Foundation of Quantum Theory: Relativistic Approach

Electromagnetic Field Quantization 1.2

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Electrodynamic degrees of freedom

Lecture- 27

So, today we will expand our discussion on the structure of electromagnetic theory and we will try to see how do we gradually move towards the usual quantization which we have learnt for other two fields namely the scalar field as well as the Dirac field.

★ $\Box A^\mu = 0$ (In vacuum)

Subject to $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$

$\Rightarrow \partial_\mu A^\mu = 0$

True in one frame ⇒ Remains true in all frames related via LT.

Quantize A^μ ?

The four set of Maxwell equations can be written in terms of A^μ.

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow -\vec{\nabla}\cdot\left(\nabla\phi + \frac{\partial \bar{A}}{\partial t}\right) = 0$

$\Rightarrow \Box\phi = 0 \Rightarrow \Box A^0 = 0$

$\vec{\nabla}\times\vec{B} = 0 \Rightarrow \Box\vec{A} = 0$

▶ $\Box A^\mu = 0$ is two set of Maxwell's eqn.

What about the remaining two sets ?

$\nabla\cdot\vec{B} = 0$ (Automatically true if $\bar{B} = \bar{\nabla}\times\bar{A}$)

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow -\bar{\nabla}\times\left(\nabla\phi + \frac{\partial \bar{A}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\times\bar{A})$

(Automatically true)

$\star \Box A^\mu = 0$ (In vacuum)
which is subject to

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \psi}{\partial t} = 0$$

$$\Rightarrow \partial_\mu A^\mu = 0$$

True in one frame ⇒Remains true in all frames related via LT.
Quantize A^μ ?

The four set of Maxwell equations can be written in terms of A^μ.

$$\nabla \dot{\vec{E}} = 0 \quad \Rightarrow \quad -\nabla \cdot (\nabla \phi + \frac{\partial \vec{A}}{\partial t}) = 0$$

$$\Rightarrow \Box \phi = 0 \Rightarrow \quad \Box A^\mu = 0$$

► $\Box A^\mu = 0$ is two set of Maxwell's eqn.

$\nabla \cdot \vec{B} = 0$ (Automatically true $if\, \vec{B} = \vec{\nabla} \times \vec{A}$)

vec ∇ × vec E = -*{∂ **vec B** } over {∂*t*}*

$$\Rightarrow \quad -\vec{\nabla} \times (\nabla \phi + \frac{\partial \vec{A}}{\partial t}) \quad = \quad \frac{-\partial (\vec{\nabla} \times \vec{A})}{\partial t}$$

(Automatically true)

So, therefore, this is a theory of vector degrees of freedom. So, A mus constitute that and we saw that in a particular gauge which is the Lorentz gauge over here, which is known named as the Lorentz gauge over here. They satisfy the wave equation, the usual Klein-Gordon equation which we have learned for scalar field as well as for Dirac field. The vector fields also seem to satisfy the same kind of Klein-Gordon equation structure subject to a condition which is the gauge condition which we are talking about is the Lorentz gauge. Okay low range gauge previously we had written in terms of a divergence of the three $\vec{A}$ and $1/c^2$ · the divergence of the time derivative of the scalar potential ϕ they should add up to zero that was the condition we obtained in order to get the wave equation for all of the amuse component now we can if you pay attention to the fact that this equation which is the gauge equation can nicely be written in a form which is $\partial_{\mu\mu}$ if you open it up you will get this equation back What is the benefit of writing it in this way?
One thing it is compact, but more than that you can see that this is a scalar equation. There is a covector and vector summed over. So ultimately it transforms like a scalar. That means it does not transform at all. So if you satisfy this condition in one frame, one inertial frame, this condition will remain true under all inertial transformations generated via Lorentz transformations. So therefore, if we have a wave equation, for the degrees of freedom A_μ then subject to one gauge condition this condition is realized in a particular gauge then the statement remains true in all inertial frame which are connected from the your initial inertial frame via Lorentz transformation okay so so far so good we have a equation which can be trusted which is transforming like a vector and the wave equation remains the wave equation because the gauge condition under which it has been obtained remains the true gauge condition even if I go to another inertial frame. Now the idea is should we go ahead and quantize A_μ just

like we had done before. We need to see whether if we quantize A_μ do I get something. which is problematic at hand for example for Klein-Gordon case the scalar field which was satisfying this equation got quantized very effortlessly and there was nothing serious about it but when we try to do the same thing for the Dirac or Spiner field there we realize that we do not have to rely on the classical structure we actually have to give it up and envisage some new kind of relations between the phase space variable which is unseen for a classical system. So, that came about as the antI commutator.

So, let us try to see whether if I go ahead and quantize this 4 degrees of freedom, do I get a meaningful consistent Lorentz invariant description without any problem, that means it is Hamiltonian and other aspects are under control or not. So, we will go towards the quantization of this, but let us spend some time in understanding this structure of the A_μ with slightly more rigor. Now we believe this wave equation is the story of electromagnetism. So somehow the whole of electromagnetism should be contained within this. It is not that such that I am just quantizing some effective equation which I have landed upon accidentally. Because in deriving this equation remember we use Maxwell's equations one after another. So, idea is whatever I am going to quantize, whatever I am going to do should be describing not just one equation of coming from the Maxwell's equations, but the whole of Maxwell's equations set should be contained by a mu. So, if I can quantize a mu, I should be getting a result that ultimately all electromagnetic field component get quantized as well. So therefore, we should be aware that how to write things which appears in electrodynamics in terms of A_μ and what does it mean to quantize A_μ whether the full action which gives rise to the equations of motion which is consistent with Maxwell's equation is getting quantized or some other Lagrangian is there. So we would hope that we get some Lagrangian whose variational approach gives me equations of motion which are just the Maxwell's equations. All right, now there is, if I pay attention towards the structure of equations of motion of this A mu, that means the physical quantity is made up out of it, electric field and magnetic field, they nicely separate into two sets of Maxwell's equation each. So, first usual way of looking at Maxwell's equation is look first for the equations of motion for electric field and then write down the same thing for magnetic field. However, if I just look on this particular structure, divergence of E and curl of B, this set of equation, this is a scalar equation, this is equal to 0 and while this is a vector equation which is three component of it. Now, look at this equation individually. If I just open it up, the divergence of electric field is equivalent to divergence of negative of a particular quantity which was gradient of ϕ and time derivative of A. So, this was the definition of electric field in electrodynamics. Remember, if electrodynamics is electrostatics, meaning there is no dynamics, then the time derivative part goes away and I get the case of electrostatics where electric field is just – of the gradient of some scalar potential. But if electric field is dynamically changing, then this equation is true, that electric field is gradient of ϕ + some temporal derivative of a vector potential, that should be 0. This is good. Now take this divergence inside divergence will go inside and hit this gradient term and convert into laplacian of Π_i will get a laplacian of Π from the first term the second term goes and hits the a so I will have a time derivative of divergence of a again I have taken the divergence inside because partial derivatives commute order doesn't match which order I do order doesn't create a matter does not matter and does not create any problem for you that first you do temporal derivative then you do partial derivative with respect to space or vice versa you get the same result. So, that is why I have taken a divergence inside the a and then I have used the gauge condition.

The gauge condition is the divergence of A is 1 by c – of $1/c^2$ and temporal derivative. that makes this temporal derivative of a divergence of a is equal to – of $1/c^2$ double derivative of a scalar field with respect to time. So, ultimately there was a Laplacian in space and then a temporal derivative with respect to the temporal derivative with respect to time came about and there was a – sign already outside. So, ultimately you will get a structure that $1/c^2\ \partial/\partial t^2$ a positive sign and then – of Laplacian everything hitting the ϕ. It should be 0 which is $\Box\phi$ is equal to 0. $\Box\phi$ remember up to a division by c was A^0, A upper index 0 which is 0th component of the vector potential, 4 vector potential. So, the 0th component satisfies this equation. And again the second Maxwell's equation second set of Maxwell's equation which we are considering is the vector class curl of b is equal to zero you can do the algebra again write down b is equal to curl of – a the curl of a use the curl of curl of a and into something and use the gauge condition again you will be able to show that that equation is also equivalent to writing that the box writing as that $\Box A\,\vec{A}$ is also 0. So, you see individually all the components of the vector potential A_μ this equation are coming just from the two Maxwell's equation two sets of Maxwell's equation.So, you see individually all the components of the vector potential A_μ this equation are coming just from the two Maxwell's equation two sets of Maxwell's equation. But Maxwell's equation is set of four equations four class of equation scalar equation for a vector equation for a scalar equation for b vector equation of b. So therefore whether if I quantize this theory up to this much, am I quantizing the full electromagnetic theory only half part of it. That is the second two Maxwell's equation as well only and what do we do with the remaining two? Actually now if we see the way we have defined electric field and other things, the gauge condition whatever we have wrote down, this does not require you to do anything. The two remaining sets of Maxwell's equation get satisfied automatically just from their geometric structure. Why is that? First remember we have decided to call magnetic field as a curl of A. So therefore divergence of B is equal to 0 is automatic.Once you write like this divergence of B cannot be anything else but it has to be 0. So the third Maxwell's equation let us say is just an identity corresponding to this definition of B. So, you do not have to do anything. You do not have to solve anything. Divergence of B will always be 0 given the structure of P. What about the last remaining Maxwell's equation? $\nabla \times E$ is equal to $-\frac{\partial \vec{B}}{\partial t}$. Again, write down the definitions of E here and definition of B over here and use for the fact that the gradient of a curl is 0. Sorry, curl of a gradient is 0. That is also geometrically true curl of a gradient is always zero you do not have to do anything or solve anything then this both sides become the same thing and they are automatically true so remaining two set of Maxwell's equations are not governed by any equation of motion like wave equation they are just geometrical identities they will be true even if the A_μ did not satisfy the wave equation in deriving this we never used the wave equation solution or anything. So the two sets of Maxwell's equations are not equations of motion they are geometrical identities. The remaining two sets of Maxwell's equation here they are physical conditions they require certain conditions on A_μ how will it propagate with what speed and what not okay very well. So, now with this information we try to go forward and try to see the structure of a particular Lagrangian which would be able to give me the first two sets of Maxwell's equation which are only the equations of motion or the box $of\,\mu$ which is made up from these two is effectively this equation and this equation collectively are this equation and that equation vector part meaning the zeroth part of the wave equation is the first Maxwell's equation.

If $A^\mu = (\phi/c, \vec{A}) \Rightarrow ?\ A_\mu = \sum \eta_{\mu\nu} A^\nu$ ✓

$= (-\phi/c, \vec{A})$

$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \Rightarrow E_x = +\left(\frac{\partial A_0}{\partial x} - \frac{\partial A_x}{\partial x^0}\right)$ ✓

$E_y = +\left(\frac{\partial A_0}{\partial y} - \frac{\partial A_y}{\partial x^0}\right)$ ✓

$E_z = +\left(\frac{\partial A_0}{\partial z} - \frac{\partial A_z}{\partial x^0}\right)$ ✓ ✓

$E_i = \partial_i A_0 - \partial_0 A_i$

$\vec{B} = \vec{\nabla} \times \vec{A}$

$B_x = \partial_y A_z - \partial_z A_y$

$B_y = \partial_z A_x - \partial_x A_z$

$B_z = \partial_x A_y - \partial_y A_x \Rightarrow B_i = \epsilon_i{}^{jk} \partial_j A_k$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$F_{0i} = -E_i \quad , \quad F_{ij} = \epsilon_{ij}{}^k B_k$$

Also $F_{\mu\nu}$ is gauge invariant

If $\vec{A}=(\frac{\phi}{c},\vec{J})\Rightarrow A_{\mu}=\sum_{\nu} A^{\mu\nu}=(\frac{-\phi}{c},\vec{J})$

vec E = - ∇ϕ - {∂ **vec A**} *over* {∂t}

$\Rightarrow$

$$E_x=+\left(\frac{\partial A_0}{\partial x}-\frac{\partial A_x}{\partial x_0}\right)$$

$$E_y=+\left(\frac{\partial A_0}{\partial y}-\frac{\partial A_y}{\partial x_0}\right)$$

$$E_z=+\left(\frac{\partial A_0}{\partial z}-\frac{\partial A_z}{\partial x_0}\right)$$

$$E_i=\partial_i A_0-\partial_0 A_i$$

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

$$B_x=\partial_y A_z-\partial_z A_y$$

$$B_x=\partial_z A_x-\partial_x A_x$$

$$B_z=\partial_x A_y-\partial_y A_x$$

$$\Rightarrow \quad B_i=\varepsilon_i^{jk}\partial_j A_k$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

$$F_{oi}=-E_i, F_{ij}=\varepsilon_{ij}^{k} B_k$$

Also $F_{\mu\nu}$ is Guage invariant.

The remaining three vector part is the third Maxwell's equation so to say. So, therefore those are only the dynamical equation remaining the second Maxwell's equation divergence of B is equal to 0 and $\vec{\nabla}\times E$ is equal to – $\frac{\partial\vec{B}}{\partial t}$, which is the fourth Maxwell's equation. They are just geometric identities. So, we should look for a Lagrangian, which generates for me correct first two equations of Maxwell's, two equations of Maxwell's, not first two, first and third, which is equivalent to finding a Lagrangian, which gives me the $\Box A_\mu$ is equal to G. I hope this is clear that all the dynamics of electrodynamics is contained in the first and the third Maxwell equation and this is covered by overall by this wave equation. The remaining two Maxwell's equations are not dynamical equations. They have to be true at all time because of a geometric identity. So, let us try to get a good meaningful Lagrangian in order to write down that and from varying that let us hope that we will get the box *of* μ is equal to 0. Now, in order to do that it is convenient to write down the variables amuse and physical quantities electric field and magnetic field in a particular appropriate relationships. You might have seen it in the covariant discussion or Lagrangian formulation of electrodynamics already. If not, I will just go through very quickly over it and you should look at any good book of electrodynamics to just revise or re-look at this idea. So we are look going for vector potential A_μ whose components are the scalar potential and the current density uh sorry the a over here I have written wrongly the the vector potential a I should have written as a somehow I have written erroneously as J but anyway this is fine I have

corrected it so I have a vector potential whose first component is scalar potential and the remaining three components are the vector potential generating the magnetic field okay so in terms of electric field how do I express electric field in terms of these variables electric field remember is gradient of I and temporal derivative of this and with a – sign. So you do that. Remember this object up to a 1 over *c* will be the that object up to a *1/c* would be the first component A_0.. So ultimately you can see that we are writing things like electric field is like $\frac{\partial A_0}{\partial x} - \frac{\partial A_x}{\partial x_0}$. This is the temporal derivative and the component x component will come from x component of this and x component of that which is equivalent to this. Similarly, if I write down what is the y component of this, I will plug in the y component of the gradient, the y component of the temporal derivative which is over here. And similarly for z component, I will just take the z component from here, the z component from here and the z component from here, the second kind temporal derivative and the spatial derivative itself. In this process I have generated things which are lower index object you see in doing so you will realize that our discussion from special relativity the lower index objects are to be obtained from a contraction of the upper index quantities through the Minkowski matrix so if a upper μ was given as ϕ over c and vector potential a I should obtain the lower index object as a lower μ as $\eta_{\mu\nu}$ a nu. It should be a ν over here. ν is being summed over. So there are a couple of typos in this. You should correct in your notes. Now we see that the electric field is made from the a with their anti-symmetric derivatives. That means if 0 appears here, x appears here. In the next term, x appears here and 0 appears here. For ey, 0 appears here, y appears here.
In the next term, y appears up, 0 appears down. And similarly for z. So, they are made from the anti-symmetric combinations of derivative terms. So, electric field is made from the derivative of x and 0, meaning indices are appearing at x and 0. For y, the indices are appearing y and 0. For z, indices are appearing as z and 0. So, overall electric field can be written in terms of A. They are made from $A_i$and A_0.. I index is the electric field index and you would have this structure. So, this is how we write down the electric field in terms of the vector potentials. What about magnetic field? Magnetic field we know they are obtainable from the curl of A directly. So, x component of magnetic field will *be* $\partial_y A_z - \partial_z A_y$. So, whatever we discuss about the electric field, whatever we discussed about the electric field, the same indices were appearing in the a and in the derivative. In the case of magnetic field, you see if x is appearing here, none of the derivatives or the components are x. They are the other two indices. Similarly, for y, the z and x will be appearing over here. None of the indices in this derivatives would be y and similarly for z. So, together due to its curled product kind of structure, you see already they are anti-symmetric in y and z exchange. So, anti-symmetric structure is maintained like electric field, but the indices structure is slightly different. For electric field, the indices appear in the derivatives and the terms. In magnetic field, the indices does not appear in the derivative and the terms. So, this is compactly written in the equation like this. This is written with something called the Levy's *cβ* tensor.
That tensor is ε_{ijk}. ijk can take various values. The idea about this that one normal ordering will be 1, 1, 2, 3. Cyclic ordering of that also be 1. For example, 3 goes to 1, 1 goes to 2 and 2 goes to 3 which will be 3, 1, 2 that will be also 1 and so on. That ε_{231}, for example, will also be 1. However, if I exchange any of them, ε_{132}, which is just 2 and 3 interchanged, that is – 1. Similarly, ε_{321}, that is all equal to – 1. So, this is anti-symmetric tensor in all of its indices. If J and k are interchanged, it will pick up a – sign. If I and J are interchanged, it will pick up a – sign. If I and k are interchanged, it will pick up – sign. So using that, you can see that these terms over here can be written in terms of Levy's evita symbol or Levy's evita tensor like this. So collectively, I see both the magnetic field and electric field are coming from anti-symmetric derivatives of the vector potential. So, I am going to define set of all anti-symmetric derivatives of the vector potential as something that is called the field strength tensor . is the collection of all anti-symmetric derivatives of A and vice versa meaning $\partial_\mu A_\mu - \partial_\mu A_\mu$. So, you see some of the components of this will be electric field because they are some components of the anti-

symmetric version and some of the components will be magnetic field because they are also anti-symmetric derivative of the same kind. You can quickly identify and realize that F_{0i}, if I put 0 here, I here, then also I would have A_0. here, I here. Then ultimately, I will get here, I here and 0 here. That is equivalent to my definition of electric field's ith component with a – sign. So f lower 0I is – ei. fij is ε_{ijk} bk, meaning F_{12}, if I write, it will be 12. The third index has to be summed over. So, if I put 1, 2, k is getting summed over. It can take value 0, 3, 1 and what not. But any repeated index for anti-symmetric tensor is 0, meaning ε 1, 2, 1 is 0, ε 1, 2, 2 is 0. Any repeated index would be 0. So, when I do this kind of things, you will see that the k will take value only different from ij. So, F_{12} will be the 3, which I have written like this. So, this way we have written, we can write down from this anti-symmetric derivatives all the terms over here, which we have. So that is one thing which we can compactly write all the electric and magnetic field component using this field strength tensor . One additional benefit of this is that apart from helping us in compactly writing electric and magnetic field components, this object is gauge free, it is gauge invariant. That if I did the gauge fixing we had done, we are supposed to do in order to get the wave equation. But suppose we are not in that gauge where wave equation is manifest, even then the definitions of does not change. So, how to see that? Remember when we were discussing about the gauge freedom, we were discussing that $\vec{A}$ was supposed to undergo a change as $\vec{A}$ + a gradient of a scalar. While the A_0. which was the ϕ the scalar potential that is under that same transformation is supposed to change by the temporal derivative of the thing with 1 over c. So, together we can see that it is asking us to change like A_μ goes to $A_\mu + \partial_\mu\phi\partial_\mu$ or something like that. So, you see A_0. will change with 0th derivative which is time derivative and there will be negative sign between up index and down index if you are careful about η_{00} is – 1 and η_{ii} is 1. And the special derivatives will change with special derivative like from this term. It so happens if I write $\partial_\mu A_\mu - \partial_\mu A_\mu$. That is the definition of , you can realize that this object particularly does not change at all under gauge transformation. So if A_μ changes like this, then the derivative of A_μ will change like derivative of first term · derivative of the second term. And similarly here, the $\partial_\nu A_\mu$ will also change in a particular way. And again an extra term will come from the derivative of f, but you will realize that both f dependent term coming from the first term and the second term are exactly the same. And partial derivatives commute using that fact you can show that the under the change of a gauge does not change at all. So, therefore it is manifestly gauge invariant object. I do not have to worry about which gauge I am writing electric field or which gauge I am writing the magnetic field once I have set down the definitions of F_{0i} as magnetic electric field and F_{ij} as magnetic field. And if I write down a Lagrangian for example in those things or equations of motion in terms of , those equations of motions will also be gauge invariant.

If A_μ transforms via (Λ^{-1})

$F_{\mu\nu}$ transforms via $(\Lambda^{-1})(\Lambda^{-1})$

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$$

The Maxwell's two equations of motion are

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \checkmark$$

The remaining two (set) are geometric identities

$$^*F^{\mu\nu} \equiv \frac{1}{2!}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

$$^*F^{01} = \frac{1}{2!}\left(\varepsilon^{0123}F_{23} + \varepsilon^{0132}F_{32}\right)$$

$$= F_{23} = B_1$$

$$^*F^{02} = -F_{13} = B_2$$

$$^*F^{03} = F_{12} = B_3$$

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow$

$$\therefore \partial_i\, {}^*F^{0i} = 0 \Rightarrow \frac{1}{2!}\partial_i\,\varepsilon^{0i\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \frac{1}{2!}\varepsilon^{0i\mu\nu}(\underbrace{\partial_i\partial_\mu}A_\nu - \underbrace{\partial_i\partial_\nu}A_\mu) = 0$$

Symmetric (Automatic)

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \varepsilon^{ijk}\partial_j(\partial_k A_0 - \partial_0 A_k) = -\partial_0\,\varepsilon^{ijk}(\partial_j A_k - \partial_k A_j)$$

Automatically true

If A_μ transforms via $(\Lambda^{-1}) F_{\mu\nu}$ transforms via $(\Lambda^{-1})(\Lambda^{-1})$

$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$
The Maxwell's two equations of motion are

$$\partial_\mu F^{\mu\nu} = 0$$

The remaining two (set) are geometric identities.

★ $F^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$

★ $F^{01} = \frac{1}{2!}\left(\varepsilon^{0123} F_{23} + \varepsilon^{0132} F_{22}\right)$

If A_μ transforms via (Λ^{-1})
$F_{\mu\nu}$ transforms via $(\Lambda^{-1})(\Lambda^{-1})$

$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$
The Maxwell's two equations of motion are

$$\partial_\mu F^{\mu\nu} = 0$$

The remaining two (set) are geometric identities.

$\star \quad F^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$

$\star \quad F^{01} = \frac{1}{2!}\left(\varepsilon^{0123} F_{23} + \varepsilon^{0132} F_{22}\right)$

$= F_{23} = B_1$

$\star F^{02} = -F_{13} = B_2$

$\star F^{03} = -F_{12} = B_2$

$\nabla \cdot \vec{B} = 0$

$\therefore \quad \partial_i \star F^{01} = 0 \quad \Rightarrow \quad \frac{1}{2!}\varepsilon^{0i\mu\nu}\left(\partial_\nu A_\nu - \partial_\nu A_\mu\right)$

$= \quad \frac{1}{2!}\left(\partial_i \partial_\nu A_\nu - \partial_i \partial_\nu A_\mu\right) = 0$

$\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$

$\Rightarrow \quad \varepsilon^{ijk} \partial_j\left(\partial_k A_0 - \partial_0 A_k\right) = \partial_0 \varepsilon^{ijk}\left(\partial_j A_k - \partial_k A_j\right)$

Automatically true.

Previously I had written the wave equation which was true only in a particular gauge. But suppose I am able to write this thing in terms of function of some , then I know whatever equation that is satisfying it is a gauge invariant equation. So, if I can express this whole equation in terms of the gauge, if I can express this equation a mu, $\Box A_\mu$ is equal to 0 in terms of , then I know that that equation of motion is also gauge invariant. So, therefore, I would not have to worry about which gauge these equations are written. So, this is a gauge invariant object. However, it is a rank two tensor. Lower is rank zero two tensor, upper is rank two zero tensor in terms of the low range transformation. So, A_μ being a vector transforms as Λ^{-1} as like this. being a second rank tensor of the lower kind, it will transform with two Λ^{-1} as we had discussed. And we had discussed as well how to make upper rank tensor from a lower rank tensor through contraction via the $\eta_{\mu\nu}$. So, that is how I will get the first index, second index here, using the first index here, second index here and the remaining indices summing over. Now, the way we have seen it through that it is a tensor quantity, it is anti-symmetric, it is gauge invariant. The last bit is to write down the wave equation which we have. So, remember we have a equation of motion for electromagnetic fields, dynamical equations of motions are these. The claim is that two equations which are dynamical equations of motions are just ∂_μ is equal to 0. You can verify before you open it up, If you open it up, use the fact that it is anti-symmetric in its structure, you will be able to prove that these two equations are one and the same thing. So therefore, whatever Lagrangian I am going to write down, either I can demand that under variation of that Lagrangian I get this equation or equivalently I can ask under the variation of that Lagrangian I should get this equation of motion. So those two will be the dynamical equations coming out from some Lagrangian variation. And as we discussed, the remaining two set of Maxwell's equations are not dynamical equations. They are not supposed to come from some Lagrangian. Rather, they come from geometry. So, therefore, you can write down something called a dual tensor to by contraction with respect to the four damaged, four index Levy's β tensor $\varepsilon\alpha_{\mu\nu\alpha\beta.}$ So, this is also the same kind of levy symmetry which we had previously seen. It is anti-symmetric in pair replacement of any of two indices and none of the pairs can, none of the indices can get repeated. It will become cyclic 1, 2, 3, 4 is 1 or rather 0, 1, 2, 3 is 1 but 0, 1, 3, 2 is – 1 and so on. So, using that I can construct something called a dual or hodge dual or a vector dual Or this is the dual to the F_{μ} Newton cell. Now this dual F_{μ} Newton cell equation which is something like this. The divergence with respect to the second index of Hodge duals F_{0I} will be 0 and this is just from the symmetry. If this is anti-symmetric, this is anti-symmetric, this is also anti-symmetric. And that anti-symmetric derivative put together overall gives you something geometrical vanishing object which is equivalent to divergence of B is equal to 0. When deriving divergence of B is equal to 0 in terms of either Hodge dual or directly with a multiplication of this geometric tensor, you will get the divergence of B is 0 automatically irrespective of any Lagrangian. For all Lagrangians this will come true. Similarly, for $\vec{\nabla} \times E$ is equal to – $\partial p/\partial t$ again I write down the equation which is over here and you can see that this is also equivalent to the another piece coming from the divergence of this Hodge dual FPU new is equal to 0. You can verify this is just a convenient way of writing do not get much dragged up into this tensor notation because this is not the topic of the discussion for today. But I am just writing in tensor notation for people to become familiar. Otherwise, we have just seen previously that this is just a manifestly zero without anything physical we have to do. This will remain always zero as long as b is a curl class vector and the definition of e what we had adopted remains there, the same. We do not care how they evolve in time, what Lagrangian drives them, nothing. These two set of equations will be true automatically. So therefore, the dynamical equations which we are going to go after is this equation ∂_{μ} . These are four equations because $_{\mu}$ has been summed over and ν is a free index. Free index can take various values, four values actually 0, 1, 2, 3. The 0 value is divergence of E Coulomb's law and the ν is equal to the spatial index would be the curl of B equation which is Lenz law let us say. So, the flux law if you want to call it. Alright, so with this information, I am just going to propose a Lagrangian just like I have done the same thing for scalar field or the Dirac field. I give you or prescribe you a Lagrangian which achieves for you whatever is required. Again, I do not claim as of now that it is the only Lagrangian which will be able to do that. But simple most, just like for scalar field or the Dirac field, we wrote down some simple most Lagrangian would be able to do that. So this proposal where I take and self-contract it, that means I take , take its upper index version and contract them together. This is like writing and this is writing $f_{\alpha\beta}$ and using double copies of ηs, MinkowskI metric, I make them raised indices so that , come about. And – 1 by 4 is put to make things consistent with the equations of motion which emerges out at the end. If you do that, you can verify that under the variation of its action. First let us do the variation with respect to a α. This a α will go in the whole action and search for wherever a α is appearing. And a α is not appearing anywhere as you can see, it is just appearing in the derivatives. So, what you can do, you can convert one of the derivative terms · this thing into a total derivative of this · this, which is , upper – The derivative hitting the which is this term and a nu. So together this two term generate for you this whole term. This · that. And similarly together these two terms generate this · that.

$$S = -\frac{1}{4} \int d^4x \, (F_{\mu\nu} F^{\mu\nu})$$

$$= -\frac{1}{4} \int d^4x \, (\partial_\mu A_\nu - \partial_\nu A_\mu) \, \eta^{\mu\alpha} \eta^{\nu\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

$$= -\frac{1}{4} \int d^4x \Big[\big[\partial_\mu (A_\nu F^{\mu\nu}) - \partial_\nu (A_\mu F^{\mu\nu}) \big] - A_\nu \partial_\mu F^{\mu\nu} + A_\mu \partial_\nu F^{\mu\nu} \Big]$$

$$\boxed{S = \frac{1}{2} \int d^4x \, [\vec{E}^2 - \vec{B}^2]}$$

$$\frac{\delta S}{\delta A_\alpha} = \partial_\mu F^{\mu\nu} \delta_{\nu\alpha} - \partial_\nu F^{\mu\nu} \delta_{\mu\alpha}$$

$$\Rightarrow \partial_\mu F^{\mu\alpha} = 0$$

$\Rightarrow$ $\partial_0 A_0$ never appears in the Lagrangian

$$\Pi^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i} = -F^{0i} = E^i = -E_i = -(\partial_0 A_i - \partial_i A_0)$$

Electric field is not a four vector

$$\Pi^0 = 0 \qquad \{ A_i(\vec{x}), E^j(\vec{x}') \} = \delta(\vec{x} - \vec{x}') \, \delta_i^j$$

$$\mathcal{H} = \Pi^i \dot{A}_i - \mathcal{L}$$

$$= \Pi^i (\Pi^i + \partial_i A_0) - \mathcal{L}$$

$$= \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + (\partial_i A_0) E^i$$

$$\boxed{H = \int d^4x \Big[\frac{1}{2} (\vec{E}^2 + \vec{B}^2) - A_0 \, \partial_i E^i \Big]}$$

$$S=\frac{-1}{4}\int d^4x(F_{\mu\nu}F^{\mu\nu})$$

$$= \frac{-1}{4}\int d^4x(\partial_\mu A_\nu-\partial_\nu A_\mu)\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta-\partial_\beta A_\alpha)$$

$$= -[\frac{1}{4}\int d^4x[(\partial_\mu(A_\nu F^{\mu\nu})-\partial_\nu(A_\mu\partial_\nu F^{\mu\nu}))]-A_\nu\partial_\nu F^{\mu\nu}+A_\mu\partial_\nu F^{\mu\nu}]$$

$$S=\frac{1}{2}\int d^4x[\bar{E}^2-\bar{B}^2]$$

$$\frac{\delta S}{\delta A^\alpha}=\partial_\mu F^{\mu\nu}\delta_{\nu\alpha}-\partial_\nu F^{\mu\nu}\delta_{\mu\alpha}$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu}=0$$

⇒∂_0 A_0 never appears in the Langrangian

$$\Pi^i=\frac{\delta L}{\delta \dot{A}_i}=-F^{0i}=E^i=-E^i=-(\partial_0 A_i-\partial_i A_0)$$

$$\{ \quad A_i(\vec{x}),E^j(\vec{x}\,') \quad \} = \quad \delta(x-x')\delta_i^j$$

$$H=\Pi^i\dot{A}_i-L$$

$$= \Pi^i(\pi^i+\partial_i A_0)-L$$

$$= \frac{1}{2}(\vec{E}^2+\vec{B}^2)+(\partial_i A_0)E^i$$

$$H=\int d^4x[\frac{1}{2}(\vec{E}^2+\vec{B}^2)+A_0\partial_i E^i]$$

So I have just conveniently written this. In order to convert things into total derivative – derivative of the first function right now it is and the derivative is hitting a *v* I use the method of integration or differentiation by parts let us say where I shifted the derivative onto the and a *v* became the free function and now I can do the derivative with respect to a *v* which will give me this kind of a structure ultimately four of this this four factor exactly cancels with the four factor which will appear on the derivative. So, this is just a simple exercise you should be able to do and you will see really that indeed the equation of motion which we were after ∂_μ or α whatever you call α is a free index is equal to 0. So, variation of this action indeed gives rise to the equation of motion of the first two or the two Maxwell equations which are dynamical equations. So, this is a good Lagrangian and this is the Lagrangian we will work with. Now if you have done a course of electrodynamics and are familiar with the Lagrangian approach or covariant approach, you would realize immediately that all these terms which are appearing over here or here can nicely be written in terms of electric field and magnetic field as well. Because ultimately tells you electric field and magnetic field. If you write it down, you will realize that what we are writing is nothing but $E^2 - B^2$. Again, those of you, you are familiar with covariant Lagrangian formulation or a covariant formulation of electrodynamics. This $E^2 - B^2$ in the units of c is equal to 1. By the way, I have put down ε_0 and c and μ_0, all things to 1 as convenient in the usual notation. So, in these units, $E^2 - B^2$ is a low range invariant quantity. That is why the Lagrangian we have written or the action we have written is Lorentz invariant. So, you do get the equation of motion correctly from this. And one more thing you will realize in this, $\partial_0 A_0$. never appears. Because $\partial_0 A_0$. will demand this index and that index to be the same. This will make this two indices also the same. None of the diagonals are supposed to appear in any of the anti-symmetric tensor. Therefore, this is a diagonal entry, $\partial_0 A_0$., it will never appear. *F* is *0, F* is *0, F* is *0*. So, therefore, when F_{00} is 0, that means this is the place where $\partial_0 A_0$. would have appeared. But it is 0, therefore it does not appear. So, two things, I have a well-defined Lagrangian and I have an appearance that $\partial_0 A_0$. never appears. That means this is not a dynamical theory for A_0.. A_0.'s derivative never appears in the Lagrangian. With this information, let us try to find out the momenta corresponding to the variable Ai. I take the Lagrangian which is this roughly or which is this over here with 1 by 4 factor. And you can do the derivative of Lagrangian with respect to A_i . Do this again just like we have done for equation of motion here. And you will find out that you get F_{0I} with a negative sign which is equivalent to electric field. So the conjugate momenta corresponding to the A_i, spatial A_i, are the electric fields. A_i and electric field are conjugate pairs, so they will have a phase-space structure. ij, because A_i will talk to its own electric field component, it will not talk to y or z if I is x. So this is ensured by the Kronecker δ and position δ function is done like that. This is what the Poisson bracket level. Again, I am rushing through, but I am just writing the Hamiltonian of the same thing in the same approach. Here also Π_0 will be 0, because in the Lagrangian there is no term corresponding to $\partial\dot{A}_0$. $\partial\dot{A}_0$. dot is never appearing, so Π_0 is 0. So, if you write down $\Pi_\mu\ \dot{A}_\mu$ and then work it out, this is equivalent to $\Pi_i\dot{A}_i$ and then working out, because Π_0 never appears in the Lagrangian. So this is the Hamiltonian of the system, again you have this structure converting $\dot{A}_{i,}$ which is appearing in this relation, writing in terms of Π_i and the special derivative of A_0., you can convert this back and you can see that the total Hamiltonian has become the $E^2 + B^2$ + a total derivative term, not total derivative, divergence term of A_0.. That is the Hamiltonian density. That means total Hamiltonian will be just integration over the spatial volume d^3x of this Hamiltonian density. And in this, I will have a $E^2 + B^2$. If you remember, this is energy density of electromagnetic wave, which integrated over d^3x should give you total energy. But I have an extra term, which is $\partial_i A_0$. initially, $\partial_i A_0$. and d^3x again use method of converting the derivatives into total derivative and on A_0. you can flip this the derivative on a A_0. into derivative on the eis eis are coming along with it so use again the same method which we did flip the derivative using a total derivative term and you will get a total Hamiltonian apart from total energy density contains an extra term which is very useful as we will see later on in identifying a proper gauge and other things and in that case what would be the total Hamiltonian. So this is the classical Hamiltonian as of now and we will try to see whether there is a case for quantum mechanics when we bring to it, whether we have reason to worry about this Hamiltonian as Spinors would have done. They got us into problem so that I have to modify the commutation relation. So, with this extra term I have to be really careful because this runs the risk of making the total Hamiltonian negative. Because other two terms are positive if this term were not there then I am through. So, therefore I have to carefully analyze this term and what it does to the Hamiltonian whether it converts it to negative quantity or not and what in quantum domain that we have to be aware about. That we will do in the next class. So, let us stop over here with this information that I have a valid Lagrangian and a Hamiltonian out of it and let us see what it gets us when we go to the quantum programming. So, let us stop here.