it will make the new appearing over here to the value α. So ultimately or you can say that it will make α appearing here taking the value new. So ultimately the end result will be this index here and this index here will become same. That is the Einstein summation coefficient. So these two are being summed over and the left hand side was also the same structure. So it is invariant. This has become new new, but new is being summed over, right? Similarly, here new was written, but new was summed over. So, there are four summation term and there are four summation term, which are exactly the same. So, this new here is a fictitious dummy index, which is summed over. You could have written this after the end of the day, new as well. So, therefore, the kinetic term is also invariant, the mass term which we had written is also invariant now and therefore, the whole Lagrangian which now we can write as $\bar{\psi}$ the kinetic term and the mass term together is an invariant Lagrangian under low-inch transformation. Okay, so once we have a valid Lagrangian at our hand, again we will do the similar exercise which we had done for the scalar field, first we will try to go to the phase space of this ,So, remember in this case we have complex field not real field, ψ is a complex, ψ had first of all it is a column matrix, it has ψ 1, ψ 2, ψ 3, ψ 4, which we had collected in terms of ϕ_0 and chi$_0$ if you remember, but anyway. So therefore, $\bar{\psi}$ appearing over here in hides a $\psi^\dagger$ which is not equal to ψ. Because it is not a real scalar field, it is a complexed spinor field. So it has real entries and imaginary entries in the ψ. So either you can treat collection of ψ as a collection of four complex fields ψ 1, ψ 2, ψ 3, ψ 4 ,that has a four real parts and four imaginary parts or equivalently you can think of the whole thing as a four complex ψ and four complex side† so this is what one can do the one can do the business with that you will say that real part on imaginary part are independent fields or you can say that ψ and side† are in independent fields both of them are equivalent. ,Now, if I try to do this, we will take this Lagrangian and first I will try to find out the momenta corresponding to the field ψ. ,Remember there are 8 fields ultimately, real part of ψ 1, real part of ψ 2, real part of ψ 3, real part of ψ 4, imaginary part of ψ 1, imaginary part of ψ 2, imaginary part of ψ 3, imaginary part of ψ 4. Equivalent description are 4 complex field ψ and four complex field $\psi^\dagger$ okay so this is how we can do the business so if we can define either the real part as a four collection and imaginary part of four correction or ψ and $\psi^\dagger$ as two separate fields so first i take the obtain the momentum corresponding to ψ Momentum corresponding to ψ, that means any component of ψ you take, its momentum corresponding to that would be ψ. Okay, ∂ ∂ ψ, this object. That means it has four components, take any of them. ψ 1 dot, it will try to look in the Lagrangian and find out where is ψ 1 dot hiding. So, this ψ and this $\bar{\psi}$ together will generatenumbers . $\bar{\psi}$ will be ψ 1†, ψ 2†. This will become a star after taking† and row will become a column. A column will become a row rather. So, ψ 3 star, ψ 4 star and then there will be a γ_0 which is under† get back to itself. And then this *i* times γ^μ derivatives of ψ. Derivatives of *i* times γ_0 derivatives of ψ will again will be ψ 1 dot, ψ 2 dot, ψ 3 dot, ψ 4 dot. So, you will see after all the multiplation, there will be term which will be ψ 1 star going to ψ 1 dot, ψ 1 star with multiplation of these things in between, giving you identity actually, it will not do anything, *i* can be taken out, γ_0 square is independent. So, therefore, ψ 2 star will get multiplied with ψ 2 dot, ψ 3 star will get multiplied to ψ 3 dot and so on so forth. So that means whenever I take ∂ L upon ∂ ψ 1 dot, I will get *i* times ψ 1 star. Similarly for ψ 2 dot, I will get *i* times ψ 2 star, ψ 3 dot I will get *i* times ψ 3 star and so on. Collectively I can write the, if *I writ*e then down the ψ, ,The end result would be $\psi^\dagger$ because $\psi^\dagger$ is component ψ 1 star, ψ 2 star, ψ 3 star, ψ 4 star. So put 1 here you will pick this one. Put 2 here you will pick the second one. So all the stars appearing over here can collectively be written as a†. So therefore I will have a momenta corresponding to ψ will be obtained as *i* times $\psi^\dagger$. So ψ and $\psi^\dagger$ were supposed to be independent fields as we discussed. However, they happen to be related in the phase space. In the phase space of the theory, they are conjugate to each other. This is momenta of ψ. And in this way I have written nowhere the derivatives of ψ stars are appearing, only the derivatives of ψ are appearing. $\psi^\dagger$ star is not appearing anywhere. So, if I compute the momenta corresponding to ∂ L upon ∂ $\psi^\dagger$ star, I am going to get 0. There is no momenta corresponding to this in the way we have written the Lagrangian. There is another cliched way of writing ,the same thing as a product of the derivative action hitting this term and a total derivative term. So, those things I am not going to discuss in this course because this is not relevant for us. But the way we are writing, the way we are writing the Lagrangian which is consistent with low-range transformation, I am going to get the momenta corresponding to ψ. is $\psi^\dagger$ with *i*. However, the momenta corresponding to $\psi^\dagger$ itself is 0, because there is no $\psi^\dagger$ star appearing in the Lagrangian the way we have written. So therefore, the total Hamiltonian density which we should be obtaining in phase space should be summation over all the P momenta Pi's and all the dynamical variables dot – L. In this case only one set of momenta survives which are the ,$\psi^\dagger$s times i which are the derivatives of ψ. $\psi^\dagger$ dot does not appear, its momenta does not appear, so therefore that pair will not appear in this summation. So, ultimately only ψ its momenta will be there and the Lagrangian which can be written like that. So, you see the Lagrangian over here has become a nice structure in which ,Only thing of interest you can see that the ψ appearing from the first term gets exactly cancelled from the first term of the Lagrangian. So, ψ terms gets wiped out under this computation. Only spatial derivative terms survive. So, this has four terms γ_0, ∂_0, γ_1, ∂_1, γ_2, ∂_2, γ_3, ∂_3 ,out of this γ_0 ∂_0 term exactly cancels this. Because this $\psi\bar{\psi}$ also has a γ_0, so γ_0 γ_0 becomes identity and these two terms exactly cancel each other for$_\mu$ is equal to 0. So only thing which survives ,Not only the three things which survive are the three spatial derivatives. The temporal derivative gets exactly cancelled out, spatial derivatives survive with respective γ and the mass term survives with a positive sign. Because Lagrangian was taken out with a – sign, in the Lagrangian already the mass term was coming with a – sign, so ultimately it becomes plus.

So overall the Hamiltonian density is size†. Spatial derivatives with appropriate γ_0 and γ_i multiplation with a – i factor and the mass term with γ_0 factor and then a ψ there. So, you see this is a column matrix, this is a row matrix, inside there are certain matrices. So, ultimately you are going to get a numbers . The Hamiltonian density has to be a number , Lagrangian density has to be a number . The Hamiltonian density has to be a number , Lagrangian density has to be a number . And once I integrate the Hamiltonian density, ,over all space I will get the total Hamiltonian just like we did for scalar field as well. So, starting with the consistent Lagrangian for chlorine transformation, I am able to generate or write down rather not generate, write down the Hamiltonian for a Dirac system. This does not look like very much like a harmonic oscillator thing which we obtained for the scalar field, but we will soon see that indeed it is hiding a harmonic oscillator structure and therefore we can use the harmonic oscillator quantization techniques which we had learnt previously as well. So, in order to see that let us do certain algebraic manipulation and massaging to the equation and then we will see how the oscillator structure emerges out naturally. ,So, look at the again the Dirac equation.

$$\left(i\gamma^\mu \partial_\mu \psi - m\psi\right) = 0. \quad (\quad)$$

Hitting it with

$$-i\gamma^\alpha \partial_\alpha = i\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_0 + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\partial_i\right]$$

$$-i\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_0 + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\partial_i\right]\begin{pmatrix} \phi_0 \\ \chi_0 \end{pmatrix} e^{\frac{-iEt + \vec{p}\cdot\vec{x}}{\hbar}}$$

$$-i\gamma^\alpha \partial_\alpha \left(i\gamma^\mu \partial_\mu \psi\right) + m\, i\gamma^\alpha \partial_\alpha \psi = 0$$

$$+\gamma^\alpha \gamma^\mu \partial_\alpha \partial_\mu \psi + m^2 \psi = 0$$

$$\frac{1}{2}\{\gamma^\alpha, \gamma^\mu\}\partial_\alpha \partial_\mu \psi + m^2\psi = 0$$

Using the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$

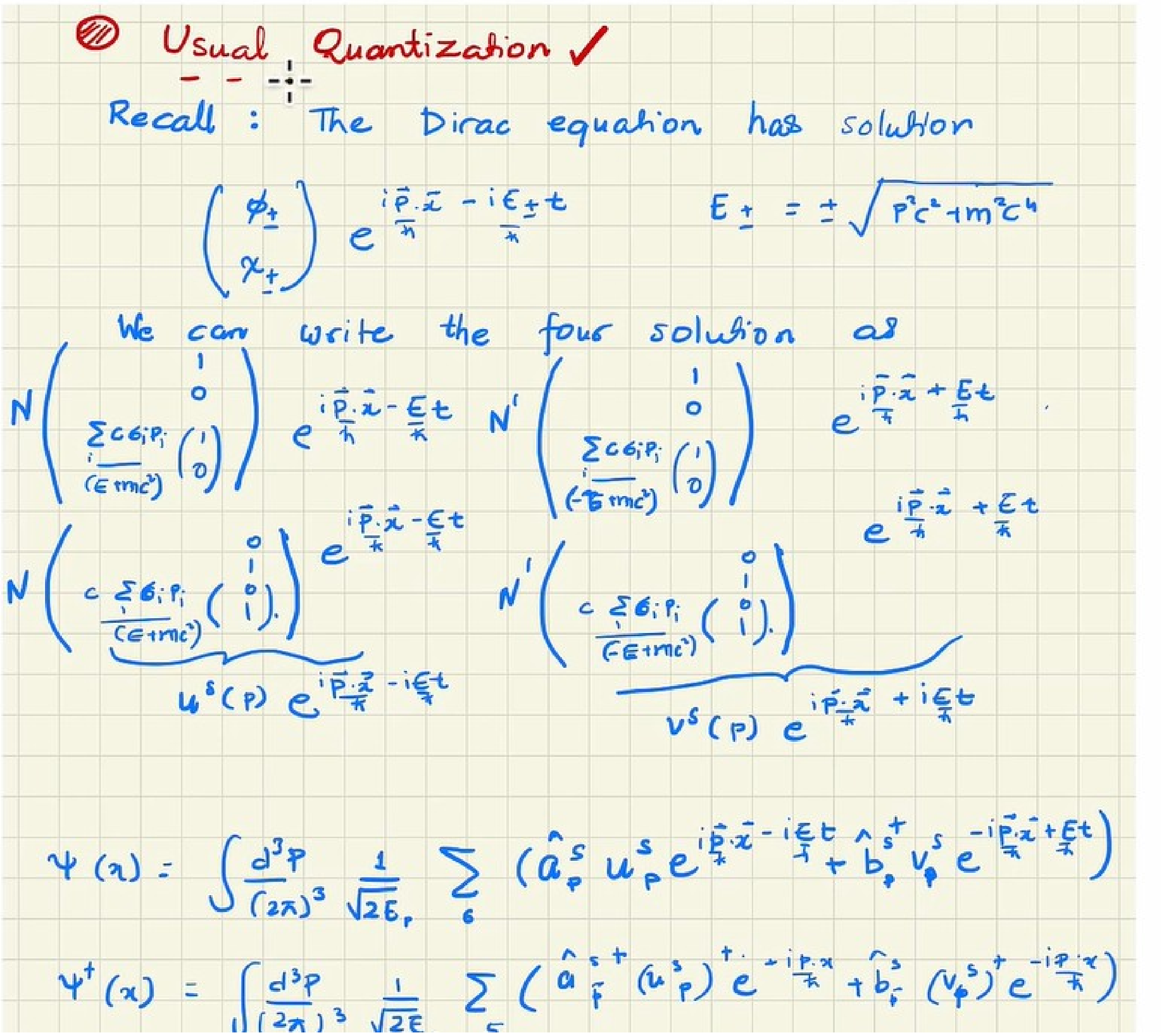
$$\Rightarrow \quad -\eta^{\alpha\mu}\partial_\alpha \partial_\mu \psi + m^2 \psi = 0$$

Thus solution of Dirac eqn. also satisfy the Klein Gordon (Harmonic oscillator) eqn.

▷ We can apply the usual (oscillator) quantization

The variation of the Lagrangian with respect to $\psi^\dagger$ or $\bar{\psi}$ is going to give me this Dirac equation. Here now on I am putting c is equal to 1 and $\hbar$ is equal to 1 because I have to maintain the c and $\hbar$s at so many places. So this is the natural units in which things are written. You know that whenever I am writing m here, it should have been mc and somewhere $\hbar$ should also be appearing because of the derivative version of the momenta which I had written. But anyway, right now I am just putting them to one just for convenience. Looking at their mass dimension, we will be able to settle where c is appearing, where $\hbar$ and where their combinations are appearing. So this gives us a nice structure to look at.$_\mu$ is getting summed over in the Dirac equation. Remember in the Hamiltonian, the 0th part has been cancelled. The summation is only over i-th part. But in the equations of motion, all the 4μs are appearing. So, let us look at the equations of motion. What I do, I take the whole equation which is a matrix equationmultiplying a 4 cross 1 matrix. So, this γ^μ is a 4 cross 1 matrix, this ψ is a, this γ^μ is rather 4 cross 4 matrix, this ψ is a 4 cross 1 matrix.This m is hiding a identity in between. So, ultimately it is again 4 cross 4multiplying of 4 cross 1. So, overall structure of this equation is some matrix which is a combination of all the derivatives and γ^μ put together acting on ψ and some m multiplied with identity acting on ψ. So, ultimately I will get a 4 cross 4 dimensional matrixmultiplying a 4 cross 1 dimensional vector and that should be 0. This is the equation telling you. What I do? Imultiply the whole equation with another matrix. Which another matrix? I take this $- i \gamma \alpha \partial_\alpha$. Again α is being summed over. That means I will take $\gamma_0 \partial_0 + \gamma_1 \partial_1 \gamma_2 \partial_2 \gamma_3 \partial_3$. So I take this matrix which is in its expanded form i ,which is coming from here, γ_0 the derivative with respect to time, γ_1 which will be σ_1, σ_1, $- \sigma_1$ here, $\delta(1)$, γ_2 which will be σ_2, $- \sigma_2$ and γ_2 or $\delta(2,)$ γ_3 which will be σ_3, $- \sigma_3$ and δ^3. Collectively I am writing σ_i, $- \sigma_i$ and δ_i. Again i has to be summed over. This is just a notational version of the full expansion. So, that matrix I take andmultiply this matrix equation of motion. So, I take the whole thing the equation of motionmultiply it through this matrix. Ultimately this solution which I am going tomultiply it through will be some will be matrix multiplation this matrixmultiplying this equation this whole thing ,This whole thing equation of motion is some 4 cross 1 dimensional matrix at the end of the day. Because 4 cross 4multiplying a 4 cross 1 should give me a 4 cross 1 which should be a 0 matrix. But ultimately this left multiplation is some 4 cross 4multiplying a 4 cross 1 already and to that I am going tomultiply this whole object which is this. So this is the mathematical formal structure. So let us see it cleanly what do I get. So I have this equation at hand, Imultiply this equation with this object and let us see what are the different terms I am going to get. The first term becomes this. An extra multiplation with $\gamma \alpha \partial_\alpha$ and the second term also becomes m times multiplation with $- i \gamma \alpha \partial_\alpha$. So – term which was already appearing in the equation of motion becomes a plus over here and ultimately I have these two terms, which is fine. This i can be pulled out,multiply this $- i$ becomes plus 1 and then I have a $\gamma \alpha$ and the derivative ,The derivative will go across because γ^μ matrices, none of the γ matrices have any space time dependence. So the derivatives will just go through and I can collect all the derivatives on one side and all the γ matrices on the other side. But in this order matters. First I have a $\gamma \alpha$ and then γ^μ. I cannot write it as γ^μ, $\gamma \alpha$ whichever way I want because these are matrices. Two matrices A and B cannot be written B and A unless they commute. And *I know* $\gamma \alpha$, $\gamma\beta$ do not commute. So I will be attentive to the fact and I will write the order right. That $\gamma \alpha$ first appears and then γ^μ appears. The derivatives can be written in any way because they are not matrices, they are just operational partial derivatives and partial derivatives commute with each other. So, that is fine. The second thing which is appearing over here, see i times $\gamma \alpha \partial_\alpha \psi$ should be equal to m of ψ already ,because of the equations of motion i times $\gamma^\mu \partial_\mu \psi$ where$_\mu$ is summed over is equal to m ψ. So, therefore, the term appearing over here which is already $i \gamma$ some index δ some index ψ and that index is being summed over should be equal to m ψ. So, using this equation of motion the second term will become m square ψ. While the first term I have a $\gamma \alpha \gamma^\mu$ and the two partial derivatives. Now you can see that what I can do this α is also summed over$_\mu$ is also summed over α are going to take value 0, 1, 2, 3$_\mu$ is also going to take value 0, 1, 2, 3. So that means I can write it in a nice way that it is $\gamma \alpha \gamma^\mu \partial_\alpha \partial_\mu$,plus one half of this the same thing again $\gamma \alpha \gamma^\mu\partial_\alpha \partial_\mu$ the same thing I can write right where α and the$_\mu$s are being summed over now in the second term I can do a relabeling so α is being summed over $\gamma \alpha \gamma^\mu$ and $\partial_\alpha \partial_\mu$ is there. So what I can see, both α and$_\mu$ are being summed over. So it does not matter, this has to take value 0, 1, 2, 3 and$_\mu$ also has to take value 0, 1, 2, 3. So it does not matter whatever I call what. So I can call for α, replace α with$_\mu$ and$_\mu$ with α. ,So this will become$_\mu$ and this will become α. This will become$_\mu$ and this will become α. However, the partial derivatives as we discussed commute with each other. So I can write it back as ∂_α ∂_μ. First I converted$_\mu$ into α then the roles of$_\mu$ and α get flipped off. But this first two γ matrices cannot be brought back in the previous form because they do not commute but partial derivatives commute. So I can write down ∂_α ∂_μ ∂_μ ∂_α which was appearing again back as ∂_α ∂_μ. So therefore I will get a structure half times γ α γ^μ. And then γ^μ γ α both coming with $\partial_\alpha\partial_\mu$. Which is the anti commutator of γ α γ^μ. So this see this thing with clarity. First what I did I just wrote it into half of twice of its pieces. And in one of the pieces I made use of the fact that this partial derivatives commute. Therefore, I can write the whole term as this. You can verify the whole term appearing over here is the same as this. Okay, you can just open it up γ α γ^μ plus γ^μ γ α and do the summation. So that means here α and$_\mu$ summation is implied. And here also α and$_\mu$ summation is implied. Do these things, obtain what are the terms you are getting.

You will see that both the terms are the same. So therefore I can write the term which is appearing over here is as γ α γ^μ anticommutator and ∂_α ∂_μ of the ψ. Now I also know that the anticommutator of γ^μ γ^ν that means γ α γ^μ is – twice of the Minkowski metric $\eta_{\mu\nu}$ that means this can be written as a ,– of η α_μ and α and$_\mu$ are being summed over. So, that means summation over α_μ is implied. And now you see this is nothing but the Klein-Gordon equation. So, therefore the Dirac equation which we obtained as a solution for Dirac equation of motion also satisfies the Klein-Gordon equation. This is not a mystery because ,The second order Klein-Gordon equation is the unique solution which is Lorentz invariant. So therefore, its second derivative version should better satisfy the Klein-Gordon equation as well. So therefore, the solution which we have obtained for the Dirac equation is a solution to a harmonic oscillator structure as well. So ψ is like a harmonic oscillator again as well. So therefore we can employ, right now it is not a harmonic oscillator, it is Klein-Gordon, but going to Fourier basis it will become harmonic oscillator equation for different k basis as we had done for harmonic, the scalar field. So therefore this can be converted back into harmonic oscillator like structure and therefore the usual quantization scheme which we adopt for a scalar field which is quantization of oscillators in Fourier space can be adopted here as well. So, *I know* the solution of harmonic oscillators. I can write down those as the operators and then quantize the full field and that is what we can do for the spinor field as well. In the next class, we will try to see how to do this for spinor field. The usual way looks very *much* settled that we can do it the way scalar field was done because both of them satisfy the same equation and we know it has our richer matrix structure scalar field was just a numbers this time it is we have 4 plus 1 matrix that means it has different components all the components satisfy the Klein-Gordon equation so therefore it is a collection of more oscillators spinors is collection of more oscillator so we can quantize it like we quantize an oscillator and that is what we will do in the next case



So, in the previous discussion session, we learned that the equations of motion for Dirac field, which was the Dirac's equations of motion, can be converted into a harmonic oscillator kind of equation of motion. Rather than a harmonic oscillator, we obtained the Klein-Gordon equation, this equation which we obtained originating from the Dirac equation of motion. What we did? We acted upon on the Dirac's equations of motion with *i* times, – *i* times $\gamma^\mu\,\partial_\mu$, and ultimately the Dirac equation therefore became the Klein-Gordon equation which we know how to quantize, courtesy the scalar fields quantization which we have learned in the previous sessions. In this case, only thing which we have is that this is not a real scalar field first of all. It is a 4 cross 1 column vector which comes with a 4 component thing, ψ 1, ψ 2, ψ 3, ψ 4. Remember, the ψ here is a representative of a collection of a 4 cross 1 vector which has 4 complex quantities at its disposal. So, we have this structure which we obtain for Klein-Gordon equation. So, now we want to know that the usual way of quantizing the scalar field in the class, in the discussion session which we had done, we discussed about the quantization of the real scalar field, but anyway a complex scalar field also can be quantized along the similar lines, which we have not done for in the discussion session, but you will see that it is not *much* different from the real scalar field. In the real scalar field, remember ,the coefficient of e to the e^{ikx} and e^{-ikx} where complex Hermitian conjugate of each other in order to get you a real ϕ. So, this d^3k gave rise to a real ϕ for this and that to become Hermitian conjugate to each other. You will do obtain ,a real ϕ but suppose you drop the requirement of having a real ϕ then there is no reason that these quantities and these quantities should be hermitian conjugate to each other in principle I can put a different b of k over here and then get my business done initially remember we had started doing the business like that and then we imposed the reality condition So, this time I am talking about Dirac's spinor and which involves four of the fields, none of them are necessarily real. So, I drop the demand that $\hat{a}_k\, e^{ikx}$ and $\hat{a}_k^\dagger\, e^{-ikx}$ should be the structure of its Fourier domain. So, let us go forward and try to see what kind of a structure we can impose and therefore, what do we learn from the standard quantization procedure. So okay so we do first the usual quantization and surprisingly or unfortunately for us we will see that this usual quantization scheme which was very well working for a harmonic oscillator structure and the scalar field despite appealing it leads us to some unphysical conclusions which we will see and therefore we have to tweak the usual quantization scheme in order to get a meaningful physics out. So let us see what is the unphysicality inherent in the usual quantization procedure. So again we are looking for a solution of this equation first classically *I know* these are the solutions while we when we were discussing the dirac equation of motion in the relativistic quantum mechanics domain there also the same story emerged that the the particles are supposed to satisfy the dirac equation and hence the klein gordon equation as well this time as well we are the same structure *I know* Despite it being a 4 cross 1 dimensional vector, its space-time dependent part satisfies the same equation which a scalar field does.

Because all the derivatives are spatial derivatives or temporal derivatives. So, therefore, its space-time dependence cannot be different from the space-time dependence of the scalar field. So, therefore, the plane wave structure should be inherent in that as well. So, *I know* the space time dependency should be through $e^{ipx/\hbar - iE+}$ – because energy comes in two signatures plus or –. This was true for even scalar field. So, this e, this e was not the energy but let us say frequency, $E/\hbar$ was the frequency, positive frequency and negative frequency which was appearing even for scalar field. What is the total energy would be known from the Hamiltonian only. In the quantum field theoretic picture but plane wave structure comes with positive frequency and negative frequency and $e^{ip\dot{x}}$ $e^{-ip\dot{x}}$ both of them are captured by a generic p vector because p vector can be positive or negative so this is the solution of the dirac equation okay. ,Now, so this part which is the finer part of this, this is the new feature which has arrived because this is not a scalar field. Scalar field part was this *much*. ,Since it is not a scalar field, a new part in terms of a 4 cross 1 dimensional column vector has appeared. And we know again in order to satisfy the Dirac equation, this column vector has to maintain a particular structure. For instance, it could be coming with definite energy, positive or negative. So it has positive energy or positive frequency solution and negative frequency solutions. As well as we realized when we were discussing about the spinal structure that it has extra degree of freedom or extra operator which quantifies that which is the spin or the helicity of that. So either it can come with one helicity spin half or spin – half with the same energy or spin half or spin – half with negative energy or frequency. So we list down both of them. , So these two, this one and this one are positive energy or positive frequency solutions. So – E $/\hbar$ appears in the exponential.

III Usual Quantization ✓

Recall : The Dirac equation has solution

$$\begin{pmatrix} \phi_\pm \\ \chi_\pm \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iE_\pm t}{\hbar}} \qquad E_\pm = \pm\sqrt{p^2c^2 + m^2c^4}$$

We can write the four solution as

$$N\begin{pmatrix} 1 \\ 0 \\ \frac{\sum_i c\sigma_i p_i}{(E+mc^2)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{Et}{\hbar}} \qquad N'\begin{pmatrix} 1 \\ 0 \\ \frac{\sum_i c\sigma_i p_i}{(-E+mc^2)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}}$$

$$N\begin{pmatrix} 0 \\ 1 \\ \frac{c\sum_i \sigma_i p_i}{(E+mc^2)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{Et}{\hbar}} \qquad N'\begin{pmatrix} 0 \\ 1 \\ \frac{c\sum_i \sigma_i p_i}{(-E+mc^2)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}}$$

$$u^s(p)\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iEt}{\hbar}} \qquad v^s(p)\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{iEt}{\hbar}}$$

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_s \left(\hat{a}^s_p u^s_p e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar} - \frac{iEt}{\hbar}} + \hat{b}^{s\dagger}_p v^s_p e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar} + \frac{Et}{\hbar}} \right)$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\hat{a}^{s\dagger}_p (u^s_p)^\dagger e^{-\frac{ip\cdot x}{\hbar}} + \hat{b}^s_p (v^s_p)^\dagger e^{-\frac{ip\cdot x}{\hbar}} \right)$$

For $[\psi(\vec{x}), \psi(\vec{x}')] = \delta^3(\vec{x} - \vec{x}')\, \mathbb{1}_{4\times 4}$

Which leads to :

$$[\hat{a}^s_{\bar{p}}, \hat{a}^{s'\dagger}_{\bar{p}'}] = \delta_{ss'}\,\delta(\bar{p}-\bar{p}')$$

$$[\hat{a}^s_{\bar{p}}, \hat{a}^s_{\bar{p}'}] = 0 = [\hat{a}^{s\dagger}_{\bar{p}}, \hat{a}^{s'\dagger}_{\bar{p}'}]$$

and

$$[\hat{b}^s_{p}, \hat{b}^{s'\dagger}_{p'}] = \delta_{ss'}\,\delta(\bar{p}-\bar{p}')$$

$$[\hat{b}^s_{p}, \hat{b}^{s'}_{p'}] = 0 = [\hat{b}^{s\dagger}_{p}, \hat{b}^{s'\dagger}_{p'}]$$

So these are positive frequency solutions because if you work this $i\partial\partial_t$ you will get a positive numbers out as a eigen function and the spinor part was remember was something called a ϕ_0 ,and a chi_0 and there was a relation between ϕ_0 and chi_0 which was related by $\sigma_i p_i$ summation over all the poly matrices and all the momentum components in the helicity operator and divided by E_+/mc^2 . So this was the structure. Go back to your Dirac equations note when we were trying to solve the Dirac equation. There this solution emerged out. In order to satisfy the Dirac equation as well as the Klein-Gordon equation, ϕ_0 and chi_0 in the doublet ,The doublet ϕ_0 and chi_0 cannot be arbitrary. They have to be related like this. And further if I demand that ϕ_0 and chi_0 individually have to be eigen functions for eigen states for the helicity operator. It can come with 1010 or 0101. 1010 structure was for plus $\hbar/2$ helicity operator and this one is $-\ \hbar/2$. Or call it loosely call the spin value. The spin half, the spin – half or as helicity Sz plus half Sz – half with positive energy. For the negative energy exponential comes with a wrong sign plus $E\ /\hbar$t, this time it will also be plus $E/\hbar t$, this is plus $\hbar/2$ in the spin and this is $-\ \hbar/2$ in the spin. The factor which will changes in the denominator of these two things as well, e goes to – e. So in the negative energy solution part, negative frequency solution part, you will get this $E + mc^2$ or E $-E$ $+mc^2$ depending upon which sector you are talking about, positive frequency or negative frequency. So keep a note that while I am calling it positive energy, just looking at the symbol E, energy ultimately will get decided by the Hamiltonian. This is field theory. So the parameter appearing over here is not ,has been not been shown to be the eigen function of or the eigen value of the Hamiltonian operator. So, I am loosely calling it energy in the spirit of the Schrodinger equation structure which we used to call it. So, this is really positive frequency and negative frequency solutions which we are talking about. So, therefore, if I go below ,So I am going to collect all the positive frequency solutions as symbol U. S will tell me about what spin value I am going to talk about. Spin can take two values, plus half or – half. Similarly, there are two negative frequency solutions, Vs, $Vs_{+\ 1/2}$ and $Vs_{-1/2}$,. Their space time dependence part are plane waves, one with +iEt and one with –iEt. So these are the four set of solution, two in the spin half sector, positive energy, spin half and spin – half and two in the negative frequency sector, spin half and – half. And in the spirit of harmonic oscillator structure with the solution, now I can write down all the possible mode function. Remember these things do satisfy the Klein-Gordon equation upstairs. The Klein-Gordon equations have now four solutions. ,positive frequency solution, positive spin solution, positive frequency solution, negative spin solution, negative frequency solution, negative spin solution, negative frequency solution, positive spin solution. So all these four are your mode functions. So these are the mode functions I am going to write, positive frequency and negative frequency. And I am going to associate operators just like harmonic oscillators mode function. Remember the same discussion. The mode functions of harmonic oscillators came with a and $a^\dagger$ with u star. I had to add $a^\dagger$ because I wanted the position operator to be Hermitian. This time I am not bounded to obtain that because the field which we are describing is not necessarily a real field. So in principle I

can write any other operator. So let me write it as a BS†. There is no relation of priory between A and B. There are four different operators for positive frequencies and different s. And similarly, there are, sorry, so there are two different operators for positive frequency with two different spins. And similarly, for negative frequencies, there are two different spins and two different operators. So, overall, I have a AP spin half, AP spin – half, BP spin half, BP spin – half.
This is the structure I have obtained you can check that this is still a solution of a klein gordon equation or the harmonic oscillator in the fourier space okay so far so good I can obtain the side† ,the side† which is Hermitian conjugate of this by just taking the conjugate of this operator. So all this A will become† B which was already I had written in $a^{\dagger}$ form this will become just B. I had written it in the spirit of the harmonic oscillator one thing comes with A and one thing comes with $a^{\dagger}$ you could have started with A and B as well but just making connection more clear with the harmonic oscillator if b becomes a it becomes a real scale field so† structure I have put in by hand there is no necessity that it has to be† over here and not the b itself but if you make this choice then your side† becomes† here and no† here, ,the course that this usp remember this usp is not a numbers now it is a column vector one of the column vectors this or that depending upon spin half this is this column vector or spin – half this is this column vector so when i take the† it will become a row vector so therefore it is not a star but a† so therefore I will get a usp† and this thing was b v sp which becomes $a^{\dagger}$ as well so therefore I have a more robust or more mathematically rich structure for the Dirac field because in addition to the oscillator operators, I have mode functions which are also vectors, row vectors or column vector depending upon whether you are talking about ψ or ψ divergence.
Okay, so now we know that ψ and $\psi^{\dagger}$ which we have obtained are not just independent quantities anymore in the phase space but i times $\psi^{\dagger}$ is supposed to be the momenta conjugate to ψ. So ψ conjugate momenta is pi which should be i times $\psi^{\dagger}$ and *I know* that the $\psi(x)$ and pi of x' should satisfy i times δ of $x - x'$. So therefore, ,There should be a non-trivial commutation relation between ψ and $\psi^{\dagger}$. So, ψ and $\psi^{\dagger}$, so here should be $a^{\dagger}$, should therefore be δ functions up to identity 4 cross 4. Why this identity 4 cross 4?
Because all the times we are going to talk about this fields not as a collection of a single object operator, but a field coming with a structure commensurate with the 4 cross 1 vectors which are appearing in the mode functions as well. So therefore, this is the structure which we are going to look for. Okay, so this will be ensured. So if *I writ*e my ψ like this and if *I writ*e my $\psi^{\dagger}$ like this and I demand ψ, $\psi^{\dagger}$ should be δ function and use the fact that $d^3pe^{ip(\dot{x}-x')}$) should be a δ function, should be a 2 pi cube times $\delta^3(x - x')$. So, you see when *I writ*e ψ and $\psi^{\dagger}$ side by side, I will get two integrals, one from d^3p from the ψ and another d^3p' integral let us say from $\psi^{\dagger}$. And collection of these two d^3p should at the end of the day ,First it should get converted into a single integral over d^3p and then after the end of the day I should get a δ function out of it. You can see ,maintaining what is the up and what is the up† what should you should get and so on so forth you will see that it would be realized that this commutation relations between ψ and $\psi^{\dagger}$ will be realized only when a and $a^{\dagger}$ satisfy the usual commutation relation that for different spin s and s† they should not be talking to each other spin numbers one has its own oscillator structure, spin numbers two has its own oscillator structure. So there are like two different oscillators. So one oscillator's A operator does not talk to another operator's A†, it talks to only itself. So first spin should be the same and then the δ function should come about. If the spins are different, that means the oscillators are different. So they should not be commuting, they should not be non-commuting, they should be commuting rather. So, spin δ function ensures that we are talking about the same. And similarly for space time dependent P also, they should be the same.
Otherwise, we are not talking about the same oscillator and therefore the operators would commute. Similarly, different A's should also commute depending whether or not P and P' are same or not and S and S' are same or not. Two different A's are supposed to always commute, two different A diagrams are always supposed to commute and the same story goes for the second set of oscillators of B. So, B's are the oscillators with let us say negative frequency, but anyway that is all right ,and they have also the

same instruction in order to get the good commutation relations and using the fact that the good commutation relation will come only from the converting the double integral over p into a single integral over p with this function in the hand. I should impose this kind of commutator structure on A and B, which is usual commutator relation between oscillators. So, this is nothing mysterious. Only thing is that we started with an oscillator kind of equation. It has more than one component. Therefore, we are getting more than one operator set. But all of the sets are talking about oscillators, since ψ talks about not one operator, but really eight operators. Operators in the sense of four complex entries. So therefore this set of operators are also different and you will have this AS for plus half – half, $a^{\dagger}$ plus half – half, BS plus half – half and BS† plus half – half. So this is the structure which emerges out. This is mathematically slightly more rich but ,nothing conceptually very different apart from the fact that we are not talking about a real scalar field but a complex oscillator kind of thing further you need to impose when you do this commutator kind of a structure you will see that this usp and this usp† will hit each other so these are vectors co-vectors hitting each other or column vector row vector hitting each other. And therefore, we should know what happens when a column vector hits a row vector. So, there should be, if I look for just the structure we had, this was the US set. The US set was here and the VS set was over here. They both come with their individual normalization thing. I do not know how to normalize this vector here. ,I have not told how to normalize this vector here. So, some normalization function should be there. It so happens for it should be independent of spin.

Further $u_s^{\dagger}(\vec{p})\cdot u_{s'}(\vec{p}) = \frac{2\delta_{ss'}E}{(E+mc^2)}|N|^2$

$v_s^{\dagger}(\vec{p})\cdot v_{s'}(\vec{p}) = \frac{-2\delta_{ss'}E\,|N'|^2}{(mc^2-E)}$

We can normalize these

$$\bar{u}_s(\vec{p})\cdot u_{s'}(\vec{p}) = \frac{2\delta_{ss'}mc^2}{(E+mc^2)}|N|^2$$

$$v_s^{\dagger}(\vec{p})\cdot u_{s'}(-\vec{p}) = 0$$

Further, $(-i\gamma^i\partial_i + m)\psi = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\hat{a}_p^s(-\gamma^i p_i + m)u_p^s e^{ip\cdot x} + \hat{b}_p^{s\dagger}(\gamma^i p_i + m)v_p^s e^{-ip\cdot x}\right]$

Using Dirac equation

$$i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi = 0$$

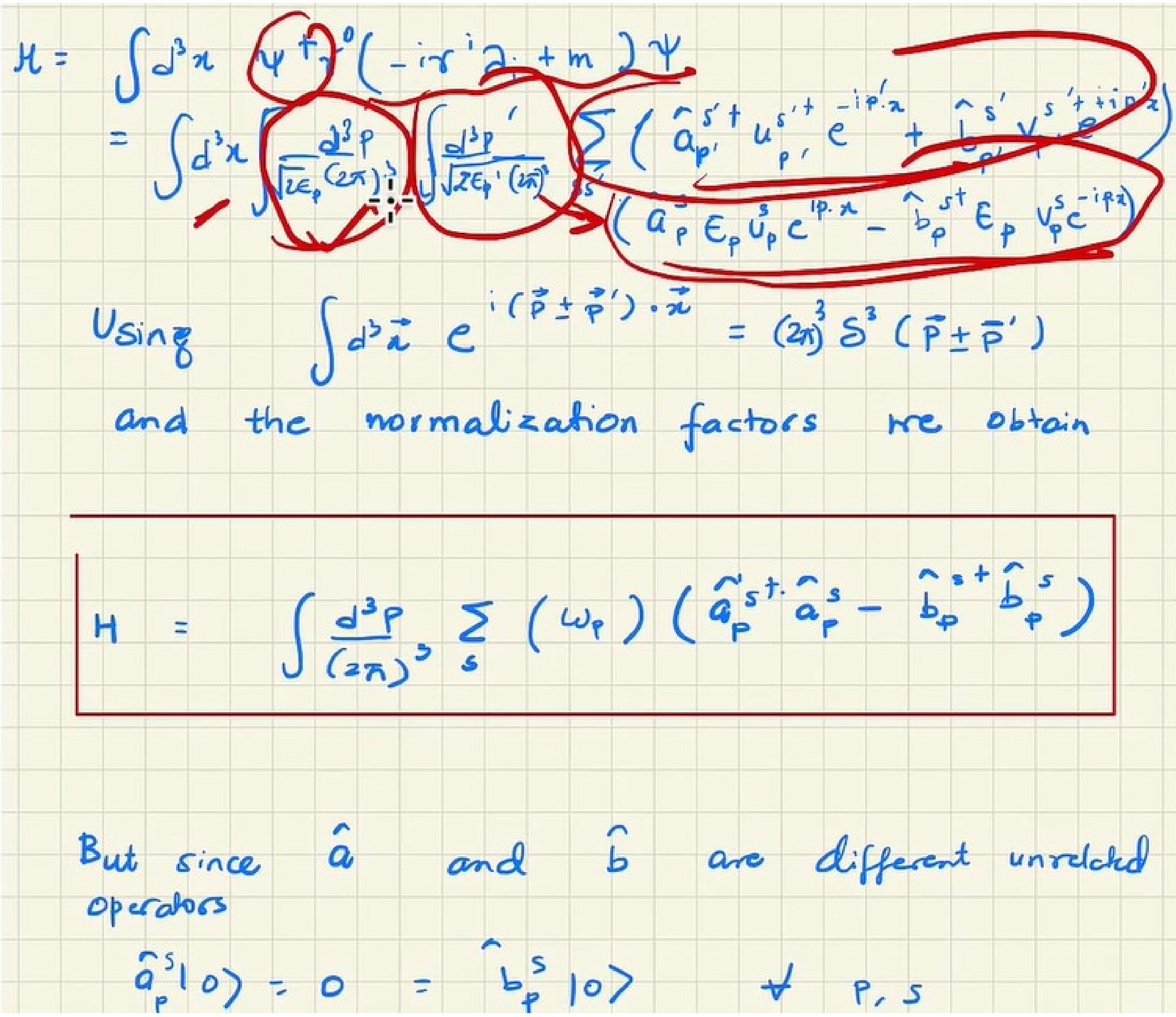
$$-i\gamma^i\partial_i\psi + m\psi = i\gamma^0\partial_0\psi$$

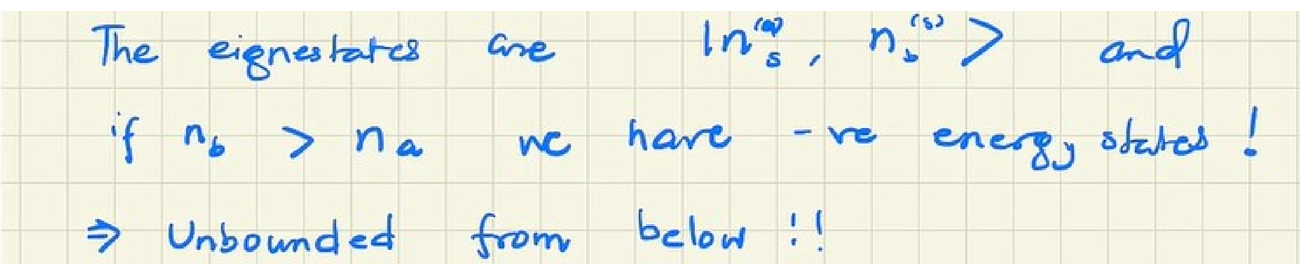
$$\therefore\ (-i\gamma^i\partial_i + m)\psi = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_s\left(\hat{a}_p^s i\gamma^0\partial_0 u_p^s e^{ip\cdot x} + \hat{b}_p^{s\dagger} i\gamma^0\partial_0 v_p^s e^{-ip\cdot x}\right)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\hat{a}_p^s\gamma^0 E_p u_p^s e^{ip\cdot x} - \hat{b}_p^{s\dagger}\gamma^0 E_p v_p^s e^{-ip\cdot x}\right)$$

This is easy to verify that the spin half and spin – half do not come with different normalization and similarly for the negative frequency sector as well. But there is no a priori region that for negative frequency and positive frequency as well you should have the same normalization. So, I am maintaining one normalization in the positive frequency sector and another normalization in negative frequency sector. And in order to obtain this set of relation there should be a good normalization which should be used and that good normalization is also useful for getting a more clear story later on as well so this is an exercise for you what kind of n one should choose in order to get This in along with this commutation relation getting this commutator between ψ and $\psi^\dagger$ what should be the n this should be fairly easy exercise impose this in your commutators and find out what are the n's and n's okay that would also become a clear in a minute when we do another computation with us and vs so now. ,We have this relations in hand. We have commutator structure in hand. So let us go ahead and operate things on this. ,So first I will want to obtain this operator which is all the spatial derivatives coming with a γ factor acting on ψ. Why do I need this operator? For that let us go back to the Hamiltonian which we had obtained for the Dirac field. Remember in the dirac fields Hamiltonian derivation the temporal derivative got exactly cancelled out and i we were left with only the spatial derivative with appropriate γ_i is multiplied to them so you see Hamiltonian is made up from there is a γ_0 terms in the first term and the second term both so I can pull that out so i would have a $\psi^\dagger$ and γ_0 and i should have a – i γ_i ∂ i and then m times i c is equal to one unit I am using so this operator's action on ψ should be known in order to get the Hamiltonian so that is what I am going to do i should be do taking this operator hitting it on the ψ and try to see what do i get so when i take this and hit on the ψ there are γ part which is matrix and the derivatives part which are action on space time. ,Space only, this is not time, this is spatial derivative. So spatial derivatives are only in the e^{ipx} part and the vector part of the matrix is only in the us. This is a column vector. So this operator and these operators together, they do not see anything else. They go right across the integration up to us. One part is behaving like vector to be hit upon by γ_i's and one part is behaving like a function to be hit upon by δx, δx_i. Similarly for the second part as well the operator will go inside and hit that as well okay so far so good but there is another uh another easier way a convenient way of obtaining this action of this operator you can do this exercise also you can find out what happens after the end of the day but a easier way is this use the dirac equation *I know* the dirac equation is i $\gamma^\mu\partial_\mu\psi$ not $\gamma_i\partial_i$ which is required over here. So, I will write it like this. So, therefore I know the part which I am interested about ,is negative of – $i\gamma_0\partial_0\psi$ so that means the part I am interested over here is just this $i\gamma_0\partial_0\psi$ so all this action of all this operator is equivalent to action of $i\gamma_0$ times the temporal derivative so why not do that why to take this operator three derivatives and three matrix multiplation better one derivative and one matrix multiplation. ,So *δ(0)* will hit only the exponential part because the temporal dependence is only there. And similarly in the second part also the derivative will hit only the exponential part. This was a positive frequency thing. So it will give me an E_p out. And this was a negative frequency thing. So it will give me a – E_p out. So there will be a sign. – sign will come between the two terms. Previously, there were no sign. They were added with a plus. After the derivative action or equivalently after the action of this whole operator, I will get a negative sign in between. And a γ_0, which I am still to act. But wait, this is the structure. But I want to know the Hamiltonian. Hamiltonian involved, apart from the operator which we have obtained, ,Another γ_0 shouldmultiply the whole operator and then a $\psi^\dagger$ should be multiplied and d^3x integration should be done. So that is what I am aiming to do.

I have obtained the action of this operator part which is this. I willmultiply a γ_0 to the whole thing. I willmultiply a $\psi^\dagger$ to the whole thing and I will do the d^3x integration. So this is what I am supposed to do and that will give rise to the Hamiltonian. So this operator ,which gives rise to a negative sign difference between these two times the d^3p integrals which is this. And the $\psi^\dagger$ which is this and this d^3p integral and γ_0 when Imultiply a γ_0 you see in this term both the terms we are having a γ_0 here and a γ_0 here. When Imultiply a γ_0 it becomes a γ_0 square which is identity. So I get rid of γ_0 in the Hamiltonian immediately.





,All I am left to do is justmultiply the $\psi^\dagger$ and take out the γ node. So, the $\psi^\dagger$ multiplation will give me this and a 1 d^3p integrals and the action of this whole operator was this term with a sign difference with this d^3p another integral and at the end of the day I have to do the d^3x integral to obtain the full Hamiltonian. Now you see the d^3x integral will just see for the spatial dependency. Spatial dependency is coming only in the exponentials. Okay. And then you can use the fact you can have a ,special integrals of the exponential give rise to $\delta^+(p + p')$ or $\delta(p - p)'$ depending upon whether you combine this with this you combine this with that or this the second term with the first of the next term and the second term of the second of the next term so you will get various combinations of exponential and all

of them doing this special integral will give rise to one of the δ functions use that put it in the δ function and do one of the momentum integral. So δ function will get rid of one of the momentum integral and then you will have just simple combinations of AP multiplied to this, AP multiplied to that, BP ,multiplied to this and BP multiplied to that. You will see the cross term will vanish under this.
There will be no term which this A and B ,multiplied versus this and this ,multiplied. Once you make use of this δ function set integration all this cross term will get cancelled all it will survive will be this term surviving with this and this term surviving with that. So as a simple exercise, you will get this structure, AS† AS and BS† BS. S takes plus half – half value and P is the integral you will be left with. One of the integration of the momentum has been taken care by δ function, but you will be left with one of the integrations at the hand. Now let us stare at the Hamiltonian, what we have. The Hamiltonian what we have is almost like the harmonic oscillator Hamiltonian, which this believed as a numbers operator and this is the frequency they come up with. However, this time we have a collection of two set of oscillator, positive frequency and negative frequency. This is set of positive frequency, A's are the positive frequency operators and B's are the negative frequency operators. So, they do come with positive frequency and negative frequency and ultimately what is happening is that we are counting all the positive frequency oscillators and all the negative frequency oscillator and combining them with their frequency weightage to get the Hamiltonian. Compare with the structure which we had for scalar field. We had just AP† AP and a plus half of ,$\delta(0)$. That $\delta(0)$ has gone away and this is being replaced by the negative frequency oscillators. But they are oscillators after all because thus the spinal field has more set of fields inside this is not one field but a collection of various fields so therefore the Hamiltonian you are going to get is also collection of various oscillators it so happens that some of the oscillators come with negative frequency because we did not demand the hermeticity or anything for the field the cost we have to pay is that you have to add up the negative frequency.
Remember, why did we not demand the hermeticity? Because we saw that they have to be complex numbers, they have to be operator in 4 cross 1 space and so on. All these things when we built our first order equations of motion story, there was inbuilt into that this can be realized only with the complex fields. It cannot be realized with the scalar fields which are the real quantities. So, in order to have a first order equation, the cost you have to pay is to come up in the Hamiltonian with a negative frequency part. But they are oscillators and they do not talk to each other. Commutators between A and B are zeros. So this is one set of oscillators and these are second set of oscillators. And we can assume that there is a ground state again or the vacuum state let us say which is annihilated by both this ground state is annihilated so this is like two oscillators joint ground state will be oscillator first ground state tensor oscillator two's ground state and therefore this combined ground state will be annihilated by either of the operators the same thing is happening here the true ground state which is ground state of all p's. So, remember in the scalar field part the ground state was supposed to be ground state of all momentum vacuum. This time it has not only the momentum vacuum, this has spin vacuum as well as negative frequency vacuums. The full vacuum is vacuum for all oscillator of a positive frequency side times vacuum of all ,negative frequency oscillators as well. So therefore, the full vacuum will be annihilated by both A's and B's. But the story is problematic even after that. Though I have a vacuum state, now I can raise as many A's positive frequency things and as many B's negative frequency thing independently. There is no reason that ,a and b should be correlated I can think of a different numbers operator in the positive frequency sector and different operator number excitation in the negative sector this is like two oscillators ground state I can have either zero or one or I can have a one or zero I can have a two here or a zero here ,just like different p's oscillator structure did not command what other oscillators excitation should be similarly positive frequency sector does not command what has to be excitation in the negative energy sector how many particles i could should keep in oscillators of one kind does not determine how many oscillators how many particles of those oscillators i should keep in the negative frequency side so therefore these and these are independent excitations. And therefore, there is a problematic possibility that I can excite the negative frequency particles more than the

positive frequency particles and I will lead to a Hamiltonian which is negative. And if I keep on increasing the negative frequency oscillator sector more and more I will get more and more negative Hamiltonian and therefore negative energy states this Hamiltonian becomes unbounded from below because in principle I can excite infinitely many particles in the negative energy sector and no particle in the positive energy sector positive frequency sector or finite particles in positive frequency sector and yet will be Hamiltonian will become – infinity. So at the end of the day doing all the field theory everything I am back to square one where my Hamiltonian has become unbounded from below. So the all clever tricks which we played to convert things into operators in the hope of getting rid of negative frequency solutions does not pay off. It did quantization for scalar field wonderfully. But for spinor field, we are led to the same falsehood, same unphysicality which we were dealing with in the case of quantum mechanics. So quantum field theory has not helped us as far as spinor fields, spinor particles are concerned. How to handle this fact, whether these spinor things really exist or not? Or there is something in which we can do in the quantization to save this story will be discussed in the next class. Or there is something in which we can do in the quantization to save this story will be discussed in the next class. The oscillator quantization does not work for these particles. These are some new particles. They are not behaving the same way scalar particles would have behaved.

So, I stop here.